

### 6.3 continued

#### Summary of Method of Shells

Slice parallel to the axis of revolution.

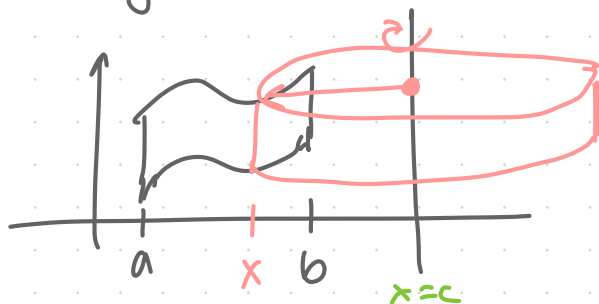
Draw an example shell and find its volume,  $dV$ .

$$dV = 2\pi (\text{radius}) \cdot (\text{height}) \, dx/dy$$

Total volume :  $V = \int_a^b dV$

and bounds  $a$  &  $b$  are determined by the region

Revolving around a vertical line  
( $y$ -axis or  $x=c$ ) *integrate in  $x$*



radius of a shell is:

$x$  if revolving about  $y$ -axis

$x-c$  /  $c-x$  if " " "  $x=c$

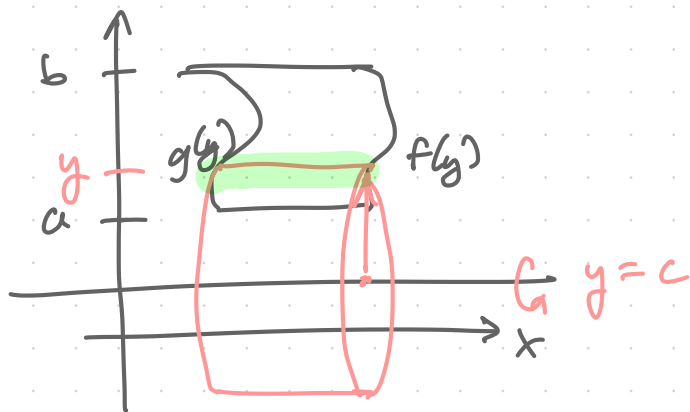
height of a shell:  $f(x)$  or  $f(x) - g(x)$

↑  
top

↑  
bottom

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revolving around a horizontal line  
( $x$ -axis or  $y=c$ )



\* radius goes from  
axis of revolution  
to slice thru  
your region

radius of a shell:

$y$  if revolving about  $x$ -axis

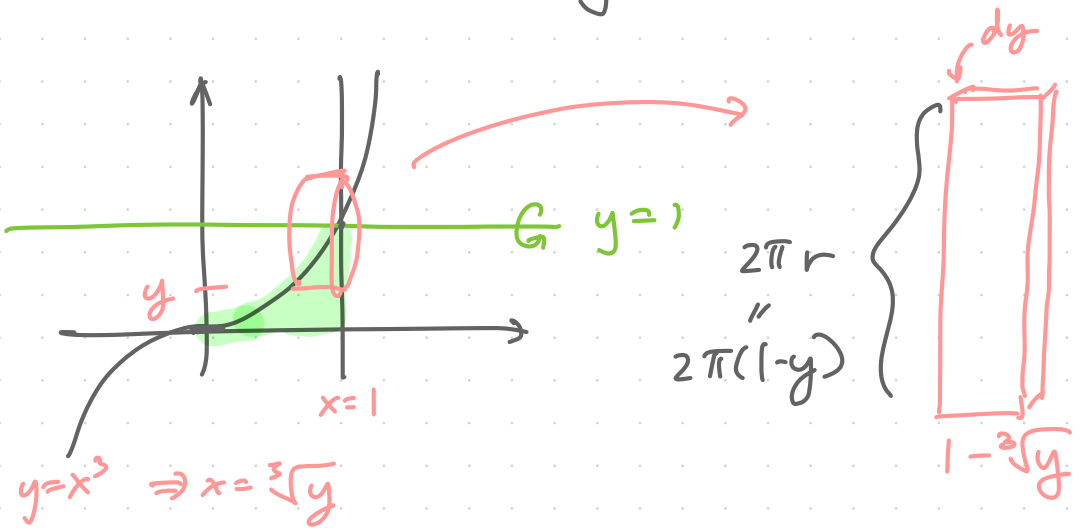
$y-c$  /  $c-y$  if " " "  $y=c$

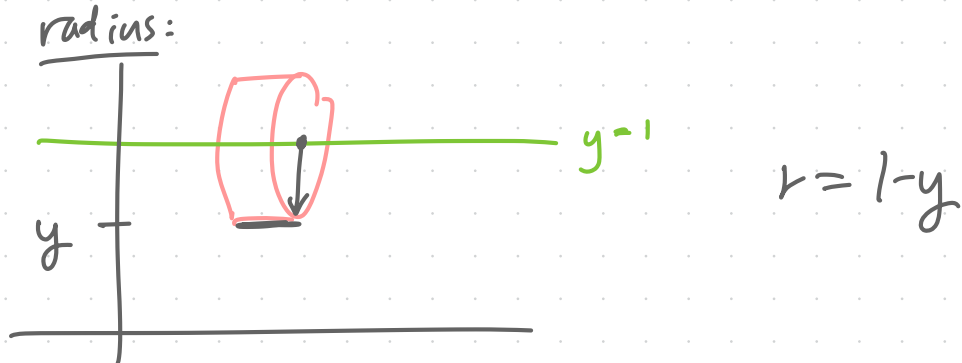
height is  $f(y)$  or  $f(y) - g(y)$

$$\left( y = \sqrt{x} \xrightarrow[\text{fcn of } y]{\text{convert to}} x = y^2 \right)$$

ex: region bounded by  $y = x^3$ ,  $y = 0$  and  $x = 1$

revolved about  $y=1$





$$dV = 2\pi(1-y)(1-\sqrt[3]{y})dy$$

$$V = \int_0^1 \underline{2\pi(1-y)(1-\sqrt[3]{y})} dy$$

$$= 2\pi \int_0^1 1 - y - y^{1/3} + y^{4/3} dy$$

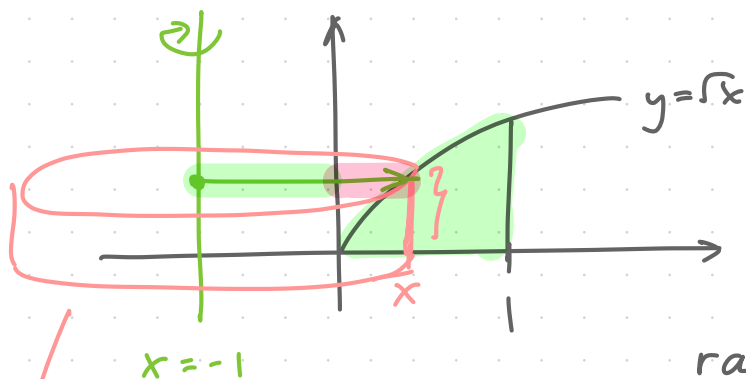
$$= 2\pi \left( y - \frac{1}{2}y^2 - \frac{3}{4}y^{4/3} + \frac{3}{7}y^{7/3} \right) \Big|_0^1$$

$$= 2\pi \left( 1 - \frac{1}{2} - \frac{3}{4} + \frac{3}{7} - 0 \right)$$

$$= 2\pi \left( \frac{28 - 14 - 3(7) + 3(4)}{28} \right)$$

$$= \frac{10\pi}{28} = \frac{5\pi}{14}$$

ex: region bdd by  $y = \sqrt{x}$ ,  $x = 1$ ,  $y = 0$   
 revolved about  $x = -1$



$$\begin{aligned} \text{radius} &= \text{right} - \text{left} \\ &= x - (-1) \\ &= x + 1 \end{aligned}$$

$\sqrt{x}$

$$dV = 2\pi(x+1)\sqrt{x} dx$$

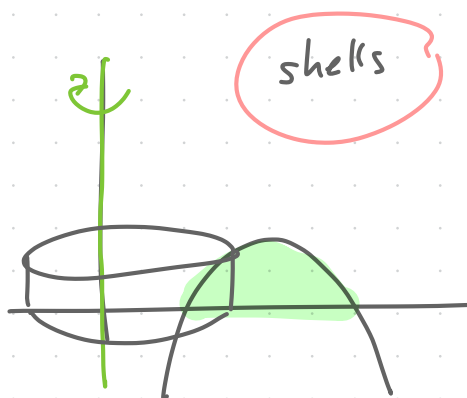
$2\pi \cdot \text{rad} = 2\pi(x+1)$

$\pi dx$

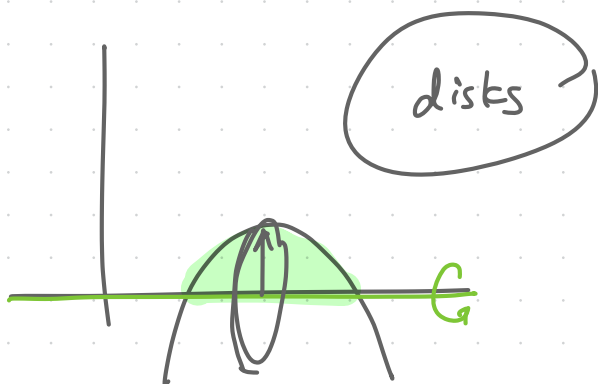
$$V = \int_0^1 2\pi(x+1)\sqrt{x} dx = \frac{32\pi}{15}$$

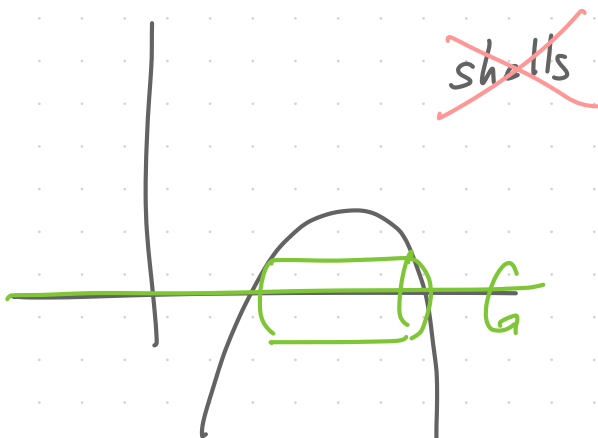
\* bounds come from the region  
 not the solid.

# Method of Disks/Washers vs. Method of Shells



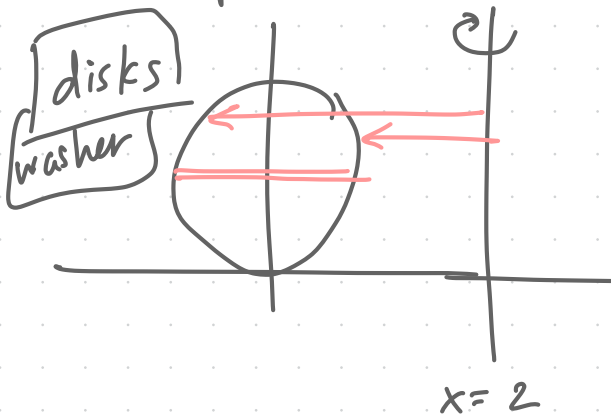
inner / outer radius  
given by same fn  
bad!





height is "right"  
minus "left" but  
they're the same?  
bad!

HW problem



$$x^2 + y^2 = 1$$

$$y = \pm \sqrt{1-x^2}$$

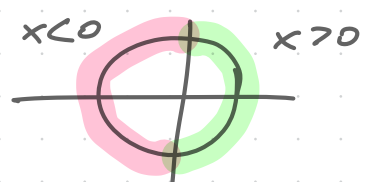
$$\begin{array}{cc} \sqrt{1-x^2} & , -\sqrt{1-x^2} \\ \text{top} & \text{bottom} \end{array}$$

$$x = \pm \sqrt{1-y^2}$$

$$\begin{array}{cc} x = \sqrt{1-y^2} & , x = -\sqrt{1-y^2} \\ \text{right} & \text{left} \end{array}$$

$$r = C - \text{right half of circle}$$

$$r = 2 - \sqrt{1-(y-1)^2}$$

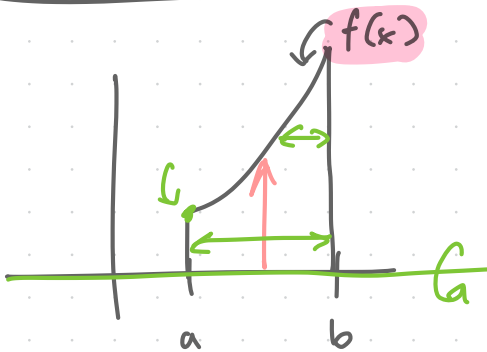


$$x^2 + (y-1)^2 = 1$$

$$x^2 = 1 - (y-1)^2$$

$$x = \pm \sqrt{1 - (y-1)^2}$$

$$R = 2 - (-\sqrt{1 - (y-1)^2}) = 2 + \sqrt{1 - (y-1)^2}$$



disks - slicing vertically?

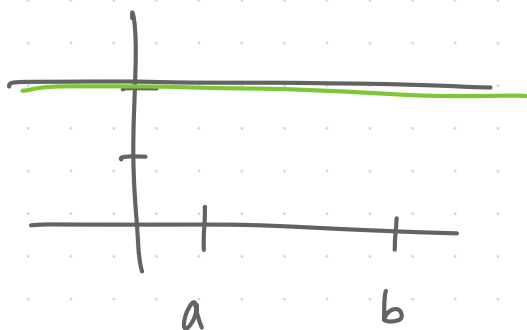
(OR)

~~shells - slicing horiz.?~~

## 6.5 - Average Value of a Function (on an interval $[a, b]$ )

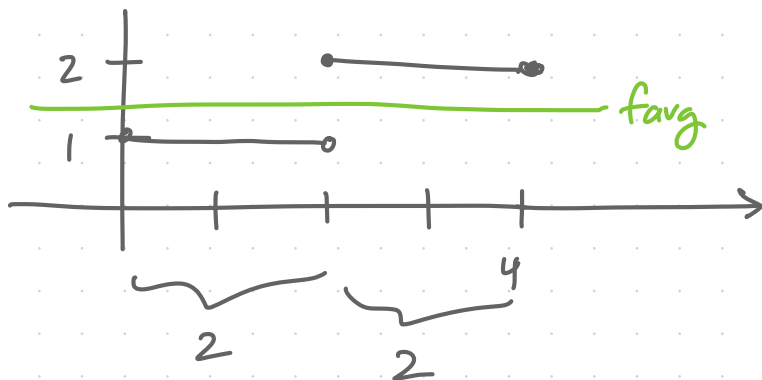
Start with some examples

ex:  $f(x) \equiv 2$



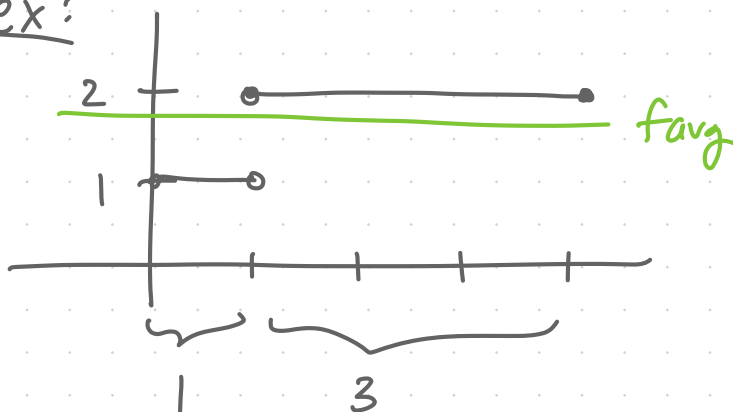
$$f_{\text{avg}} \text{ or } f_{\text{ave}} = 2$$

ex:

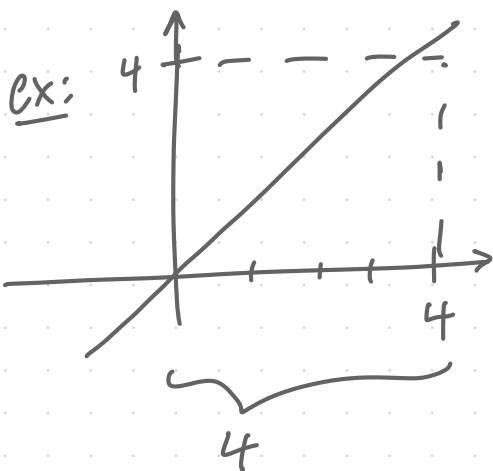


$$\begin{aligned} \text{avg value on } [0, 4] \\ = \frac{1+2}{2} = \frac{3}{2} \end{aligned}$$

ex:



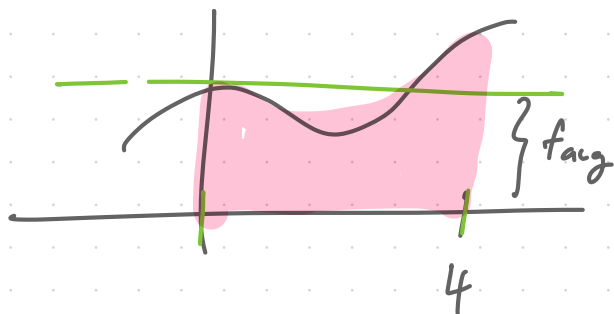
$$\begin{aligned}
 \text{avg value} &= 1 \cdot \left(\frac{1}{4}\right) + 2 \cdot \left(\frac{3}{4}\right) \\
 &= 1 \cdot (0.25) + 2 \cdot (.75) \\
 &= \frac{7}{4} = 1.75
 \end{aligned}$$



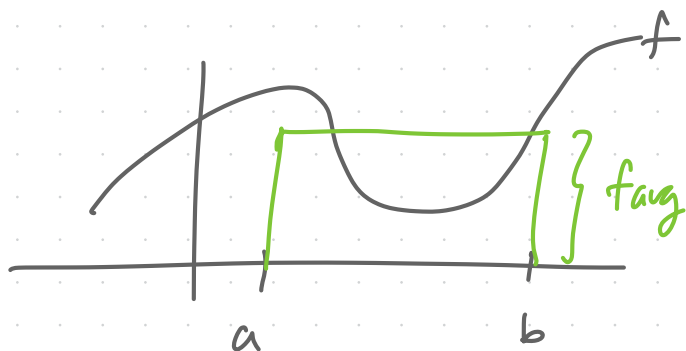
$f_{\text{avg}} = ?$   
on  $[0, 4]$

guess for  $f_{\text{avg}}$ ?  
what things are relevant?

"height" of  $f$ , length of interval



What height would a constant function be if it had the same average value?



Area under  $f$  = area of rectangle of height  $f_{avg}$  and width  $b-a$  }

$$\int_a^b f(x) dx = f_{avg} \cdot (b-a)$$

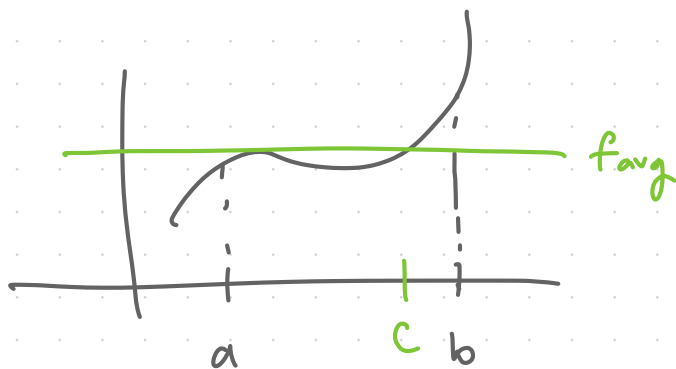
$$f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx$$

↙ this formula is what you need to know from this section!

If you were to replace  $f$  with a constant function, its height would be  $f_{\text{avg}}$ .

### MVT for Integrals

If  $f$  is continuous on  $[a, b]$ , then there exists a number  $c$  in  $(a, b)$  such that  $f(c) = f_{\text{avg}}$  (i.e.  $f(c) = \frac{1}{b-a} \int_a^b f(x) dx$ )



\* MVT shows up on thw, but is usually not an exam topic

ex: (a) Find  $f_{\text{avg}}$  of  $f(x) = \ln x$  on  $[1, 3]$

(b) Find any points  $c$  in  $(1, 3)$  with  $f(c) = f_{\text{avg}}$

(c) Sketch  $f$  and a line representing  $f_{\text{avg}}$ .

Solution

$$(a) \quad f_{\text{avg}} = \frac{1}{3-1} \int_1^3 \ln x \, dx$$

|                       |           |
|-----------------------|-----------|
| $u = \ln x$           | $dv = dx$ |
| $du = \frac{1}{x} dx$ | $v = x$   |

$$= \frac{1}{2} \left( x \ln x \Big|_1^3 - \int_1^3 x \cdot \frac{1}{x} dx \right)$$

$$= \frac{1}{2} \left( 3 \ln(3) - \cancel{\ln(1)}^0 - \int_1^3 1 \, dx \right)$$

$$= \frac{1}{2} \left( 3 \ln(3) - x \Big|_1^3 \right)$$

$$= \frac{1}{2} (3 \ln 3 - 2)$$

$$= \frac{3}{2} \ln(3) - 1 \approx 0.65$$

$$(b) \quad f(c) = f_{avg}$$

$$\ln(c) = \frac{3}{2} \ln(3) - 1$$

$$e^{\ln(c)} = e^{\frac{3}{2} \ln(3) - 1}$$

$$c = e^{\frac{3}{2} \ln(3) - 1} \approx 1.91$$

