

Lecture Notes on **Conformal Field Theory**

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The following set of notes on conformal field theory (CFT) are based on the book *Conformal Field Theory* by Philippe Di Francesco, Pierre Mathieu and David Sénéchal. It is an attempt at making the foundational chapters (4-6) in the book more palatable for a first read; advanced content has been omitted if it was deemed unnecessary to understand the basics of CFT, verification of identities and other problems left to the reader have been explicitly worked out and brief reviews of relevant prerequisite topics in QFT have been added before sections that require some understanding of them. No pretense of originality is made, these notes borrow conventions, definitions and even explanations verbatim; the policy is to have a clear and concise introduction to CFT!

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1 Global Conformal Invariance

1.1 The Conformal Group

Consider the metric tensor $g_{\mu\nu}(x)$ of a d dimensional spacetime. A coordinate transformation $x \rightarrow x'$ will change the form of the tensor; a conformal transformation is such an invertible coordinate mapping that leaves the metric invariant up to a scale factor $\Lambda(x) \neq 0$,

$$\boxed{g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x)} \quad (1)$$

It is clear that the set of all conformal transformations form a group; evidently, the action of two successive conformal transformations is also a conformal transformation, the outcome of multiple conformal transformations is associative (the scale factors just multiply), the inverse of a coordinate transformation is just the conformal transformation with the reciprocal scale factor, and the identity is just the identity transformation. Notice how the Poincaré group is a subgroup of the conformal group, it corresponds to the transformations with scale factor equal to unity.

1.1.1 Calculating the Scale Factor

Let us now consider what are the consequences of the definition for conformal transformations. Consider an infinitesimal coordinate transformation

$$x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu(x) \quad (2)$$

The change in the metric due to the change in coordinates is

$$\begin{aligned} g_{\mu\nu} &= \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} g'_{\alpha\beta} \\ &= (\delta_\mu^\alpha + \partial_\mu \epsilon^\alpha)(\delta_\nu^\beta + \partial_\nu \epsilon^\beta) g'_{\alpha\beta} \end{aligned} \quad (3)$$

and thus, to first order in ϵ , we have

$$\boxed{g'_{\mu\nu} = g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)} \quad (4)$$

If we want this to be a conformal transformation, then this change in the metric should be proportional to the metric, that is,

$$\boxed{\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x)g_{\mu\nu}} \quad (5)$$

where $f(x)$ is some appropriate (smooth) function. Taking the trace on both sides, we find that

$$\boxed{f(x) = \frac{2}{d} \partial_\mu \epsilon^\mu} \quad (6)$$

Further simplification of Eq. 5 is possible if we assume that the we are deforming flat Minkowski or Euclidean spacetime ($g_{\mu\nu} = \eta_{\mu\nu}$). Take an extra derivative ∂_ρ on both sides of the expression to get

$$\partial_\rho(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \partial_\rho(f(x)\eta_{\mu\nu}) = \eta_{\mu\nu}\partial_\rho f \quad (7)$$

Now, notice that various permutations of the three indices μ, ν, ρ are possible. Enumerating them, we get

$$\begin{aligned} \partial_\rho \partial_\mu \epsilon_\nu + \partial_\rho \partial_\nu \epsilon_\mu &= \eta_{\mu\nu} \partial_\rho f \\ \partial_\mu \partial_\nu \epsilon_\rho + \partial_\mu \partial_\rho \epsilon_\nu &= \eta_{\nu\rho} \partial_\mu f \\ \partial_\nu \partial_\rho \epsilon_\mu + \partial_\nu \partial_\mu \epsilon_\rho &= \eta_{\rho\mu} \partial_\nu f \end{aligned} \quad (8)$$

which gives us

$$2\partial_\mu\partial_\nu\epsilon_\rho = \eta_{\rho\mu}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f \quad (9)$$

Contracting with $\eta^{\mu\nu}$ on both sides, we get

$$2\partial_\mu\partial^\mu\epsilon_\rho = (2-d)\partial_\rho f \quad (10)$$

Taking a ∂_ν on both sides to get

$$2\partial_\nu\Box\epsilon_\mu = (2-d)\partial_\nu\partial_\mu f \quad (11)$$

and comparing this with Eq. 5 with a $\Box$ on both sides, we can substitute Eq. 11 in it to get

$$\eta_{\mu\nu}\Box f(x) = (2-d)\partial_\mu\partial_\nu f(x) \quad (12)$$

which we rewrite in a more indicative form

$$\boxed{\{\eta_{\mu\nu}\partial_\rho\partial^\rho + (d-2)\partial_\mu\partial_\nu\}f(x) = 0} \quad (13)$$

Finally, contracting with $\eta^{\mu\nu}$ on both sides gives us

$$\boxed{(d-1)\partial_\mu\partial^\mu f(x) = 0} \quad (14)$$

and so we have a nice set of equations that determine $f(x)$, and consequently the scale factor $\Lambda(x)$, as

$$\Lambda(x) = 1 + f(x) + O(\epsilon^2) \quad (15)$$

1.1.2 Form of Conformal Coordinate Transformations

Postponing the more involved study of the conformal group for $d = 2$ dimensions for later; we use the constraint equations developed in the previous section to understand which coordinate transformations are allowed for $d \geq 3$ dimensions. According to Eqs. 13 and 14,

$$\partial_\mu\partial^\mu f(x) = 0 \quad (16)$$

for $d \geq 3$, which means that the function $f(x)$ can be at most linear in x ,

$$f(x) = A + B_\mu x^\mu \quad (17)$$

But, if we plug Eq. 17 into Eq. 10, we see that $\partial_\mu\partial_\nu\epsilon_\rho$ is a constant; which means that ϵ is at most quadratic in x ,

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \quad (18)$$

Since all constraint equations for ϵ (that is, Eqs. 5 and 6) have a power series on both sides, we can deal with each power of x separately. With this, we make the following observations:

1. Notice how the constant a_μ has to follow no constraint, as all constraint equations of ϵ are with a derivative. Thus, this is a constraint free constant term, and it amounts to an infinitesimal translation.

2. Plugging the linear term in Eq. 5 and 6, we get

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b_\rho{}^\rho \eta_{\mu\nu} \quad (19)$$

The L.H.S gets rid of the antisymmetric part of the tensor, while the expression itself tells us that the symmetric part is proportional to the metric. Thus, we can write it as

$$b_{\mu\nu} = \alpha \eta_{\mu\nu} + \omega_{\mu\nu} \quad (20)$$

The symmetric part represents an infinitesimal scale transformation; while the antisymmetric part is the usual infinitesimal rigid rotation.

3. Finally, if we turn to the quadratic term, after plugging it into the constraint equation Eq. 10, we get

$$c_{\mu\nu\rho} = \frac{1}{d} (\eta_{\rho\mu} c^\sigma{}_{\sigma\mu} + \eta_{\nu\rho} c^\sigma{}_{\sigma\mu} - \eta_{\mu\nu} c^\sigma{}_{\sigma\rho}) \quad (21)$$

and the corresponding transformation is

$$x'^\mu = x^\mu + 2(x \cdot b)x^\mu - b^\mu x^2 \quad \text{where} \quad b^\mu = \frac{1}{d} c^\sigma{}_{\sigma\mu} \quad (22)$$

This is a type of coordinate transformation that we usually do not encounter; it is given the name *special conformal transformation* (SCT).

This concludes our analysis of types of coordinate transformations allowed under the paradigm of a conformal coordinate transformation for $d \geq 3$ dimensions. A summary of the finite transformations is given below.

(translations)	$x'^\mu = x^\mu + a^\mu$
(dilations)	$x'^\mu = \alpha x^\mu$
(rotations)	$x'^\mu = M^\mu{}_\nu x^\nu$
(SCT)	$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}$

1.1.3 Generators of the Transformations

Now that we know the coordinate transformations, we can easily find out the generators of these transformations. First we recall that the generator of an infinitesimal coordinate transformation parametrized by ω_a is given by (this is just a partial expression, the full generator form will be discussed in the next section)

$$G_a = -i \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \quad (23)$$

Using this, and the various coordinate transformations enumerated in the previous section, we can write down the generators of the coordinate transformations.

1. Consider an infinitesimal translation

$$x'^\mu = x^\mu + \omega^\mu \quad (24)$$

Clearly, the parameter here is ω^μ (the parameter index a is now a spacetime index μ). Thus, we get

$$\frac{\delta x^\mu}{\delta \omega^\nu} = \delta_\nu^\mu \quad (25)$$

$$\boxed{G_a \equiv P_\mu = -i\partial_\mu} \quad (26)$$

2. Consider an infinitesimal Lorentz/rotation transformation given by

$$x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu = x^\mu + \omega_{\rho\nu} \eta^{\rho\mu} x^\nu \quad (27)$$

where the parameter is $\omega_{\alpha\beta}$, the components of the antisymmetric rotation matrix. Using this antisymmetry, we can write

$$\begin{aligned} \delta x^\mu &= \frac{1}{2}(\omega_{\rho\nu} \eta^{\rho\mu} x^\nu - \omega_{\nu\rho} \eta^{\rho\mu} x^\nu) \\ &= \frac{1}{2}(\omega_{\rho\nu} \eta^{\rho\mu} x^\nu - \omega_{\rho\nu} \eta^{\nu\mu} x^\rho) \end{aligned} \quad (28)$$

which leads to

$$\frac{\delta x^\mu}{\delta \omega^{\rho\nu}} = \frac{1}{2}(\eta^{\rho\mu} x^\nu - \eta^{\nu\mu} x^\rho) \quad (29)$$

which means that

$$\boxed{G_a \equiv L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)} \quad (30)$$

is the rotation generator. (Note that the double counting of the indices cancels out the factor of half, more will be elaborated later)

3. Consider an infinitesimal scale transformation

$$x'^\mu = (1 + \omega)x^\mu \quad (31)$$

Clearly, we have

$$\frac{\delta x^\mu}{\delta \omega} = x^\mu \quad (32)$$

which gives

$$\boxed{G_a \equiv D = -ix^\mu \partial_\mu} \quad (33)$$

4. Consider an infinitesimal SCT

$$x'^\mu = x^\mu + 2(x \cdot b)x^\mu - b^\mu x^2 \quad (34)$$

With a little bit of complicated algebra, one can show that

$$\boxed{G_a \equiv K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)} \quad (35)$$

We now define

$$\boxed{\begin{aligned} J_{\mu\nu} &= L_{\mu\nu} & J_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu) \\ J_{-1,0} &= D & J_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu) \end{aligned}} \quad (36)$$

and $J_{ab} = -J_{ba}$ with $a, b \in \{-1, 0, 1, \dots, d\}$. These “new” generators obey the $SO(d+1, 1)$ commutation relations

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} - \eta_{bd}J_{ac} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd}) \quad (37)$$

This shows that there is an isomorphism between the d dimensional conformal group and $SO(d+1, 1)$. We can see that the Poincaré group is a subgroup of this conformal group; which is good, as theories invariant under conformal transformations are automatically Lorentz invariant too.

With this, we conclude the brief introduction to the conformal group.

1.2 Conformal Invariance in Classical Field Theory

If the action of a classical field theory

$$S[\Phi] = \int d^d x \mathcal{L}(\Phi, \partial_\mu \Phi) \quad (38)$$

remains invariant under a transformation, which in general affects both the coordinates and the fields as

$$\begin{aligned} x &\rightarrow x' \\ \Phi(x) &\rightarrow \Phi'(x') \end{aligned} \quad (39)$$

then such a transformation is called a symmetry of the classical theory. Thus, for conformal symmetry, we need to have invariance of the action under conformal transformations.

1.2.1 Infinitesimal Transformations and Generators in Classical Field Theory

Consider the infinitesimal transformations of the coordinates and fields

$$\begin{aligned} x^\mu &\rightarrow x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a} \\ \Phi(x) &\rightarrow \Phi'(x') = \Phi(x) + \omega_a \frac{\delta \mathcal{F}(x)}{\delta \omega_a} \end{aligned} \quad (40)$$

where $\Phi'(x') = \mathcal{F}(\Phi(x))$ and $\{\omega_a\}$ are the infinitesimal parameters characterizing the transformation. The *generator* G_a of a symmetry transformation is defined by

$$\delta_\omega \Phi(x) \equiv \Phi'(x) - \Phi(x) \equiv -i\omega_a G_a \Phi(x) \quad (41)$$

Basically, we figure out what the difference is between the field values *at the same point* x before and after the transformation, and we attribute it to the generator of the transformation acting on the original field at x .

To calculate this, first we write

$$\Phi(x') = \Phi(x) + \omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \Phi(x') \quad (42)$$

as a simple Taylor expansion in x' up to first order in ω . Next, we substitute this in Eq. 40 to get

$$\Phi'(x') = \Phi(x') - \omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \Phi(x') + \omega_a \frac{\delta \mathcal{F}(x)}{\delta \omega_a} \quad (43)$$

and so we rearrange this to get

$$\Phi'(x') - \Phi(x') = -\omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \Phi(x') + \omega_a \frac{\delta \mathcal{F}(x')}{\delta \omega_a} = -i\omega_a G_a \Phi(x) \quad (44)$$

Thus, we get that the expression for the generator is

$$\boxed{iG_a \Phi \equiv \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \Phi + \frac{\delta \mathcal{F}}{\delta \omega_a}} \quad (45)$$

Notice the similarity with Eq. 23; indeed the expression there is the same as the one above restricted to the case where the fields do not transform.

1.2.2 Representations of the Conformal Group in d dimensions

Under an infinitesimal conformal transformation parameterized by ω_g , the field $\Phi(x)$ transforms as

$$\Phi'(x') = (1 - i\omega_g T_g) \Phi(x) \quad (46)$$

The generator T_g will have some representation, and to get the full generator of conformal symmetry, we need to add this to the spacetime part that we computed in Sec. 1.1.3. We will postpone that analysis for now, as we want to move onto the more approachable aspects of conformal theory in two dimensions.

However, we will quote the results of this analysis for a scalar field $\phi(x)$. Under a conformal transformation $x \rightarrow x'$, a scalar field transforms as

$$\phi(x) \rightarrow \phi'(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\frac{\Delta}{d}} \phi(x) \quad (47)$$

where the Jacobian of the conformal transformation is related to the scale factor of the transformation by

$$\left| \frac{\partial x'}{\partial x} \right| = \Lambda(x)^{-\frac{d}{2}} \quad (48)$$

A field transforming this way is called *quasi-primary*.

1.2.3 The Energy-Momentum Tensor

Consider an arbitrary infinitesimal transformation $x^\mu \rightarrow x^\mu + \epsilon^\mu$, under this, the action will change as

$$\delta S = \int d^d x T^{\mu\nu} \partial_\mu \epsilon_\nu \quad (49)$$

The reason this is true is because we know that for any transformation parametrized by ω_a ,

$$\delta S = \int d^d x \partial_\mu j_a^\mu \omega_a = - \int d^d x j_a^\mu \partial_\mu \omega_a \quad (50)$$

which is the Noether current for that transformation. For an infinitesimal spacetime transformation, the energy-momentum tensor becomes a “conserved current” (d currents for d spacetime indices); and so we verify that Eq. 49 is indeed true.

Now, consider a conformal transformation; the change in the action can be written as (assuming a symmetric energy-momentum tensor can be found, in most cases, such a tensor can be constructed)

$$\begin{aligned}
\delta S &= \int d^d x T^{\mu\nu} \partial_\mu \epsilon_\nu \\
&= \frac{1}{2} \int d^d x T^{\mu\nu} (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \\
&= \frac{1}{2} \int d^d x T^{\mu\nu} \left(\frac{2}{d} \partial_\mu \epsilon^\mu \right) \eta_{\mu\nu} \\
&= \frac{1}{d} \int d^d x T^\mu_\mu \partial_\nu \epsilon^\nu
\end{aligned} \tag{51}$$

where we have substituted Eqs 5 and 6. Now, notice that if the energy-momentum tensor is traceless, then it implies that the variation in the action vanishes, that is, the classical theory is invariant under conformal transformations. However, this is only a sufficient condition; it is therefore not the only way in which the presence of a conformal symmetry can be verified with.

It can be shown that under certain conditions, the energy-momentum tensor of a theory with scale invariance can be made traceless; much like how it can be made symmetric for theories with rotational invariance (the latter are called *Belinfante* energy-momentum tensors). In this case, the above analysis immediately implies that such a theory has full conformal symmetry, and it is a consequence of the scale and Poincaré invariance.

1.3 Conformal Invariance in Quantum Field Theory

For a quantum field theories to have a symmetry, the invariance of the action associated to the quantum theory is not a sufficient condition; for it is the correlation functions that play an important role here. Thus, constraints on these correlation functions need to be studied to understand (conformal) symmetries in the quantum theory. These are usually expressed in the form of Ward identities.

1.3.1 Correlation Functions

Recall that the form of the correlation function of a theory involving the fields $\Phi(x)$ with an action $S[\Phi]$ is given by the path integral as

$$\langle \Phi(x_1) \dots \Phi(x_n) \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \Phi(x_1) \dots \Phi(x_n) e^{-S[\Phi]} \tag{52}$$

where

1. Z is the path integral without insertions

$$Z = \int \mathcal{D}\Phi e^{-S[\Phi]} \tag{53}$$

2. We will be working in the imaginary time prescription $t \rightarrow -i\tau$, which means that this is the Euclidean path integral. It is more suitable as path integrals are better defined in terms of an exponentially suppressing factor ($e^{-S[\Phi]}$) as compared to an oscillatory factor ($e^{iS[\Phi]}$).

If this action is invariant under a transformation given by Eq. 39; then we expect that the correlation functions should also remain invariant, namely

$$\langle \Phi(x'_1) \dots \Phi(x'_n) \rangle = \langle \mathcal{F}(\Phi(x_1)) \dots \mathcal{F}(\Phi(x_n)) \rangle \quad (54)$$

To derive the aforementioned result, we proceed as follows

$$\begin{aligned} \langle \Phi(x'_1) \dots \Phi(x'_n) \rangle &= \frac{1}{Z} \int \mathcal{D}\Phi \, \Phi(x'_1) \dots \Phi(x'_n) e^{-S[\Phi]} \\ &= \frac{1}{Z} \int \mathcal{D}\Phi' \, \Phi'(x'_1) \dots \Phi'(x'_n) e^{-S[\Phi']} \\ &= \frac{1}{Z} \int \mathcal{D}\Phi' \, \mathcal{F}(\Phi(x_1)) \dots \mathcal{F}(\Phi(x_n)) e^{-S[\mathcal{F}(\Phi)]} \end{aligned} \quad (55)$$

To proceed, firstly, we observe that since the action is invariant under this transformation, we have $S[\Phi] = S[\mathcal{F}(\Phi)]$; and secondly, we assume that the measure is trivially changed under the change of variables (that is, the Jacobian is not dependent on the fields Φ). This assumption is not a very good one; in fact, it is known that the non-trivial change in the measure is the source of anomalies in quantum field theory. However, if our hypothesis is correct, then it leads to the identity Eq. 54.

Having completed a review of correlation functions, we will now examine the consequences of conformal invariance on two-point and three-point correlation functions of quasi-primary fields.

1.3.2 Conformal Invariance of two-point and three-point correlation functions

Consider the two-point function

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \, \phi_1(x_1) \phi_2(x_2) e^{-S[\Phi]} \quad (56)$$

of two quasi-primary fields ϕ_1, ϕ_2 ; Φ denotes the set of all functionally independent fields in this theory (to which ϕ_1, ϕ_2 may or may not belong). The action $S[\Phi]$ is conformally invariant.

Here, we will first clarify the convention that features in most conformal theory literature; the objects that we refer to as *fields* are quite general, any *local operator* with coordinate dependence will be referred to as a *field*. These may be the fields that appear independently in the functional integral and integrated over (like the scalar field ϕ) or their derivatives (for scalar fields, $\partial_\mu \phi$), or composite objects like the energy-momentum tensor.

Coming back to our two-point function, using the identity Eq. 54; and the transformation law for a quasi-primary field given by Eq. 47, we get that

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\frac{\Delta_1}{d}} \left| \frac{\partial x'}{\partial x} \right|_{x=x_2}^{\frac{\Delta_2}{d}} \langle \phi_1(x'_1) \phi_2(x'_2) \rangle \quad (57)$$

In particular, for a scale transformation $x \rightarrow x' = \lambda x$, we have

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \lambda^{\Delta_1 + \Delta_2} \langle \phi_1(\lambda x_1) \phi_2(\lambda x_2) \rangle \quad (58)$$

Now, we have two other properties of conformal transformations left to constraint the form of the two-point function. Invariance under rotations and translations implies that

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = f(|x_1 - x_2|) \quad (59)$$

and so, substituting Eq. 59 in Eq. 58, we get

$$f(r) = \lambda^{\Delta_1 + \Delta_2} f(\lambda r) \implies f(r) \propto \frac{1}{r} \quad (60)$$

Thus, we can write the two-point function as

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} \quad (61)$$

Lastly, we use the invariance under SCT, where we have

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{1}{(1 - 2b \cdot x + b^2 x^2)^d} \quad (62)$$

It can be shown that distances transform under an SCT as follows

$$|x'_i - x'_j| = \frac{|x_i - x_j|}{(1 - 2b \cdot x_i + b^2 x_i^2)^{-\frac{1}{2}} (1 - 2b \cdot x_j + b^2 x_j^2)^{-\frac{1}{2}}} \quad (63)$$

which, along with the covariance of Eq. 61 implies that

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} = \frac{C_{12}}{\gamma_1^{\Delta_1} \gamma_2^{\Delta_2}} \frac{(\gamma_1 \gamma_2)^{\frac{\Delta_1 + \Delta_2}{2}}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} \quad (64)$$

with

$$\gamma_i = (1 - 2b \cdot x_i + b^2 x_i^2) \quad (65)$$

This constraint is satisfied if $\Delta_1 = \Delta_2$; and thus, we see that two dimensional quasi-primary fields are correlated (have a non-zero correlation function) only if they have the same scaling dimension

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \begin{cases} \frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}} & \Delta_1 = \Delta_2 \\ 0 & \Delta_1 \neq \Delta_2 \end{cases} \quad (66)$$

A similar analysis can be made for three-point correlation functions; we will just quote the result here

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \rangle = \frac{C_{123}}{x_{12}^{\Delta_1 + \Delta_2 - \Delta_3} x_{23}^{\Delta_2 + \Delta_3 - \Delta_1} x_{13}^{\Delta_3 + \Delta_1 - \Delta_2}} \quad (67)$$

It seems like we can find the expression for a general n -point function, but that is not the case. In general, conformal invariance can constrain the form of the correlation function only for $n = 2, 3$ and for others, we can only build functions of anharmonic ratios.

1.3.3 Ward Identities

The consequence of a symmetry of the action and the path integral measure on correlation function is usually expressed in the form of *Ward identities*. First, we write down the infinitesimal transformation of the fields in terms of the generator as

$$\Phi'(x) = \Phi(x) - i\omega_a G_a \Phi(x) \quad (68)$$

where as usual, $\{\omega_a\}$ are the infinitesimal parameters associated with the symmetry transformation. Consider now the n -point correlation function

$$\langle \Phi(x_1) \dots \Phi(x_n) \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \Phi(x_1) \dots \Phi(x_n) e^{-S[\Phi]} \quad (69)$$

which we will now denote by $X \equiv \Phi(x_1) \dots \Phi(x_n)$. Under our symmetry transformation, we have that

$$\delta S = \int d^d x \partial_\mu j_a^\mu \omega_a(x) \quad (70)$$

At the same time, the fields also will transform; and so, in terms of these new fields Φ' , we have

$$\langle X \rangle = \frac{1}{Z} \int \mathcal{D}\Phi' (X + \delta_\omega X) e^{-S[\Phi]} e^{-\int d^d x \partial_\mu j_a^\mu \omega_a(x)} \quad (71)$$

Note that Z does not change to first order in ω_a , and thus, we need not worry about the denominator.

Now, we once again make the assumption that the path integral measure remains invariant under this local transformation; this implies that to first order in ω , we have

$$\begin{aligned} \langle X \rangle &= \frac{1}{Z} \int \mathcal{D}\Phi (X + \delta X) e^{-S[\Phi]} \left(1 - \int d^d x \partial_\mu j_a^\mu \omega_a(x) + O(\omega^2) \right) \\ &= \langle X \rangle + \langle \delta X \rangle - \int d^d x \langle X \partial_\mu j_a^\mu \omega_a(x) \rangle + O(\omega^2) \\ &\implies \langle \delta X \rangle = \int d^d x \partial_\mu \langle j_a^\mu(x) \Phi(x_1) \dots \Phi(x_n) \rangle \omega_a(x) \end{aligned} \quad (72)$$

Now, we know how the fields transform from Eq. 68; and so, we can find out the first order variation in X , namely

$$\begin{aligned} \delta X &= -i \sum_{i=1}^n \Phi(x_1) \dots \omega_a(x_i) G_a \Phi(x_i) \dots \Phi(x_n) \\ &= -i \int d^d x \omega_a(x) \delta(x - x_i) \sum_{i=1}^n \Phi(x_1) \dots G_a \Phi(x_i) \dots \Phi(x_n) \end{aligned} \quad (73)$$

Comparing Eqs 73 and 68; we deduce the Ward identity for the current $j_a^\mu(x)$

$$\partial_\mu \langle j_a^\mu(x) \Phi(x_1) \dots \Phi(x_n) \rangle = -i \sum_{i=1}^n \delta(x - x_i) \langle \Phi(x_1) \dots G_a \Phi(x_i) \dots \Phi(x_n) \rangle \quad (74)$$

Notice that we can add any divergenceless quantity to the current, and we would not change the statement of the Ward identity; this is consistent with our derivations of the canonical form of the

current, and it is important as we sometimes we need to add terms to the energy-momentum tensor to make it divergenceless or traceless. Also, observe that when we have no field insertions, the Ward identity reduces to the statement that the expectation value of the current is zero; this is consistent with Ehrenfest's theorem of correspondence between classical dynamical equations and quantum operator expectation values.

We can obtain another form of the Ward identity by integrating both sides over a spacetime region containing all $\{x_i\}$; and then taking the limit of this region going to infinity. The L.H.S of the equation goes to zero due to Stoke's theorem; whereas the R.H.S simply integrates out the delta function, which leaves us with

$$-i \sum_{i=1}^n \langle \Phi(x_1) \dots G_a \Phi(x_i) \dots \Phi(x_n) \rangle = 0 \quad (75)$$

But, this is just the first order variation in the n -point correlation function; as given by Eq. 73. This means that an infinitesimal transformation does not change the correlation function up to first order, which is what we expect.

One can also show that the conserved charge

$$Q_a = \int d^{d-1}x j_a^0 \quad (76)$$

is the generator of the symmetry transformation in the Hilbert space of quantum states, namely

$$[Q_a, \Phi] = -iG_a \Phi = \delta \Phi \quad (77)$$

using the Ward identity.

1.3.4 Ward Identities of Conformal Invariance

We can now enumerate the Ward identities implied by the conformal invariance by using Eq. 74 and the generators of the field transformations. The three Ward identities associated to conformal invariance are computed as follows:

1. For translations; the generator is $-i\partial_\mu$, while the current is given by the energy-momentum tensor (each index gives one current conservation). Thus, for translations

$$\partial_\mu \langle T^\mu_\nu(x) \Phi(x_1) \dots \Phi(x_n) \rangle = - \sum_i \delta(x - x_i) \langle \Phi(x_1) \dots \frac{\partial}{\partial x_i^\nu} \Phi(x_i) \dots \Phi(x_n) \rangle \quad (78)$$

2. For Lorentz transformations (or rotations), the symmetrized conserved current is given $j^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$ while the generator is given by $L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) + S^{\mu\nu}$, where $S^{\mu\nu}$ is the associated spin operator. Then, the Ward identity for this is given by

$$\partial_\mu \langle (T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu) \Phi(x_1) \dots \Phi(x_n) \rangle = -i \sum_i \delta(x - x_i) \langle \Phi(x_1) \dots L_i^{\mu\nu} \Phi(x_i) \dots \Phi(x_n) \rangle \quad (79)$$

Expanding this out, we find that the product rules gives us one set of terms proportional to T and another proportional to ∂T . We can eliminate the latter by using the first Ward identity Eq. 78, and we are left with

$$\langle (T^{\mu\nu} - T^{\mu\rho}) \Phi(x_1) \dots \Phi(x_n) \rangle = -i \sum_i \delta(x - x_i) \langle \Phi(x_1) \dots S_i^{\mu\nu} \Phi(x_i) \dots \Phi(x_n) \rangle \quad (80)$$

3. For scale invariance, the dilatation current is given by $j_D^\mu = T^\mu_\nu x^\nu$ and the energy-momentum tensor is assumed to be in traceless form. The generator is given by $D = -ix^\nu \partial_\nu - i\Delta$ for a field of scaling dimension Δ . The Ward identity is thus,

$$\partial_\mu \langle T^\mu_\nu x^\nu \Phi(x_1) \dots \Phi(x_n) \rangle = -i \sum_i \delta(x - x_i) \langle \Phi(x_1) \dots D\Phi(x_i) \dots \Phi(x_n) \rangle \quad (81)$$

where again we can eliminate the ∂T terms using the first Ward identity as before. We then get

$$\boxed{\langle T^\mu_\mu(x) \Phi(x_1) \dots \Phi(x_n) \rangle = - \sum_i \delta(x - x_i) \langle \Phi(x_1) \dots \Delta_i \Phi(x_i) \dots \Phi(x_n) \rangle} \quad (82)$$

2 Conformal Invariance in Two Dimensions

Conformal invariance in two dimensional is different because the constraint equations, especially Eq. 10 are different for $d = 2$. Thus, we will restart the analysis of the conformal group in two dimensions from scratch.

2.1 The Conformal Group in Two Dimensions

2.1.1 Conformal Mappings

Consider a two dimensional plane mapped by the coordinates (z^0, z^1) . Under a coordinate transformation $z^\mu \rightarrow w^\mu(x)$, the metric will transform as

$$g^{\mu\nu} \rightarrow g'^{\mu\nu} = \frac{\partial w^\mu}{\partial z^\alpha} \frac{\partial w^\nu}{\partial z^\beta} g^{\alpha\beta} \quad (83)$$

Remember that we are only working everything out for flat spacetime; in fact, now we will restrict our case to Euclidean flat spacetime. Thus, expanding out the conformal condition $g'^{\mu\nu}(w) \propto g^{\mu\nu}(z)$, we have

$$\begin{aligned} & \frac{\partial w^\mu}{\partial z^\alpha} \frac{\partial w^\nu}{\partial z^\beta} g^{\alpha\beta} \propto g^{\mu\nu} \\ \implies & \frac{\partial w^\mu}{\partial z^0} \frac{\partial w^\nu}{\partial z^0} + \frac{\partial w^\mu}{\partial z^1} \frac{\partial w^\nu}{\partial z^1} \propto g^{\mu\nu} \end{aligned} \quad (84)$$

which then implies

$$\begin{aligned} \left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 &= \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2 \\ \frac{\partial w^0}{\partial z^0} \frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1} \frac{\partial w^1}{\partial z^1} &= 0 \end{aligned} \quad (85)$$

which is basically equating the diagonal terms and setting the non-diagonal terms to zero. These are equivalent to the relations

$$\boxed{\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1} \quad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1}} \quad (86)$$

or

$$\boxed{\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1} \quad \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1}} \quad (87)$$

We recognize these as the Cauchy-Riemann conditions for holomorphic functions; while the second set is those for antiholomorphic functions. For the analogy to be clear do the following substitutions

$$\begin{aligned} z^0 &\rightarrow x \\ z^1 &\rightarrow y \\ w^0 &\rightarrow u(x, y) \\ w^1 &\rightarrow v(x, y) \end{aligned} \quad (88)$$

where $f(z = x + iy) = u(x, y) + iv(x, y)$, and we recognize these to be the criteria for differentiability of complex functions.

This motivates us to use complex variables $\{z, \bar{z}\}$ instead of $\{z^0, z^1\}$. We will now switch to using these, and the rules to go from one set to another are as follows,

$$\boxed{\begin{aligned} z &= z^0 + z^1 & z^0 &= \frac{1}{2}(z + \bar{z}) \\ \bar{z} &= z^0 - z^1 & z^1 &= \frac{1}{2i}(z - \bar{z}) \\ \partial_z &= \frac{1}{2}(\partial_0 - i\partial_1) & \partial_0 &= (\partial_z + \partial_{\bar{z}}) \\ \partial_{\bar{z}} &= \frac{1}{2}(\partial_0 + i\partial_1) & \partial_1 &= (\partial_z - \partial_{\bar{z}}) \end{aligned}} \quad (89)$$

The metric tensor in terms of these coordinates is

$$g_{\mu\nu} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad g^{\mu\nu} = 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (90)$$

The raising and lowering operations now correspond to taking a covariant holomorphic index to a contravariant antiholomorphic index and vice versa.

With the analogy of complex variables complete, we can now simplify the constraint equation on conformal mappings in two dimensions. The holomorphic Cauchy-Riemann equations can be written in a more simple form,

$$\partial_{\bar{z}} w(z, \bar{z}) = 0 \quad (91)$$

which has a simple solution, namely, any holomorphic mapping, or in other words, a coordinate transformation that has no $\bar{z}$ dependence, given by $z \rightarrow w(z)$. To summarize, what we have learnt is that a conformal mapping in two dimensions is equivalent to a holomorphic coordinate transformation. The conformal group in two dimensions is therefore the set of all analytic maps (on the complex plane); the product rule simply being composition of functions.

An important point to be noted is that $z, \bar{z}$ are independent variables, as we have extended the range of the Cartesian coordinates into the complex plane. However, the physical space is the sub-manifold $\bar{z} = z^*$, that is, where the complex conjugate is equal to this “new coordinate” $\bar{z}$!

2.1.2 Global Conformal Transformations

When we wrote down Eqs. 86 and 87; we did not worry about the *global* properties of the coordinate mappings. More precisely, we did not impose any conditions of invertibility or that it should be defined everywhere. Strictly speaking, the mappings can only form a group if they are invertible and they map the Riemann sphere onto itself (the Riemann sphere is formed when we consider a stereographic projection of the *extended complex plane* onto a sphere). Thus, such *global* transformations differ from *local* transformations, which may not be defined everywhere.

The set of global conformal transformations form the *special conformal group*, which comprises of all the mappings of the form

$$f(z) = \frac{az + b}{cz + d} \quad \text{with } ad - bc = 1 \quad (92)$$

We can associate with them a matrix of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (93)$$

and noting that the function composition $f_1 \circ f_2$ is equivalent to the matrix multiplication of the associated matrices $A_2 \cdot A_1$, we see that this global conformal group in two dimensions is actually isomorphic to $SL(2, \mathbb{C})$, which itself is isomorphic to the Lorentz group in four dimensions, $SO(3, 1)$. Thus, we have rediscovered the same thing that followed from Eq. 37!

2.1.3 Conformal Generators

We will now try to understand the local conformal mappings, the set of all holomorphic mappings which may not be invertible, or defined everywhere. The best way to proceed would be to analyze the algebra of its generators. Since the mapping is holomorphic (and thus analytic), we can express the infinitesimal transformation $z \rightarrow z' = z + \epsilon(z)$ (note, we are interested in *local* transformations) as a Laurent series

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} c_n z^{n+1} \quad (94)$$

where we have assumed that the mapping has a Laurent expansion around $z = 0$. (Note that there is also a similar expression for $\bar{z}$ that we will not mention every time, but assume implicitly) Consider the effect of such a mapping on a scalar field $\phi(z, \bar{z})$ which is also dimensionless (in terms of scaling),

$$\begin{aligned} \phi'(z', \bar{z}') &= \phi(z, \bar{z}) \\ &= \phi(z', \bar{z}') - \epsilon(z)' \partial_{z'} \phi(z', \bar{z}') - \epsilon(\bar{z}') \partial_{\bar{z}'} \phi(z', \bar{z}') \end{aligned} \quad (95)$$

which implies that

$$\begin{aligned} \delta\phi &= -(\epsilon(z) \partial_z + \epsilon(\bar{z}) \partial_{\bar{z}}) \phi \\ &= \sum_n (c_n z^{n+1} \partial_z + \bar{c}_n \bar{z}^{n+1} \partial_{\bar{z}}) \phi \end{aligned} \quad (96)$$

and so, we can introduce the generators

$$\boxed{l_n = -z^{n+1} \partial_z \quad \bar{l}_n = -\bar{z}^{n+1} \partial_{\bar{z}}} \quad (97)$$

so that Eq. 96 becomes

$$\delta\phi = \sum_n (c_n l_n + \bar{c}_n \bar{l}_n) \phi(z, \bar{z}) \quad (98)$$

We can study the algebra of the generators, they follow the commutation relations

$$\boxed{\begin{aligned} [l_n, l_m] &= (n - m) l_{n+m} \\ [\bar{l}_n, \bar{l}_m] &= (n - m) \bar{l}_{n+m} \\ [l_n, \bar{l}_m] &= 0 \end{aligned}} \quad (99)$$

which are trivial to verify. Note that the conformal algebra is the direct sum of two isomorphic algebras; the total algebra is sometimes called the Witt algebra.

Further, we observe that these two infinite dimensional algebras contain a finite sub-algebra generated by $l_{-1} = -\partial$, $l_0 = -z\partial$ and $l_1 = -z^2\partial$ (and of course, their $\bar{z}$ counterparts!). These are

associated with the global conformal group; comparing with generators we found out it Sec. 1.1.3, we see that l_{-1} generates the translations, l_0 the scale transformations and rotations while l_1 generates the special conformal transformations.

2.1.4 Primary Fields

Recall the definition of a quasi-primary field, as given in Sec. 1.2.2. For a field with scaling dimension Δ and spin s , we define the *holomorphic conformal dimension* h and its antiholomorphic counterpart $\bar{h}$ as

$$h = \frac{1}{2}(\Delta + s) \quad \text{and} \quad \bar{h} = \frac{1}{2}(\Delta - s) \quad (100)$$

and with this, we can generalize the transformation of a quasi-primary field given by Eq. 47 as

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz} \right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}} \right)^{-\bar{h}} \phi(z, \bar{z}) \quad (101)$$

Thus, a quasi-primary field with conformal dimensions $(h, \bar{h})$ transforms as if it is a covariant tensor of rank $h + \bar{h}$, with h “ z ” indices and $\bar{h}$ “ $\bar{z}$ ” indices.

Now, consider the infinitesimal mapping $z \rightarrow w = z + \epsilon(z)$; we find the variation in a field with conformal dimensions $(h, \bar{h})$

$$\delta\phi = \phi'(z, \bar{z}) - \phi(z, \bar{z}) \quad (102)$$

For that, first, we calculate

$$\begin{aligned} \phi'(w, \bar{w}) &= \left(\frac{dw}{dz} \right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}} \right)^{-\bar{h}} \phi(z, \bar{z}) \\ &= \phi(z, \bar{z}) - h\phi(z, \bar{z})\partial\epsilon - \bar{h}\phi(z, \bar{z})\partial_{\bar{z}}\bar{\epsilon} \\ &= \phi(w, \bar{w}) - \epsilon(w)\partial'\phi(w, \bar{w}) - \bar{\epsilon}(\bar{w})\partial_{\bar{z}}'\phi(w, \bar{w}) - h\phi(z, \bar{z})\partial\epsilon - \bar{h}\phi(z, \bar{z})\partial_{\bar{z}}\bar{\epsilon} \end{aligned} \quad (103)$$

where we have used Eq. 95, and then substituting $w \equiv w(z)$, we have

$$\delta\phi = -(h\phi\partial\epsilon + \bar{h}\phi\partial_{\bar{z}}\bar{\epsilon}) - (\epsilon\partial\phi + \bar{\epsilon}\partial_{\bar{z}}\phi) \quad (104)$$

If a field under *any local conformal transformation* in two dimensions has a variation given by the Eq. 104; then such a field is called a *primary field*. All primary fields are thus, also quasi-primary; but the converse is not true. This is because a field may transform as mentioned before under an element of the *global* conformal group $SL(2, \mathbb{C})$ only; in which case it does not satisfy the definition of a primary field: it does not transform under *any local* conformal transformation.

2.1.5 Correlation Functions

Consider the n -point correlation function of n primary fields ϕ_i (conformal dimensions $(h, \bar{h})$) in terms of holomorphic and anti-holomorphic coordinates. Under a transformation $z \rightarrow w$, we have, using Eq. 101

$$\langle \phi'_1(w_1, \bar{w}_1) \dots \phi'_1(w_n, \bar{w}_n) \rangle = \prod_{i=1}^n \left(\frac{dw}{dz} \right)^{-h_i}_{w=w_i} \left(\frac{d\bar{w}}{d\bar{z}} \right)^{-\bar{h}_i}_{\bar{w}=\bar{w}_i} \langle \phi_1(z_1, \bar{z}_1) \dots \phi_1(z_n, \bar{z}_n) \rangle \quad (105)$$

We can fix the form of the two-point and three-point function just as we did in Sec. 1.3.2. In fact, we will get the exact same form, which we will now express in terms of the complex coordinates.

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \rangle = \frac{C_{12}}{(z_1 - z_2)^{2h} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \text{ if } \begin{cases} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h} \end{cases} \quad (106)$$

where we have used the fact $|x_1 - x_2| = (z_1 - z_2)^{\frac{1}{2}} (\bar{z}_1 - \bar{z}_2)^{\frac{1}{2}}$. The condition on the scaling dimensions are inherited by the conformal dimensions, primary fields are only correlated if they have the same conformal dimensions, plus the additional requirement of rotational invariance; the total spin within a correlation function must be zero. (An expression for the three-point function can be found out in the same way as in Sec. 1.3.2.)

2.2 Ward Identities

2.2.1 Delta Functions in Holomorphic/Antiholomorphic coordinates

Before we begin, we first take a slight detour to verify an important identity that will be useful later,

$$\delta(x) = \frac{1}{\pi} \partial_{\bar{z}} \frac{1}{z} = \frac{1}{\pi} \partial \frac{1}{\bar{z}} \quad (107)$$

To proceed further, first we consider a vector F^μ whose divergence is integrated over a region M of the complex plane, bounded by a contour ∂M . We can apply Gauss's theorem to equate a volume integral and a surface integral as

$$\int_M d^2x \partial_\mu F^\mu = \oint_{\partial M} d\xi_\mu F^\mu \quad (108)$$

where ξ_μ is the outward pointing (normal to the boundary ∂M) differential of the circumference. We can express this in terms of a counterclockwise differential ds^ρ which runs parallel to ∂M by the relation

$$d\xi_\mu = \epsilon_{\mu\rho} ds^\rho \quad (109)$$

Further, we can leverage the fact that we can use complex coordinates, for then the integral along the boundary takes the familiar form of a contour integral. The holomorphic component of ds^ρ is simply dz (while the antiholomorphic one is $d\bar{z}$), and so, in these coordinates, we have

$$\begin{aligned} \int_M d^2x \partial_\mu F^\mu &= \oint_{\partial M} d\xi_\mu F^\mu \\ &= \oint_{\partial M} ds^\rho \epsilon_{\mu\rho} F^\mu \\ &= \oint_{\partial M} (dz \epsilon_{\bar{z}z} F^{\bar{z}} + d\bar{z} \epsilon_{z\bar{z}} F^z) \end{aligned} \quad (110)$$

We now use the fact the antisymmetric tensor $\epsilon_{\mu\nu}$ in terms of (anti)holomorphic coordinates is given by

$$\epsilon_{\mu\nu} = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (111)$$

which then implies that

$$\int_M d^2x \partial_\mu F^\mu = \frac{i}{2} \oint_{\partial M} (-dz F^{\bar{z}} + d\bar{z} F^z) \quad (112)$$

Finally, consider the vector F^μ such that $F^z = 0, F^{\bar{z}} = \frac{f(z)}{\pi z}$, where $f(z)$ is a holomorphic function (the motivation for picking this particular form will be made clear as we set to verify the expression of the identity). Plugging this in the above equation, we get

$$\int_M d^2x \partial_\mu F^\mu = -\frac{i}{2} \oint_{\partial M} dz \frac{f(z)}{\pi z} \implies \frac{1}{\pi} \int_M d^2x \partial_{\bar{z}} \left(\frac{f(z)}{z} \right) = \frac{1}{2\pi i} \oint_{\partial M} dz \frac{f(z)}{z} \quad (113)$$

With this in place, we can finally verify the identity; integrate the holomorphic function $f(z)$ against the delta function within the region M around the origin, and then use the identity to write it in terms of the complex coordinates. We get

$$\begin{aligned} \int_M d^2x \delta(x) f(z) &= \frac{1}{\pi} \int_M d^2x f(z) \partial_{\bar{z}} \frac{1}{z} \\ &= \frac{1}{\pi} \int_M d^2x \partial_{\bar{z}} \left(\frac{f(z)}{z} \right) \\ &= \frac{1}{2\pi i} \oint_{\partial M} dz \frac{f(z)}{z} \\ &= f(0) \end{aligned} \quad (114)$$

wherein we have used Eq. 113 and the Cauchy theorem in complex analysis in the final two steps. We can use the same arguments to get the second, antiholomorphic form of the delta function.

2.2.2 Holomorphic Form of the Conformal Ward Identities

We can now study the holomorphic form of the Ward identities as follows (note that we will use X to denote the string of n primary fields $\Phi(x_1) \dots \Phi(x_n)$ henceforth)

1. For translations, we have

$$\partial_\mu \langle T^\mu_\nu(x) X \rangle = - \sum_i \delta(x - x_i) \frac{\partial}{\partial x_i^\nu} \langle X \rangle \quad (115)$$

Using both parts of Eq. 107 for different indices $z, \bar{z}$, we get

$$\boxed{\begin{aligned} 2\pi \partial_z \langle T_{\bar{z}z} X \rangle + 2\pi \partial_{\bar{z}} \langle T_{zz} X \rangle &= - \sum_{i=0}^n \partial_{\bar{z}} \frac{1}{z - w_i} \partial_{w_i} \langle X \rangle \\ 2\pi \partial_z \langle T_{\bar{z}\bar{z}} X \rangle + 2\pi \partial_{\bar{z}} \langle T_{z\bar{z}} X \rangle &= - \sum_{i=0}^n \partial_z \frac{1}{\bar{z} - \bar{w}_i} \partial_{\bar{w}_i} \langle X \rangle \end{aligned}} \quad (116)$$

where the extra factor of 2 comes from lowering the indices in the derivative.

2. For Lorentz transformations (or rotations), we have

$$\epsilon_{\mu\nu} \langle T^{\mu\nu}(x) X \rangle = -i \sum_{i=0}^n \delta(x - x_i) s_i \langle X \rangle \quad (117)$$

where we have used the specific form of the two dimensional spin generators $S_{\mu\nu}^i = s_i \epsilon_{\mu\nu}$ (s_i is the spin of the field ϕ_i and $\epsilon_{\mu\nu}$ is the two dimensional antisymmetric tensor). Again, substitution of the relevant quantities leads to

$$\boxed{-2 \langle T_{z\bar{z}} X \rangle + 2 \langle T_{\bar{z}z} X \rangle = - \sum_{i=0}^n \delta(x - x_i) s_i \langle X \rangle} \quad (118)$$

3. For the scale transformations, we have

$$\langle T^\mu_\mu(x)X \rangle = -i \sum_{i=0}^n \delta(x - x_i) \Delta_i \langle X \rangle \quad (119)$$

which becomes

$$2\langle T_{z\bar{z}}X \rangle + 2\langle T_{\bar{z}z}X \rangle = - \sum_{i=0}^n \delta(x - x_i) \Delta_i \langle X \rangle \quad (120)$$

The reason we keep the final two equations in the same form on the R.H.S is to simplify them further. Observed that adding and subtracting Eqs. 118 and 120 lead to two equations

$$\begin{aligned} 2\pi \langle T_{\bar{z}z}X \rangle &= - \sum_{i=0}^n \partial_{\bar{z}} \frac{1}{z - w_i} h_i \langle X \rangle \\ 2\pi \langle T_{z\bar{z}}X \rangle &= - \sum_{i=0}^n \partial_z \frac{1}{\bar{z} - \bar{w}_i} \bar{h}_i \langle X \rangle \end{aligned} \quad (121)$$

where we have used the appropriate form of the delta identity depending whether we get the holomorphic or antiholomorphic conformal dimension on the R.H.S. Substituting these equations in Eq. 116, we get

$$\begin{aligned} \partial_{\bar{z}} \left\{ \langle T(z, \bar{z}) \rangle - \sum_{i=0}^n \left[\frac{1}{z - w_i} \partial_{w_i} \langle X \rangle + \frac{h_i}{(z - w_i)^2} \langle X \rangle \right] \right\} &= 0 \\ \partial_z \left\{ \langle \bar{T}(\bar{z}, z) \rangle - \sum_{i=0}^n \left[\frac{1}{\bar{z} - \bar{w}_i} \partial_{\bar{w}_i} \langle X \rangle + \frac{\bar{h}_i}{(\bar{z} - \bar{w}_i)^2} \langle X \rangle \right] \right\} &= 0 \end{aligned} \quad (122)$$

where we introduce the rescaled energy-momentum tensor components

$$T = -2\pi T_{zz} \quad \bar{T} = -2\pi T_{\bar{z}\bar{z}} \quad (123)$$

Evidently, the terms inside the derivatives are holomorphic and antiholomorphic respectively, and thus, we can write

$$\begin{aligned} \langle T(z)X \rangle &= \sum_{i=0}^n \left[\frac{1}{z - w_i} \partial_{w_i} \langle X \rangle + \frac{h_i}{(z - w_i)^2} \langle X \rangle \right] + f(z) \\ \langle \bar{T}(\bar{z})X \rangle &= \sum_{i=0}^n \left[\frac{1}{\bar{z} - \bar{w}_i} \partial_{\bar{w}_i} \langle X \rangle + \frac{\bar{h}_i}{(\bar{z} - \bar{w}_i)^2} \langle X \rangle \right] + \bar{f}(\bar{z}) \end{aligned} \quad (124)$$

where $f(z)$ is a holomorphic function of z , regular at $z = w_i$ (and the same is true for the antiholomorphic counterpart).

2.2.3 The Conformal Ward Identity

Consider an arbitrary conformal coordinate variation $\epsilon^\nu(x)$; we can write

$$\begin{aligned} \partial_\mu (\epsilon_\nu T^{\mu\nu}) &= \epsilon_\nu \partial_\mu T^{\mu\nu} + \partial_\mu \epsilon_\nu T^{\mu\nu} \\ &= \epsilon_\nu \partial_\mu T^{\mu\nu} + \frac{1}{2} (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) T^{\mu\nu} + \frac{1}{2} (\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu) T^{\mu\nu} \\ &= \epsilon_\nu \partial_\mu T^{\mu\nu} + \frac{1}{2} (\partial_\rho \epsilon^\rho) \eta_{\mu\nu} T^{\mu\nu} + \frac{1}{2} \epsilon^{\alpha\beta} \partial_\alpha \epsilon_\beta T^{\mu\nu} \end{aligned} \quad (125)$$

where we have used the tensor identities involving the metric $\eta^{\mu\nu}$ and the antisymmetric tensor $\epsilon^{\mu\nu}$. Notice how the three terms in the equation are exactly the same as the ones appearing on the L.H.S of the three Ward identities that we derived Sec.1.3.4, and thus, by multiplying by $X = \Phi(x_1) \dots \Phi(x_n)$ and integrating both sides, we have (for a region M containing all the spacetime positions of the fields in X)

$$\delta_\epsilon \langle X \rangle = \int_M d^2x \partial \langle T^{\mu\nu}(x) \epsilon_\nu(x) X \rangle \quad (126)$$

where $\delta_\epsilon \langle X \rangle$ is the variation of X under a local conformal transformation. (It is best to relate both expressions term by term in both the equations to find the equivalence. An important fact to remember is that $\partial_\rho \epsilon^\rho$ is proportional to the local scale factor $f(x)$, while $\frac{1}{2} \epsilon^{\alpha\beta} \partial_\alpha \epsilon_\beta$ is a rotation angle.)

We can use Gauss's theorem (in the complex coordinate form as in Eq. 112) to get

$$\delta_{\epsilon, \bar{\epsilon}} \langle X \rangle = \frac{i}{2} \int_{\partial M} [-dz \langle T^{\bar{z}\bar{z}} \epsilon_{\bar{z}} X \rangle + d\bar{z} \langle T^{zz} \epsilon_z X \rangle] \quad (127)$$

We have dropped the cross $T^{z\bar{z}}$ type terms already because they do not contribute to the contour integrals (as they vanish unless the contour passes exactly through one of the spacetime positions of a field contained in X , refer to Eq. 82). Finally, substituting Eq. 123 and $\epsilon = \epsilon^z, \bar{\epsilon} = \epsilon^{\bar{z}}$, we have the *conformal Ward identity*

$$\delta_{\epsilon, \bar{\epsilon}} \langle X \rangle = -\frac{1}{2\pi i} \oint_C dz \epsilon(z) \langle T(z) X \rangle + \frac{1}{2\pi i} \oint_C d\bar{z} \bar{\epsilon}(\bar{z}) \langle \bar{T}(\bar{z}) X \rangle \quad (128)$$

where we can now choose any contour C as long as it encloses all the positions of the fields in X .

2.3 Free Fields and the Operator Product Expansion

It is known to us from quantum field theory that correlation functions have singularities when the positions of two or more fields coincide. (For instance, the two-point function of a scalar field theory; it has an ultraviolet divergence that comes from the Feynman propagator $D_F(x - y = 0)$.) This is indicative of the infinite fluctuations of a quantum field taken at a precise point. A way to deal with this analytically is to represent a product of operators (at positions, say z and w) by a sum of terms which can quantify some level of well-definedness, as well as the singular nature. Each of these terms is a single operator which is well-defined as $z \rightarrow w$, multiplied by a function of $z - w$, which is possibly divergent as $z \rightarrow w$. Such a representation is called the *Operator Product Expansion* (OPE).

An illustration of the aforementioned concept has already been done when we were studying the conformal Ward identity in the previous section, namely, the behaviour of the energy-momentum tensor along with primary fields ϕ_i had an OPE, given by Eq. 124. The OPE of the energy-momentum tensor $T(z)$ with the a primary field $\phi_i(w, \bar{w})$ (conformal dimensions $(h, \bar{h})$), which captures the singular behaviour as $z \rightarrow w$ is

$$T(z) \phi(w, \bar{w}) \sim \frac{h}{(z - w)^2} \phi_i(w, \bar{w}) + \frac{1}{z - w} \partial_w \phi_i(w, \bar{w}) \quad (129)$$

where it is understood that this expression really makes sense only within a correlation function, that is, within $\langle \dots \rangle$. Further, we use $\sim$ to denote equality modulo expressions regular as $z \rightarrow w$, in fact, there is an infinite number of regular terms which cannot be obtained just from the conformal

Ward identity.

In general, we would write the OPE of two fields $A(z)$ and $B(w)$ as

$$A(z)B(w) = \sum_{n=-\infty}^N \frac{(AB)_n(w)}{(z-w)^n} \quad (130)$$

where the composite field $(AB)_n(w)$ is regular at $z = w$. Let us now look at specific examples in order to understand the basic procedures and techniques for simple systems.

2.3.1 The Free Boson

Consider the theory of a free massless boson ϕ in two dimensions, given by the action

$$S = \frac{1}{2}g \int d^2x \partial_\mu \phi \partial^\mu \phi \quad (131)$$

where g is some unspecified (for the moment) normalization parameter. First, we need to calculate the two-point function or propagator $\langle \phi(x)\phi(y) \rangle$. For this, we recall that we can write the action as

$$S = \frac{1}{2} \int d^2x \int d^2y \phi(x) O(x, y) \phi(y) \quad (132)$$

where $O(x, y) = g\delta(x - y)(-\partial^2)$, then the propagator is simply $\langle \phi(x)\phi(y) \rangle = O^{-1}(x, y)$. This, by definition, means

$$g(-\partial_x^2) \langle \phi(x)\phi(y) \rangle = \delta(x - y) \quad (133)$$

To solve this, we first notice that the correlation function must be a function of $r = |x - y|$ only, as it has to be invariant under rotations and translations. Integrating the Green's function relation over a disc of radius r centered around y to get

$$\begin{aligned} 1 &= 2\pi g \int_0^r d\rho \rho \left(-\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho K'(\rho)) \right) \\ &= -2\pi g r K'(r) \\ \implies K(r) &= -\frac{1}{2\pi g} \ln r + \text{const.} \end{aligned} \quad (134)$$

where $K(r) = \langle \phi(x)\phi(y) \rangle$, and $K'(r) = \frac{dK}{dr}$. This means that

$$\boxed{\langle \phi(x)\phi(y) \rangle = -\frac{1}{4\pi g} \ln (x - y)^2 + \text{const.}} \quad (135)$$

In terms of complex coordinates, this is

$$\langle \phi(z, \bar{z})\phi(w, \bar{w}) \rangle = -\frac{1}{4\pi g} \{ \ln(z - w) + \ln(\bar{z} - \bar{w}) \} + \text{const.} \quad (136)$$

Taking a derivatives $\partial_z \phi, \partial_{\bar{z}} \phi$, we can separate out the holomorphic and antiholomorphic parts

$$\boxed{\begin{aligned} \langle \partial_z \phi(z, \bar{z}) \partial_w \phi(w, \bar{w}) \rangle &= -\frac{1}{4\pi g} \frac{1}{(z - w)^2} \\ \langle \partial_{\bar{z}} \phi(z, \bar{z}) \partial_{\bar{w}} \phi(w, \bar{w}) \rangle &= -\frac{1}{4\pi g} \frac{1}{(\bar{z} - \bar{w})^2} \end{aligned}} \quad (137)$$

We will now consider only the holomorphic part, keeping in mind that the same analysis will hold true for the antiholomorphic counterpart.

The calculation above shows that the OPE of this field ($\partial\phi$) with itself is

$$\partial_z\phi(z)\partial_w\phi(w) \sim -\frac{1}{4\pi g} \frac{1}{(z-w)^2} \quad (138)$$

Observe how the OPE also inherits the bosonic character of the field; it remains symmetric under the exchange of the fields.

The energy-momentum tensor for the free massless boson is given by

$$T_{\mu\nu} = g \left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi \right) \quad (139)$$

The rescaled quantum version of this (as defined in Eq. 123) in complex coordinates is

$$T(z) = -2\pi g : \partial\phi\partial\phi : (z) = -2\pi g \lim_{w \rightarrow z} (\partial\phi(z)\partial\phi(w) - \langle \partial\phi(z)\partial\phi(w) \rangle) \quad (140)$$

where the “normal-ordering” is to ensure a zero vacuum expectation value. Now, consider the OPE of $T(z)$ with $\partial\phi$. We first remember that an OPE only makes sense when we put it inside a correlation function, or equivalently, it is useful only when we put take the expectation value of the time-ordered product. But notice that when there is a time-ordering required, it immediately invokes Wick’s theorem for the normal-ordering. Thus, we get

$$\begin{aligned} T(z)\partial\phi(w) &= -2\pi g : \partial\phi(z)\partial\phi(z) : \partial\phi(w) \\ &\sim -4\pi g : \overline{\partial\phi(z)\partial\phi(z)} : \partial\phi(w) \\ &\sim \frac{\partial\phi(z)}{(z-w)^2} \end{aligned} \quad (141)$$

Finally, we expand $\partial\phi(z)$ around w , as the OPE as the regular terms in the OPE have to be in terms of w . Consequently, we get

$$T(z)\partial\phi(w) \sim \frac{\partial_w\phi(w)}{(z-w)^2} + \frac{\partial_w^2\phi(w)}{(z-w)} \quad (142)$$

which, when compared to Eq. 129, shows us that $\partial\phi$ is a field with conformal dimension $h = 1$.

We can also find out the OPE for the energy-momentum tensor with itself, this involves another use of Wick’ theorem to give

$$\begin{aligned} T(z)T(w) &= 4\pi^2 g^2 : \partial\phi(z)\partial\phi(z) : : \partial\phi(w)\partial\phi(w) : \\ &= 16\pi^2 g^2 : \overline{\partial\phi(z)\partial\phi(z)} : : \partial\phi(w)\partial\phi(w) : + 8\pi^2 g^2 : \overline{\partial\phi(z)\partial\phi(z)} : : \overline{\partial\phi(w)\partial\phi(w)} : \\ &\sim -\frac{4\pi g : \partial\phi(z)\partial\phi(w) :}{(z-w)^2} + \frac{1}{2} \frac{1}{(z-w)^4} \\ &\sim \frac{1}{2} \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} \end{aligned} \quad (143)$$

where again, we have considered all possible contractions, and then expanded around $z = w$. We observe that due to the anomalous $(z-w)^{-4}$ term, the energy-momentum tensor is not a primary field, at least in the case of a free boson.

2.4 The Central Charge

The general form of the OPE of the energy-momentum tensor is given by

$$T(z)T(w) \sim \frac{c}{2} \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \quad (144)$$

where the constant c is a model dependent quantity. For instance, a free boson has $c = 1$ (we can verify this from Eq. 143), a free fermion has $c = \frac{1}{2}$ and so on. This model dependent constant is called the *central charge*. Notice that without the central charge term, T would have been a quasi-primary field with conformal dimension $h = 2$. (Note that there is an analogous antiholomorphic central charge $\bar{c}$ which we do not describe explicitly, but the exact same analysis holds for it)

The central charge is determined by the short-distance behaviour of the theory; symmetry considerations can help in constraining the regime of the magnitude of the charge, but in general, that may not be true. For free fields, as we have seen in the previous section, one gets that anomalous term due to the specifics of Wick contraction of the time-ordered fields.

For decoupled systems (say two free fields with no interaction terms) will have energy-momentum tensors that will simply add, and so the central charge of such a combined system will simply be the sum of the central charges of each system. In some sense, the central charge acts like an extensive measure of the number of degrees of freedom of the system.

2.4.1 Transformation of the Energy-Momentum Tensor

The anomalous term in the OPE for the energy-momentum tensor means that it does not transform like a primary field. To find out what the transformation rule for this field is, we first recall the conformal Ward identity,

$$\delta_{\epsilon, \bar{\epsilon}} \langle X \rangle = -\frac{1}{2\pi i} \oint_C dz \epsilon(z) \langle T(z) X \rangle \quad (145)$$

which means that for $X = T(w)$, we have

$$\begin{aligned} \delta_{\epsilon, \bar{\epsilon}} T(w) &= -\frac{1}{2\pi i} \oint_C dz \epsilon(z) T(z) T(w) \\ &\sim -\frac{1}{2\pi i} \oint_C dz \epsilon(z) \left(\frac{c}{2} \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} \right) \\ &= -\frac{c}{12} \partial_w^3 \epsilon(w) - 2T(w) \partial_\epsilon(w) - \epsilon(w) \partial_w T(w) \end{aligned} \quad (146)$$

where we have used the OPE of the energy-momentum tensor given by Eq. 144, and the generalization of the Cauchy differentiation formula for holomorphic functions

$$f^n(a) = \frac{n!}{2\pi i} \oint_C dz \frac{f(z)}{(z-a)^{n+1}} \quad (147)$$

Exponentiation of this infinitesimal case then gives the variation for the finite transformation $z \rightarrow w(z)$, which can be shown to be

$$T'(w) = \left(\frac{dw}{dz} \right)^{-2} \left[T(z) - \frac{c}{12} \{w; z\} \right] \quad (148)$$

where we introduce the *Schwarzian derivative* given by

$$\{w; z\} = \frac{d^3w/dz^3}{dw/dz} - \frac{3}{2} \left(\frac{d^2w/dz^2}{dw/dz} \right)^2 \quad (149)$$

2.4.2 Physical Meaning of the Central Charge c

The central charge c is also known as the *conformal anomaly*, and is related to the “soft” breaking of conformal symmetry due to the introduction of a macroscopic scale in the system. An illustration related to a macroscopic length scale introduced due to boundary conditions is as follows:

Consider a conformal theory living on the complex plane, and we map this theory on a cylinder with circumference L by the transformation

$$z \rightarrow w(z) = \frac{L}{2\pi} \ln z \quad (150)$$

We can then find how the energy-momentum tensor changes by using Eq. 148, to get

$$\begin{aligned} T_{\text{cylinder}}(w) &= \left(\frac{dw}{dz} \right)^{-2} \left[T_{\text{plane}}(z) - \frac{c}{12} \{w; z\} \right] \\ &= \left(\frac{2\pi z}{L} \right)^2 \left[T_{\text{plane}}(z) - \frac{c}{12} \left(\frac{2/z^3}{1/z} - \frac{3}{2} \left(\frac{-1/z^2}{1/z} \right)^2 \right) \right] \\ &= \left(\frac{2\pi z}{L} \right)^2 \left[T_{\text{plane}}(z) - \frac{c}{24z^2} \right] \\ &= \left(\frac{2\pi}{L} \right)^2 \left[T_{\text{plane}}(z) z^2 - \frac{c}{24} \right] \end{aligned} \quad (151)$$

Now, consider the vacuum expectation of both sides of the expression; for the correct normal-ordered form for $T_{\text{plane}}(z)$, it vanishes. Thus, we have

$$\langle T_{\text{cylinder}}(w) \rangle = -\frac{c\pi^2}{6L^2} \quad (152)$$

The central charge is seen to be proportional to the *Casimir energy*, the change in the vacuum energy density brought about by the periodicity condition on the cylinder. It naturally goes to zero as the macroscopic scale L goes to infinity.

The central charge also arises when a conformal field theory is defined on a curved two dimensional manifold; the curvature defines the macroscopic length scale then. Here, we are introduced to the *trace anomaly*, the expectation value of the trace of the energy-momentum tensor no longer vanishes. Instead, it is proportional to the central charge and the curvature R

$$\langle T^\mu_\mu(x) \rangle_g = \frac{c}{24} R(x) \quad (153)$$

3 The Operator Formalism

Till now, the consequences of conformal field theory on two dimensional field theories were understood by looking at the constraints imposed by the Ward identities on correlation functions. These were expressed by considering the OPEs of various local fields with the energy-momentum tensor. However, these OPEs were understood to be occurring inside correlation functions, and no mention of operators or Hilbert spaces was made. In fact, this followed entirely from a path integral approach.

Nonetheless, Hilbert spaces and operators are still very useful in conformal field theory because of the powerful algebraic and group-theoretic structures present in this formalism, which is why we study this formalism as being complementary to what has already been developed in the sections preceding this one.

3.1 The Operator Formalism of Conformal Field Theory

3.1.1 Radial Quantization

The operator formalism distinguishes a time direction from a space direction, which means that it is no longer manifestly Lorentz invariant. This is natural in a Minkowski spacetime, due to the signature of the metric. However, it is somewhat arbitrary in Euclidean spacetime; and so we motivate the conventional choice of space and time axes in Euclidean spacetime by the following Minkowski space point of view (particularly in the context of string theory):

Initially, define the theory on an infinite spacetime cylinder, with time t going from $-\infty$ to $+\infty$ along the axial direction of the cylinder; while space is compactified with a coordinate x going from 0 to L . We identify the point $(0, t)$ with (L, t) . (This is more natural if one thinks of this as a compactified worldsheet). Next, if we analytically continue to Euclidean space, the cylinder is described by a single complex coordinate $\xi = t + ix$ (or the complex conjugate). Finally, we can map the cylinder to the complex plane (or rather, the Riemann sphere) via

$$z = e^{\frac{2\pi\xi}{L}} = e^{\frac{2\pi t}{L}} e^{\frac{2\pi ix}{L}} \quad (154)$$

Thus, we have the complex plane spanned by concentric circles of time with radius $e^{\frac{2\pi t}{L}}$, that is, we have a radial time direction and an angular space direction! In particular, the *remote past* ($t \rightarrow -\infty$) is mapped to the origin $z = 0$, while the *remote future* ($t \rightarrow +\infty$) is mapped to the point at infinity

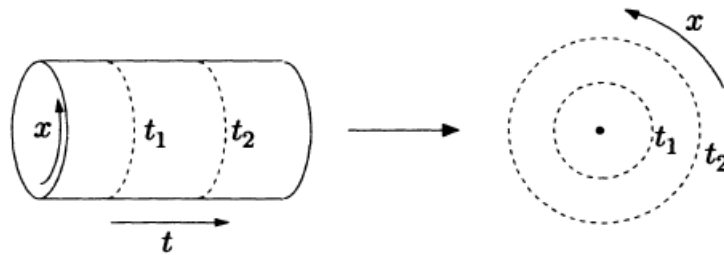


Figure 1: Mapping from the spacetime cylinder to the complex plane

This choice of space and time directions leads to *radial quantization* of two dimensional conformal field theories. For that, we must first assume the existence of a vacuum state $|0\rangle$ from which the Hilbert space can be constructed by application of creation operators (or something akin to them). For free field theories, we define the vacuum as the state that is annihilated by the positive frequency part (the one proportional to the annihilation operator) of the field; while for an interacting theory, we assume that the Hilbert space is the same as the free field, except that the energy eigenstates are different from the free field ones.

3.1.2 The Hermitian Product and In/Out States

We suppose that the interaction vanishes at $t \rightarrow \pm\infty$, and the asymptotic field (henceforth we shall assume that ϕ is a quasi-primary field with conformal dimensions $(h, \bar{h})$)

$$\phi_{\text{in}} \propto \lim_{t \rightarrow -\infty} \phi(x, t) \quad (155)$$

is free. Further, after radial quantization, this field reduces to a single operator, which, upon acting on the vacuum state gives the “in” state

$$|\phi_{\text{in}}\rangle = \lim_{z, \bar{z} \rightarrow 0} \phi(z, \bar{z})|0\rangle \quad (156)$$

Next, we must define a bilinear product on this Hilbert space, which we do indirectly by first defining an asymptotic “out” state is defined by demanding that

$$\langle \phi_{\text{out}} | = |\phi_{\text{in}}\rangle^\dagger \quad (157)$$

and the action of Hermitian conjugation on conformal fields

$$\phi(z, \bar{z})^\dagger = \bar{z}^{-2h} z^{-2\bar{h}} \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) \quad (158)$$

In Minkowski spacetime, Hermitian conjugation does not affect the spacetime coordinates; but in the Euclidean continuation, we recall that Euclidean time $\tau = it$ will be affected ($\tau \rightarrow -\tau$) upon conjugation. In term of the mapping given by Eq. 154, it amounts to $z \rightarrow \frac{1}{z^*}$. Thus, the aforementioned definition of Hermitian conjugation is justified, at least on the real surface where $\bar{z} = z^*$. If we consider the inner product

$$\begin{aligned} \langle \phi_{\text{out}} | \phi_{\text{in}} \rangle &= \lim_{z, \bar{z}, w, \bar{w} \rightarrow 0} \langle 0 | \phi(z, \bar{z})^\dagger \phi(w, \bar{w}) | 0 \rangle \\ &= \lim_{z, \bar{z}, w, \bar{w} \rightarrow 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0 | \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) \phi(w, \bar{w}) | 0 \rangle \\ &= \lim_{\xi, \bar{\xi} \rightarrow \infty} \bar{\xi}^{2h} \xi^{2\bar{h}} \langle 0 | \phi(\bar{\xi}, \xi) \phi(0, 0) | 0 \rangle \end{aligned} \quad (159)$$

Now, recall the form of the two-point correlation function given by Eq. 106, plugging it in here tell us that the expression on the L.H.S is independent of ξ and thus well-defined at $\xi \rightarrow \infty$. This justifies the appearance of the prefactors appearing in the definition of Hermitian conjugation. (It is important to observe that the time-ordering was correct in the expression on the R.H.S, that is how we could consider the correlation function directly!)

3.1.3 Mode Expansions

A conformal field $\phi(z, \bar{z})$ can be mode expanded as

$$\phi(z, \bar{z}) = \sum_{m,n \in \mathbb{Z}} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n} \quad (160)$$

and the inverse relation is

$$\phi_{m,n} = \frac{1}{2\pi i} \oint dz z^{m+h-1} \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{n+\bar{h}-1} \phi(z, \bar{z}) \quad (161)$$

The Hermitian conjugation can be done in two ways; by a straightforward conjugation of Eq.160 on the real surface, or by using Eq. 158. Comparing the equations, we get

$$\begin{aligned} \sum_{m,n \in \mathbb{Z}} \bar{z}^{-m-h} z^{-n-\bar{h}} \phi_{m,n}^\dagger &= \bar{z}^{-2h} z^{-2\bar{h}} \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) \\ &= \bar{z}^{-2h} z^{-2\bar{h}} \sum_{m,n \in \mathbb{Z}} \bar{z}^{m+h} z^{n+\bar{h}} \phi_{m,n} \\ &= \sum_{m,n \in \mathbb{Z}} \bar{z}^{-m-h} z^{-n-\bar{h}} \phi_{-m,-n} \end{aligned} \quad (162)$$

which is consistent if

$$\phi_{m,n}^\dagger = \phi_{-m,-n} \quad (163)$$

This is the usual expression for the Hermitian conjugate of the modes, and so, it further justifies the various powers in the conformal dimensions appearing in the definition of the mode expansion. Finally, the vacuum must satisfy the annihilation condition

$$\phi_{m,n}|0\rangle = 0 \quad (m > -h, n > -\bar{h}) \quad (164)$$

To simplify the notation, we will henceforth drop the antiholomorphic dependence of the fields, and simply write

$$\begin{aligned} \phi(z) &= \sum_{m \in \mathbb{Z}} z^{-m-h} \phi_m \\ \phi_m &= \frac{1}{2\pi i} \oint dz z^{m+h-1} \phi(z) \end{aligned} \quad (165)$$

It must be kept in mind that the antiholomorphic dependence is always there, and owing to the decoupling between the holomorphic and antiholomorphic degrees of freedom, we can always drop them knowing that they can be reconstructed in the same form as the holomorphic dependence.

3.1.4 Radial Ordering and Operator Product Expansion

Due to the way space and time coordinates are mapped on the complex plane, the time-ordering in correlation functions becomes *radial-ordering* in the complex plane, given by

$$\mathcal{R}\Phi(z)\Phi(w) = \begin{cases} \Phi(z)\Phi(w) & \text{if } |z| > |w| \\ \Phi(w)\Phi(z) & \text{if } |z| < |w| \end{cases} \quad (166)$$

If the fields are fermions, then we need to put an extra negative sign when we exchange the two fields. Note that this means that all the OPEs we have written previously all have an operator meaning only when $|z| > |w|$; or more precisely, the L.H.S makes sense only if it radially-ordered.

Consider the integral of two holomorphic fields $a(z)$ and $b(z)$

$$\oint_w dz a(z)b(z) \quad (167)$$

where the subscript w denotes that the contour circles around w in a counterclockwise manner. This expression will have an operator meaning within a correlation function if we radially-order it. Accordingly, we split the contour as the sum of two fixed time circles going in opposite directions (see Fig. 2)

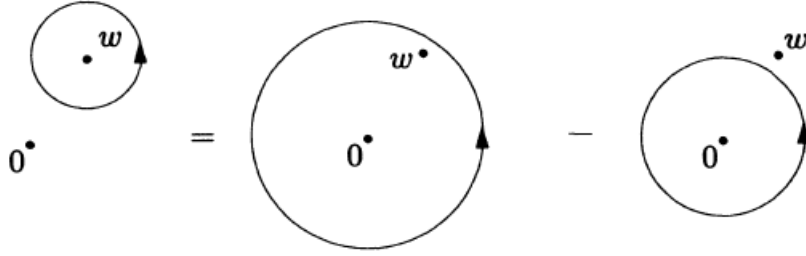


Figure 2: Subtraction of two contours going in the opposite direction

Thus our integral now becomes

$$\begin{aligned} \oint_w dz a(z)b(z) &= \oint_{C_1} dz a(z)b(w) - \oint_{C_2} dz b(w)a(z) \\ &= \oint dz a(z) b(w) - b(w) \oint dz a(z) \\ &= [A, b(w)] \end{aligned} \quad (168)$$

where the operator A is the integral over space at fixed time (which implies a contour integral on the complex plane) of the field $a(z)$

$$A = \oint dz a(z) \quad (169)$$

The two contours C_1 and C_2 are fixed time contours (circles centered around the origin) of radii $|w|+\epsilon$ and $|w|-\epsilon$ respectively for an infinitesimal ϵ . The limit $\epsilon \rightarrow 0$ also means that the commutator obtained is an *equal time* commutator (and remember, for fermions, it is the anticommutator). Thus, for the operators (each the integral of a holomorphic field)

$$A = \oint dz a(z) \quad B = \oint dz b(z) \quad (170)$$

have the commutator

$$[A, B] = \oint_0 dw \oint_w dz a(z)b(w) \quad (171)$$

where integral over z is taken around w , and the integral over w is taken around the origin.

In practice, we compute such integrals by substituting the OPE of $a(z)$ with $b(w)$ in the integral, and then use the residue theorem. This also means that we really need to know only the $\frac{1}{(z-w)}$ accurately, as that is the term that contributes via the residue theorem. Thus, we can now relate OPEs to commutation relations, and this allow us to translate dynamical or symmetry statements (like the Ward identities) contained in the OPE into the operator language.

Before we continue, we also note the following (important) convention: Whenever we write a contour integral without a specified contour, it is understood that the integral is to be calculated along a circle centered at the origin (in other words, it is a fixed time integral). Otherwise, the point around which we integrate is indicated by the subscript of the integral sign.

3.2 The Virasoro Algebra

3.2.1 Conformal Generators

Recall the conformal Ward identity (given by Eq. 128)

$$\delta_{\epsilon, \bar{\epsilon}} \langle X \rangle = -\frac{1}{2\pi i} \oint_C dz \epsilon(z) \langle T(z) X \rangle + \frac{1}{2\pi i} \oint_C d\bar{z} \bar{\epsilon}(\bar{z}) \langle \bar{T}(\bar{z}) X \rangle \quad (172)$$

We define the conformal charge for the holomorphic part $\epsilon(z)$ of an infinitesimal conformal change of coordinates to be

$$Q_\epsilon = \frac{1}{2\pi i} \oint dz \epsilon(z) T(z) \quad (173)$$

Then, with the help of Eq. 168, we have

$$\begin{aligned} \delta_\epsilon \Phi(w) &= -\frac{1}{2\pi i} \oint_C dz \epsilon(z) T(z) \Phi(w) \\ &= -[Q_\epsilon, \Phi(w)] \end{aligned} \quad (174)$$

which implies that the operator Q_ϵ is the generator of conformal transformations (which we expect from our knowledge of QFT).

Next, we find the mode expansion of the energy-momentum tensor according to Eq. 160 to get

$$\begin{aligned} T(z) &= \sum_{n \in \mathbb{Z}} z^{-n-2} L_n & L_n &= \frac{1}{2\pi i} \oint dz z^{n+1} T(z) \\ \bar{T}(\bar{z}) &= \sum_{n \in \mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n & \bar{L}_n &= \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z}) \end{aligned} \quad (175)$$

where we remember that the antiholomorphic part is implied, as was noted before. If we also expand the conformal change $\epsilon(z)$ as

$$\epsilon(z) = \sum_{n \in \mathbb{Z}} z^{n+1} \epsilon_n \quad (176)$$

then, we can plug it into the definition of Q_ϵ to get

$$\begin{aligned}
 Q_\epsilon &= \frac{1}{2\pi i} \oint dz \epsilon(z) T(z) \\
 &= \frac{1}{2\pi i} \oint dz \sum_{n \in \mathbb{Z}} z^{n+1} \epsilon_n \sum_{m \in \mathbb{Z}} z^{-m-2} L_m \\
 &= \sum_{m, n \in \mathbb{Z}} \frac{1}{2\pi i} \oint dz \epsilon_n L_m z^{n-m-1} \\
 &= \sum_{m, n \in \mathbb{Z}} \epsilon_n L_m \delta_{nm} \\
 &= \sum_{n \in \mathbb{Z}} \epsilon_n L_n
 \end{aligned} \tag{177}$$

These mode operators $\{L_n, \bar{L}_n\}$ are exactly the same as the generators $\{l_n, \bar{l}_n\}$ that we found in Sec. 2.1.3; those were the generators of the conformal transformation on the space of functions, whereas these are the local conformal transformation generators on the Hilbert space! In fact, we can use this equivalence even further to deduce that the generators of $SL(2, \mathbb{C})$ in the Hilbert space are L_{-1}, L_1 and L_0 (and their antiholomorphic components). In particular, the operator $L_0 + \bar{L}_0$ generates the dilations $(z, \bar{z}) \rightarrow \lambda(z, \bar{z})$, which is nothing but time translations according to the radial quantization convention. This implies that $L_0 + \bar{L}_0 \propto H$, the Hamiltonian of the system!

We can now derive the celebrated *Virasoro Algebra* by using Eqs. 175, 144 and 171 to get

$$\begin{aligned}
 [L_m, L_n] &= \frac{1}{2\pi i} \oint_0 dz z^{n+1} T(z) \frac{1}{2\pi i} \oint_z dw w^{m+1} T(w) \\
 &= \left(\frac{1}{2\pi i} \right)^2 \oint_0 dz z^{n+1} \oint dw_z w^{m+1} T(z) T(w) \\
 &= \left(\frac{1}{2\pi i} \right)^2 \oint_0 dz z^{n+1} \oint_z dw w^{m+1} \left\{ \frac{c}{2} \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{reg.} \right\} \\
 &= \frac{1}{2\pi i} \oint_0 dz z^{n+1} \left\{ \frac{c}{12} (m+1)m(m-1)z^{m-2} + 2(m+1)z^m T(z) + z^{n+1} \partial T(z) \right\} \\
 &= \frac{c}{12} (m^2 - 1)m \delta_{n+m,0} + 2(m+1) \frac{1}{2\pi i} \oint_0 dz z^{m+n+1} T(z) + \frac{1}{2\pi i} \oint_0 dz z^{n+m+2} \partial T(z) \\
 &= \frac{c}{12} (m^2 - 1)m \delta_{n+m,0} + 2(m+1)L_{n+m} + \frac{1}{2\pi i} \oint_0 dz z^{n+m+2} \partial T(z) \\
 &= \frac{c}{12} (m^2 - 1)m \delta_{n+m,0} + 2(m+1)L_{n+m} - \frac{1}{2\pi i} \oint_0 dz (n+m+2)z^{n+m+1} T(z) \\
 &= \frac{c}{12} (m^2 - 1)m \delta_{n+m,0} + 2(m+1)L_{n+m} - (n+m+2)L_{n-m} \\
 &= \frac{c}{12} (m^2 - 1)m \delta_{n+m,0} + (m-n)L_{n+m}
 \end{aligned} \tag{178}$$

where we have used the generalization of the Cauchy formula, and integration by parts. Note that the antiholomorphic commutation relation follow from the exact same calculations. Finally, because of the trivial OPE $T(z)\bar{T}(w) \sim 0$, we get

$$[L_m, \bar{L}_n] = 0 \tag{179}$$

To summarize, we have derived the Virasoro algebra for the quantum generators L_n with a central charge c ,

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\ [L_m, \bar{L}_n] &= 0 \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)m\delta_{n+m,0} \end{aligned} \quad (180)$$

3.2.2 The Hilbert Space

The Hilbert space of a conformal field theory can have an intricate structure in general. However, we can still understand a few common underlying features of it.

The vacuum state ought to be invariant under global conformal transformations, which implies that it is annihilated by L_{-1} , L_1 and L_0 and their antiholomorphic components. Further, consider the modes of the energy-momentum tensor, specifically the relation

$$L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z) \quad (181)$$

Since $T(z)$ is well-defined as $z \rightarrow 0$, we should have

$$L_n|0\rangle = \frac{1}{2\pi i} \oint dz z^{n+1} T(z)|0\rangle = 0 \quad \text{for } n \geq -1 \quad (182)$$

This leads us to the conditions

$$\begin{aligned} L_n|0\rangle &= 0 \\ \bar{L}_n|0\rangle &= 0 \end{aligned} \quad (n \geq -1) \quad (183)$$

which also includes as a subcondition the invariance of the vacuum under the global conformal group (as mentioned before). It also implies the vanishing of the vacuum expectation of the energy-momentum tensor

$$\langle 0|T(z)|0\rangle = \langle 0|\bar{T}(\bar{z})|0\rangle = 0 \quad (184)$$

Primary fields when acting on the vacuum create asymptotic states, which are eigenstates of the Hamiltonian. We demonstrate this below: Consider the OPE of $T(z)$ and a primary field $\phi(z, \bar{z})$ of conformal dimensions $(h, \bar{h})$, which we can translate into the operator formalism using the commutation relation identity as follows

$$\begin{aligned} [L_n, \phi(w, \bar{w})] &= \frac{1}{2\pi i} \oint_w dz z^{n+1} T(z) \phi(w, \bar{w}) \\ &= \frac{1}{2\pi i} \oint_w dz z^{n+1} \left[\frac{h\phi(w, \bar{w})}{(z-w)^2} + \frac{1}{z-w} \partial\phi(w, \bar{w}) + \text{reg.} \right] \\ &= h(n+1)w^n \phi(w, \bar{w}) + w^{n+1} \partial\phi(w, \bar{w}) \quad (n \geq -1) \end{aligned} \quad (185)$$

There is a similar expression that we get for the antiholomorphic counterpart of this equation, namely

$$[\bar{L}_n, \phi(w, \bar{w})] = \bar{h}(n+1)\bar{w}^n \phi(w, \bar{w}) + \bar{w}^{n+1} \bar{\partial}\phi(w, \bar{w}) \quad (n \geq -1) \quad (186)$$

We can then apply these relations to the asymptotic state defined by

$$|h, \bar{h}\rangle \equiv \phi(0, 0)|0\rangle \quad (187)$$

to get

$$\begin{aligned}
 L_0|h, \bar{h}\rangle &= L_0\phi(0, 0)|0\rangle \\
 &= \phi(0, 0)L_0|0\rangle + [L_0, \phi(0, 0)]|0\rangle \\
 &= 0 + h(0+1)0^0\phi(0, 0) + 0^{0+1}\partial\phi(0, 0)|0\rangle \\
 &= h\phi(0, 0)|0\rangle \\
 &= h|h, \bar{h}\rangle
 \end{aligned} \tag{188}$$

where we have used Eqs. 183 and 186 to simplify the commutation relations. Using the antiholomorphic form, we conclude that

$$\boxed{L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle \quad \bar{L}_0|h, \bar{h}\rangle = \bar{h}|h, \bar{h}\rangle} \tag{189}$$

which also implies that $|h, \bar{h}\rangle$ is an eigenstate of the Hamiltonian $H \propto L_0 + \bar{L}_0$, with an eigenvalue proportional to $h + \bar{h}$. In general, for $n > 0$, we have

$$\begin{aligned}
 L_n|h, \bar{h}\rangle &= L_n\phi(0, 0)|0\rangle \\
 &= \phi(0, 0)L_n|0\rangle + [L_n, \phi(0, 0)]|0\rangle \\
 &= 0 + h(n+1)0^n\phi(0, 0) + 0^{n+1}\partial\phi(0, 0)|0\rangle \\
 &= 0
 \end{aligned} \tag{190}$$

and so, we have (the antiholomorphic counterpart is the exact same calculation)

$$\boxed{\begin{aligned} L_n|h, \bar{h}\rangle &= 0 \\ \bar{L}_n|h, \bar{h}\rangle &= 0 \end{aligned} \quad (n > 0)} \tag{191}$$

We now know an eigenstate of the Hamiltonian, and it is an asymptotic state. We obtain the excited states above $|h, \bar{h}\rangle$ by applying ladder operators. For that, first, we need to expand the field $\phi(w)$ into its modes as given by Eq. 165. With that, we find the commutation relation

$$\begin{aligned}
 [L_n, \phi_m] &= \left(\frac{1}{2\pi i}\right)^2 \oint_0 dw \oint_w dz z^{n+1} T(z) w^{m+h-1} \phi(w) \\
 &= \left(\frac{1}{2\pi i}\right)^2 \oint_0 dw \oint_w dz z^{n+1} w^{m+h-1} T(z) \phi(w) \\
 &= \left(\frac{1}{2\pi i}\right)^2 \oint_0 dw \oint_w dz z^{n+1} w^{m+h-1} \left[\frac{h\phi(w, \bar{w})}{(z-w)^2} + \frac{1}{z-w} \partial\phi(w, \bar{w}) + \text{reg.} \right] \\
 &= \frac{1}{2\pi i} \oint_0 dw \left[h(n+1) w^{m+n+h-1} \phi(w) + w^{m+n+h} \partial\phi(w) \right] \\
 &= \frac{1}{2\pi i} \oint_0 dw [h(n+1) - (m+n+h)] w^{m+n+h-1} \phi(w) \\
 &= [n(h-1) - m] \frac{1}{2\pi i} \oint dw w^{m+n+h-1} \phi(w) \\
 &= [n(h-1) - m] \phi_{n+m}
 \end{aligned} \tag{192}$$

of which, we have the special case,

$$[L_0, \phi_m] = -m\phi_m \tag{193}$$

Note again that there is an antiholomorphic field, which acts as a spectator because it is decoupled from the holomorphic part. Evidently, ϕ_m acts like a raising or lowering operator for the eigenstates (for $m > 0$); application of the ϕ_{-m} operator results in a change of the conformal dimension by m (and the opposite for the lowering operator ϕ_m).

We also observe that by virtue of the Virasoro algebra, we have the commutation relation (for $m > 0$)

$$[L_0, L_{-m}] = mL_{-m} \quad (194)$$

which implies that the generators also increase the conformal dimension. This means that excited states may be obtained by successive applications of these operators on the asymptotic state $|h, \bar{h}\rangle \equiv |h\rangle$ (we keep making the notation lighter by dropping decoupled antiholomorphic components!). Thus, we have

$$|\text{excited}\rangle = L_{-k_1} L_{-k_2} \dots L_{-k_n} |h\rangle \quad (1 \leq k_1 \leq k_2 \leq \dots \leq k_n) \quad (195)$$

The increasing order of the k_i 's is just to have a straightforward ordering convention of the excited states, one can always have a different ordering by taking linear combinations of the aforementioned states and then applying the commutation rules of the Virasoro algebra as necessary. The state in Eq. 195 is an eigenstate of L_0 with eigenvalue $h' = h + k_1 + k_2 + \dots + k_n \equiv h + N$. Such states are called *descendants* of the asymptotic state $|h\rangle$; and N is called the level of the descendant. The number of distinct, linearly independent states at level N is simply the *partition number* of the integer N , given by $p(N)$.

3.3 The Free Boson

We will now all the operator machinery we have developed so far in this section to analyze the canonical quantization of the free boson on the (spacetime) cylinder.

3.3.1 Canonical Quantization on the Cylinder

Consider a free Bose field $\psi(x, t)$ on a cylinder of circumference L ; this enforces the periodic boundary condition $\psi(x + L, t) = \psi(x, t)$. This prompts us to do a Fourier expansion as follows:

$$\begin{aligned} \psi(x, t) &= \sum_n e^{\frac{2\pi i n x}{L}} \psi_n(t) \\ \psi_n(t) &= \frac{1}{L} \int dx e^{-\frac{2\pi i n x}{L}} \psi(x, t) \end{aligned} \quad (196)$$

The free field Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}g \int dx \partial_\mu \psi \partial^\mu \psi \quad (197)$$

which can be written in terms of the Fourier components as

$$\begin{aligned}
 \mathcal{L} &= \frac{1}{2}g \int dx \partial_\mu \psi \partial^\mu \psi \\
 &= \frac{1}{2}g \int dx [(\partial_t \psi)^2 - (\partial_x \psi)^2] \\
 &= \frac{1}{2}g \int dx [(\partial_t \psi)^2 - (\partial_x \psi)^2] \\
 &= \frac{1}{2}g \int dx \sum_{n,m} e^{\frac{2\pi i n x}{L}} \left[\dot{\psi}_n(t) \dot{\psi}_m(t) - \left(\frac{2\pi i n}{L} \right) \left(\frac{2\pi i m}{L} \right) \psi_n(t) \psi_m(t) \right] \\
 &= \frac{1}{2}g \sum_{n,m} L \delta_{n+m,0} \left[\dot{\psi}_n(t) \dot{\psi}_m(t) - \left(\frac{2\pi i n}{L} \right) \left(\frac{2\pi i m}{L} \right) \psi_n(t) \psi_m(t) \right] \\
 \mathcal{L} &= \frac{1}{2}gL \sum_n \left[\dot{\psi}_n \dot{\psi}_{-n} - \left(\frac{2\pi n}{L} \right)^2 \psi_n \psi_{-n} \right]
 \end{aligned} \tag{198}$$

The momentum conjugate to ψ_n is

$$\pi_n = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_n} = gL \dot{\psi}_{-n} \tag{199}$$

and we set up the commutation relation

$$[\psi_n, \pi_m] = i\delta_{nm} \tag{200}$$

as part of the canonical quantization. Finally, we can compute the Hamiltonian H as,

$$\begin{aligned}
 H &= \sum_n \pi_n \dot{\psi}_n - \mathcal{L} \\
 &= \frac{1}{gL} \sum_n \pi_n \pi_{-n} - \frac{1}{2}gL \sum_n \left[\dot{\psi}_n \dot{\psi}_{-n} - \left(\frac{2\pi n}{L} \right)^2 \psi_n \psi_{-n} \right] \\
 H &= \frac{1}{2gL} \sum_n [\pi_n \pi_{-n} + (2\pi n g)^2 \psi_n \psi_{-n}]
 \end{aligned} \tag{201}$$

We can see that this represents a collection of decoupled harmonic oscillators with frequencies $\omega_n = \frac{2\pi|n|}{L}$.

Notice that the $n = 0$ case is a special one, the frequency of that oscillator vanishes. This means that there is no mass term there, which, with the boundary conditions chosen, indicates conformal invariance. In fact, this mode makes us use a different definition of creation and annihilation operators. Instead of the usual annihilation operator

$$\tilde{a}_n = \frac{1}{\sqrt{4\pi g|n|}} (2\pi g|n| \psi_n + i\pi_{-n}) \tag{202}$$

and its Hermitian conjugate, the creation operator $\tilde{a}_n^\dagger$, we define the operators

$$a_n = \begin{cases} -i\sqrt{n}\tilde{a}_n & (n > 0) \\ i\sqrt{-n}\tilde{a}_{-n}^\dagger & (n < 0) \end{cases} \quad \bar{a}_n = \begin{cases} -i\sqrt{n}\tilde{a}_{-n} & (n > 0) \\ i\sqrt{-n}\tilde{a}_n^\dagger & (n < 0) \end{cases} \tag{203}$$

and then, we treat the zero mode ψ_0 differently. The commutation relations of these “new” operators follow from the commutation relations of the “original” operators

$$\begin{array}{ll} [\tilde{a}_n, \tilde{a}_m] = 0 & [a_n, a_m] = n\delta_{nm} \\ [\tilde{a}_n^\dagger, \tilde{a}_m^\dagger] = 0 & \implies [a_n, \bar{a}_m] = 0 \\ [\tilde{a}_n, \tilde{a}_m^\dagger] = \delta_{nm} & [\bar{a}_n, \bar{a}_m] = n\delta_{nm} \end{array} \quad (204)$$

We can then express the Hamiltonian as

$$H = \frac{1}{2gL}\pi_0^2 + \frac{2\pi}{L} \sum_{n \neq 0} (a_{-n}a_n + \bar{a}_{-n}\bar{a}_n) \quad (205)$$

and then, we find that the commutation relation of H with a_{-m} is

$$\begin{aligned} [H, a_{-m}] &= \left[\frac{1}{2gL}\pi_0^2 + \frac{2\pi}{L} \sum_{n \neq 0} (a_{-n}a_n + \bar{a}_{-n}\bar{a}_n), a_{-m} \right] \\ &= \left[\frac{1}{2gL}\pi_0^2, a_{-m} \right] + \frac{2\pi}{L} \sum_{n \neq 0} [(a_{-n}a_n + \bar{a}_{-n}\bar{a}_n), a_{-m}] \\ &= 0 + \frac{2\pi}{L} \sum_{n \neq 0} (a_{-n}[a_n, a_{-m}] + [a_{-n}, a_{-m}]a_n + \bar{a}_{-n}[\bar{a}_n, a_{-m}] + [\bar{a}_{-n}, a_{-m}]\bar{a}_n) \\ &= \frac{2\pi}{L} \sum_{n \neq 0} (na_{-n}\delta_{n-m,0} - na_n\delta_{n+m,0}) \\ &= \frac{2\pi}{L} ma_{-m} \end{aligned} \quad (206)$$

which means that for $m > 0$, a_{-m} acts like a raising operator; it produces a state with energy $E + \frac{2m\pi}{L}$ when applied to an eigenstate of H with energy E .

We now want to find the out the expression for $\psi(x, t)$. For that, we first note that the Heisenberg equations for the operators are

$$\begin{array}{ll} \psi_0(t) = \psi_0(0) + \frac{1}{gL}\pi_0 t & a_n(t) = a_n(0)e^{-\frac{2\pi i n t}{L}} \\ & \bar{a}_n(t) = \bar{a}_n(0)e^{-\frac{2\pi i n t}{L}} \end{array} \quad (207)$$

Further, since

$$\psi_n = \frac{i}{n\sqrt{4\pi g}}(a_n - \bar{a}_{-n}) \quad (208)$$

we can write down the full expression for $\psi(x, t)$ as

$$\begin{aligned} \psi(x, t) &= \sum_n e^{\frac{2\pi i n x}{L}} \psi_n(t) \\ &= \psi_0(t) + \sum_{n \neq 0} e^{\frac{2\pi i n x}{L}} \frac{i}{n\sqrt{4\pi g}} (a_n(t) - \bar{a}_{-n}(t)) \\ &= \psi_0 + \frac{1}{gL}\pi_0 t + \sum_{n \neq 0} \frac{i}{n\sqrt{4\pi g}} e^{\frac{2\pi i n x}{L}} (a_n e^{-\frac{2\pi i n t}{L}} - \bar{a}_{-n} e^{\frac{2\pi i n t}{L}}) \\ &= \psi_0 + \frac{1}{gL}\pi_0 t + \frac{i}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{1}{n} (a_n e^{-\frac{2\pi i n (x-t)}{L}} - \bar{a}_{-n} e^{\frac{2\pi i n (x+t)}{L}}) \end{aligned} \quad (209)$$

where we have defined the constant operators ψ_0, π_0, a_n and $\bar{a}_n$ to be the their expression at $t = 0$. Thus, we have the mode expansion of the field for arbitrary time; we can further “simplify” this by doing a Euclidean continuation ($t \rightarrow -i\tau$) and using the conformal coordinates

$$z = e^{\frac{2\pi(\tau-ix)}{L}} \quad \bar{z} = e^{\frac{2\pi(\tau+ix)}{L}} \quad (210)$$

we get the mode expansion of the field on the complex plane to be

$$\psi(z, \bar{z}) = \psi_0 - \frac{i}{4\pi g} \pi_0 \ln(z\bar{z}) + \frac{i}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{1}{n} (a_n z^{-n} + \bar{a}_n \bar{z}^{-n}) \quad (211)$$

Now, we know that ψ itself is not a primary field, but its derivatives $\partial\psi, \bar{\partial}\psi$ are. Consider $\partial\psi$, which we can calculate from the mode expansion to be

$$\begin{aligned} \partial\psi(z) &= -\frac{i}{4\pi g} \frac{\pi_0}{z} + \frac{i}{\sqrt{4\pi g}} \sum_{n \neq 0} a_n z^{-n-1} \\ \Rightarrow i\partial\psi(z) &= \frac{1}{\sqrt{4\pi g}} \sum_n a_n z^{-n-1} \end{aligned} \quad (212)$$

where we have now absorbed the zero mode into the sum by defining

$$a_0 = \bar{a}_0 = \frac{\pi_0}{\sqrt{4\pi g}} \quad (213)$$

The periodicity condition on the field ψ is what decouples the holomorphic and antiholomorphic components. Thus, the operators a_n are associated with “right-moving” excitations, while $\bar{a}_n$ are associated with “left-moving” ones. This is particularly consistent with the idea of vibrations on a string; in the context of string theory, these boundary conditions describe a closed string and the analogy is exact! In this case, the zero mode (ψ_0, π_0) is associated with the center-of-mass (COM) position and momentum respectively.

3.3.2 Vertex Operators

It is possible to construct an infinite variety of local fields related to ψ without introducing a scale, as there is no longer any canonical scaling dimension of the boson ψ . These called *vertex operators*, given by

$$V_\alpha(z, \bar{z}) = :e^{i\alpha\psi(z, \bar{z})}: \quad (214)$$

The normal-ordering has a very specific meaning in terms of the operators in the mode expansion of ψ ; namely (note that the operators within each exponential commute)

$$\begin{aligned} V_\alpha(z, \bar{z}) &= \exp \left\{ i\alpha\psi_0 + \frac{\alpha}{\sqrt{4\pi g}} \sum_{n>0} \frac{1}{n} (a_{-n} z^n + \bar{a}_{-n} \bar{z}^{-n}) \right\} \\ &\times \exp \left\{ \frac{\alpha}{4\pi g} \pi_0 - \frac{\alpha}{\sqrt{4\pi g}} \sum_{n>0} \frac{1}{n} (a_n z^{-n} + \bar{a}_n \bar{z}^n) \right\} \end{aligned} \quad (215)$$

Consider the OPE of $\partial\phi$ with V_α ,

$$\begin{aligned}
\partial\phi(z)V_\alpha(w,\bar{w}) &= \partial\phi(z) \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} : \phi(w,\bar{w})^n : \\
&= \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} n \times [\overline{\partial\phi(z)\phi(w,\bar{w})} : \phi(w,\bar{w})^{n-1} :] : \\
&\sim -\frac{1}{4\pi g} \frac{1}{z-w} \sum_{n=1}^{\infty} \frac{(i\alpha)^n}{(n-1)!} \partial\phi(z) : \phi(w,\bar{w})^{n-1} : \\
&\sim -\frac{1}{4\pi g} \frac{V_\alpha(w,\bar{w})}{z-w}
\end{aligned} \tag{216}$$

where we have used Eq. 136 to plug in the expression for the contraction for the n possible contractions. Similarly, we can also find out the OPE of the vertex operator with the energy-momentum tensor

$$\begin{aligned}
T(z)V_\alpha(w,\bar{w}) &= -2\pi g : \partial\phi(z)\partial\phi(z) : \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} : \phi(w,\bar{w})^n : \\
&= -2\pi g \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} : \partial\phi(z)\partial\phi(z) : : \phi(w,\bar{w})^n : \\
&= -2\pi g \sum_{n=1}^{\infty} \frac{(i\alpha)^n}{n!} \left[2n \times : \overline{\partial\phi(z)\phi(w,\bar{w})} \phi(w,\bar{w})^{n-1} : \right] \\
&\quad - 2\pi g \sum_{n=2}^{\infty} \frac{(i\alpha)^n}{n!} \left[n(n-1) \times : \overline{\partial\phi(z)\phi(w,\bar{w})} \overline{\partial\phi(z)\phi(w,\bar{w})} \phi(w,\bar{w})^{n-2} : \right] \\
&\sim \frac{1}{z-w} \sum_{n=1}^{\infty} \frac{(i\alpha)^n}{n!} n : \partial\phi(z)\phi(w,\bar{w})^{n-1} : - \frac{1}{8\pi g} \frac{(i\alpha)^2}{(z-w)^2} \sum_{n=2}^{\infty} \frac{(i\alpha)^n}{(n-2)!} : \phi(w,\bar{w})^{n-2} : \\
&\sim \frac{\alpha^2}{8\pi g} \frac{V_\alpha(w,\bar{w})}{(z-w)^2} + \frac{\partial_w V_\alpha(w,\bar{w})}{z-w}
\end{aligned} \tag{217}$$

which means that the vertex operators are primary fields with conformal dimensions

$$h = \bar{h} = \frac{\alpha^2}{8\pi g} \tag{218}$$

3.3.3 The Fock Space

The Hamiltonian of the system, given by Eq. 205 is independent of ψ_0 , and so π_0 is a conserved quantity; alternatively, we can say that it is a “good” quantum number to label the eigenstates of H . Since π_0 commutes with all $\{a_n, \bar{a}_n\}$, we can think of the Fock state as being built upon a one-parameter family of vacua $|\alpha\rangle$, where α is the continuous eigenvalue of $a_0 = \frac{\pi_0}{\sqrt{4\pi g}}$.

In the context of string theory, this just implies that the vacuum is spanned by different values of the COM momentum; and the other excited states are generated by using the creation and annihilation operators on vacuum states with different COM values. Ideally, we should label the vacuum states as $|0, \alpha\rangle$, but we usually drop one of the labels when it is obvious from context.

As mentioned above, we have the conformal creation and annihilation operators which act on the conformal vacuum as follows

$$a_n|\alpha\rangle = \bar{a}_n|\alpha\rangle = 0 \quad (n > 0) \quad (219)$$

with

$$a_0|\alpha\rangle = \bar{a}_0|\alpha\rangle = \alpha \quad (220)$$

They are creation operators when $n < 0$, as is consistent with the mode expansion of the field.

Next, we consider the energy-momentum tensor, given by

$$\begin{aligned} T(z) &= -2\pi g; \partial\phi(z)\partial\phi(z): \\ &= \frac{1}{2} \sum_{n,m \in \mathbb{Z}} z^{-n-m-2} :a_n a_m: \end{aligned} \quad (221)$$

where we have used Eq. 212. But, using the inverse relation of the energy-momentum modes, we have

$$\begin{aligned} L_l &= \frac{1}{2\pi i} \oint dz z^{l+1} T(z) \\ &= \frac{1}{2\pi i} \oint dz z^{l+1} \left(\frac{1}{2} \sum_{n,m \in \mathbb{Z}} z^{-n-m-2} :a_n a_m: \right) \\ &= \frac{1}{2} \sum_{n,m \in \mathbb{Z}} \left(\frac{1}{2\pi i} \oint dz z^{l-n-m-1} \right) :a_n a_m: \\ &= \frac{1}{2} \sum_{n,m \in \mathbb{Z}} \delta_{l-n-m,0} :a_n a_m: \\ &= \frac{1}{2} \sum_{m \in \mathbb{Z}} :a_{l-m} a_m: \end{aligned} \quad (222)$$

and so, we obtain

$$\boxed{\begin{aligned} L_n &= \frac{1}{2} \sum_m a_{n-m} a_m \\ \bar{L}_n &= \frac{1}{2} \sum_m \bar{a}_{n-m} \bar{a}_m \end{aligned}} \quad (223)$$

Particularly, we observe that the Hamiltonian can be rewritten in terms of the zero modes as

$$\boxed{H = \frac{2\pi}{L} (L_0 + \bar{L}_0)} \quad (224)$$

which is consistent with our previous arguments based on the generator algebra.

With this, we can now carefully construct the state space of the this theory. First, we note that an excited state can be generated using the creation operators a_{-n} ($n > 0$) due to the commutation relation $[L_0, a_{-n}] = na_{-n}$ and its antiholomorphic counterpart (recall that $H \propto L_0 + \bar{L}_0$). Thus, we get that the general excited state is of the form

$$\boxed{|\Psi\rangle = a_{-1}^{n_1} a_{-2}^{n_2} \dots \bar{a}_{-1}^{m_1} \bar{a}_{-2}^{m_2} \dots |\alpha\rangle \quad (n_i, m_i \geq 0)} \quad (225)$$

These are all eigenstates of $L_0, \bar{L}_0$ with conformal dimensions given by the eigenvalues of these operators, namely

$$L_0(a_{-1}^{n_1} a_{-2}^{n_2} \dots \bar{a}_{-1}^{m_1} \bar{a}_{-2}^{m_2} \dots |\alpha\rangle) = a_{-1}^{n_1} a_{-2}^{n_2} \dots \bar{a}_{-1}^{m_1} \bar{a}_{-2}^{m_2} \dots (L_0 + \sum_j j n_j) |\alpha\rangle \quad (226)$$

wherein we have used the commutation relation $[L_0, a_{-n}] = n a_{-n}$ consecutively; and finally, we compute (using the definition of a_0)

$$\begin{aligned} L_0 |\alpha\rangle &= \frac{1}{2} \sum_{m \in \mathbb{Z}} a_{-m} a_m |\alpha\rangle \\ &= \frac{1}{2} a_0^2 |\alpha\rangle \\ &= \frac{1}{2} \alpha^2 |\alpha\rangle \end{aligned} \quad (227)$$

to get the conformal dimensions of the general excited state $|\Psi\rangle$ to be

$$\boxed{h = \frac{1}{2} \alpha^2 + \sum_j j n_j \quad \bar{h} = \frac{1}{2} \alpha^2 + \sum_j j m_j} \quad (228)$$

The calculation for the antiholomorphic part $\bar{L}_0$ is the exact same as the one carried above.

Finally, we will elaborate on the fact that there are two vacuum states, the “absolute” vacuum labelled by $|0\rangle$, and the vacuum with a non-zero COM momentum (or alternatively, the one-parameter family labelled by the eigenvalue α of π_0) given by $|\alpha\rangle$. To obtain these vacuum states from $|0\rangle$, we use the vertex operators $V_\alpha(z, \bar{z})$. For that, we will show that

$$\begin{aligned} \pi_0(V_\alpha(0, 0)|0\rangle) &= \pi_0(e^{i\alpha\psi(0,0)}|0\rangle) \\ &= \sum_n \pi_0 \frac{(i\alpha\psi(0,0))^n}{n!} |0\rangle \\ &= \sum_n \frac{(i\alpha)^n}{n!} \pi_0 \psi^n |0\rangle \\ &= \sum_n \frac{(i\alpha)^n}{n!} ([\pi_0, \psi] + \psi \pi_0) \psi^{n-1} |0\rangle \\ &= \left(\sum_n \frac{(i\alpha)^n}{n!} n \times (-i\psi^{n-1}) + \sum_n \frac{(i\alpha)^n}{n!} \psi^n \pi_0 \right) |0\rangle \iff [\pi_0, \psi(0,0)] = [\pi_0, \psi_0] = -i \\ &= \alpha V_\alpha(0, 0) |0\rangle \end{aligned} \quad (229)$$

Thus, we have

$$\boxed{|\alpha\rangle = V_\alpha(0, 0) |0\rangle} \quad (230)$$

With this, we conclude the analysis of the free boson on a spacetime cylinder. We have “solved” our first illustration of a quantum conformal field theory!

3.4 Normal Ordering

Up to now, our prescription for normal ordering has been very simple: subtract the corresponding expectation value. This has been useful because the OPEs of various operator products usually contain a single singular term, which vanishes when the corresponding singular term from the expectation value is subtracted from it. In terms of the modes, this is equivalent to putting all the annihilation operators towards the rightmost positions so that they annihilate the vacuum.

However, this works only for free fields. For instance, consider the regularization of $T(z)T(w)$. The OPE of this has 3 singular terms; and the expectation value only gets rid of the term proportional to the central charge; this still leaves two subleading singular terms, and we have not regularized this expression completely. Nonetheless, this motivates us to generalize the regularization prescription: subtract all singular terms of the OPE instead of just subtracting the vacuum expectation value as $z \rightarrow w$.

More explicitly, if the OPE of $A(z)$ and $B(z)$ is

$$A(z)B(w) = \sum_{n=-\infty}^N \frac{\{AB\}_n(w)}{(z-w)^n} \quad (231)$$

then the normal-ordered version is given by

$$\boxed{(AB)(w) = \{AB\}_0(w)} \quad (232)$$

With this definition, we generalize the definition of a contraction, so as to preserve the form of Wick's theorem, namely, we define a contraction as

$$\boxed{\overline{A(z)B(w)} = \sum_{n=1}^N \frac{\{AB\}_n(w)}{(z-w)^n}} \quad (233)$$

to include all the singular terms in an OPE. Thus, we can write the normal-ordering prescription as

$$(AB)(w) = \lim_{z \rightarrow w} \left[A(z)B(w) - \overline{A(z)B(w)} \right] \quad (234)$$

or alternatively, the OPE of $A(z)$ with $B(w)$ is expressed as

$$A(z)B(w) = \overline{A(z)B(w)} + (A(z)B(w)) \quad (235)$$

wherein the last term stands for the complete sequence of regular terms whose explicit forms can be extracted from the Taylor series expansion of $A(z)$ around w ,

$$(A(z)B(w)) = \sum_{k \geq 0} \frac{(z-w)^k}{k!} (\partial^k AB)(w) \quad (236)$$

Contour integration provides us a useful representation of normal ordering, namely

$$\boxed{(AB)(w) = \frac{1}{2\pi i} \oint_w \frac{dz}{z-w} A(z)B(w)} \quad (237)$$

which can be verified as follows

$$\begin{aligned}
 \frac{1}{2\pi i} \oint_w \frac{dz}{z-w} A(z) B(w) &= \frac{1}{2\pi i} \oint_w \frac{dz}{z-w} \sum_{n=-\infty}^N \frac{\{AB\}_n(w)}{(z-w)^n} \\
 &= \sum_{n=-\infty}^N \frac{1}{2\pi i} \oint_w dz \frac{\{AB\}_n(w)}{(z-w)^{n+1}} \\
 &= \sum_{n=-\infty}^N \delta_{n,0} \{AB\}_n(w) \\
 &= \{AB\}_0(w) \\
 &= (AB)(w)
 \end{aligned} \tag{238}$$

We can translate this expression into one involving the modes of the operators $A(z), B(z)$. We quote the result here directly, namely,

$$(AB)_m = \sum_{n \leq -h_A} A_n B_{m-n} + \sum_{n > -h_A} B_{m-n} A_n \tag{239}$$

where

$$\begin{aligned}
 A(z) &= \sum_n (z-x)^{-n-h_A} A_n(x) \\
 B(w) &= \sum_n (w-x)^{-n-h_B} B_n(x) \\
 (AB)(y) &= \sum_n (y-x)^{-n-h_A-h_B} (AB)_n(x)
 \end{aligned} \tag{240}$$

This makes the non-commutivity and non-associativity of the normal-ordering manifest, as $(AB) \neq (BA)$ and $((AB)C) \neq (A(BC))$. Indeed, for interacting theories, a generalization of Wick's theorem is needed, and though we will not go into the detail here, we have motivated part of the reformulation here.