# 3-MANIFOLDS SEMINAR

## Heegaard Splittings

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Departamento de Matemática - ICEx - UFMG

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# Introduction

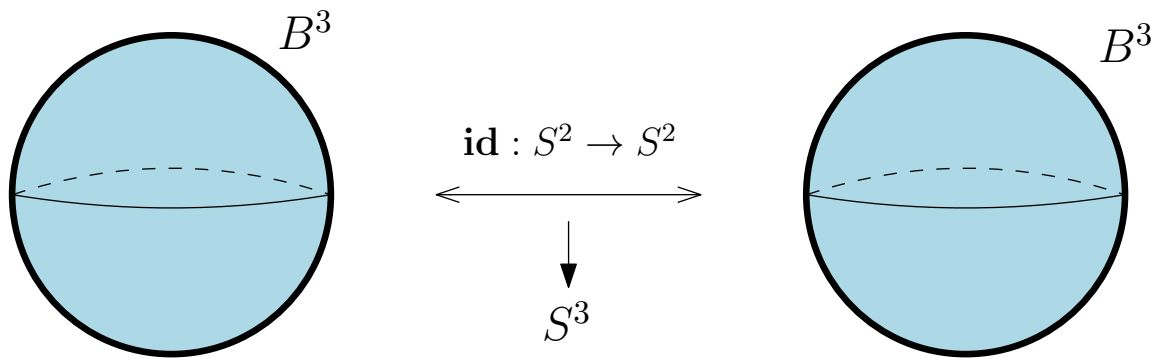
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4. *The Heegaard genus of $M$ is the smallest possible genus of a possible Heegaard splitting of $M$.*

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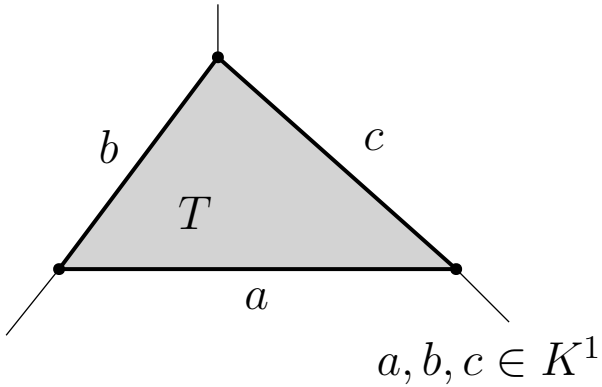
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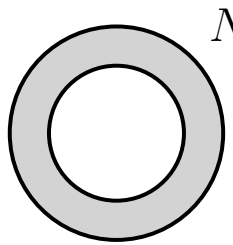
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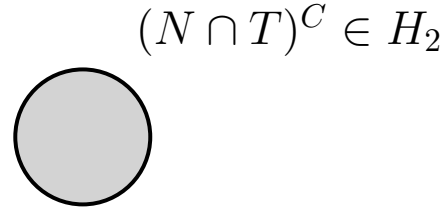
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- Let $H_2$ be the complement of $N$.

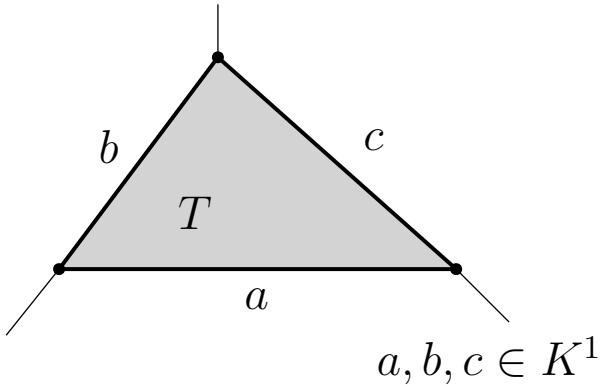
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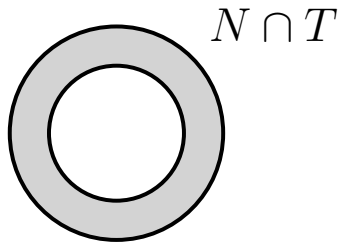


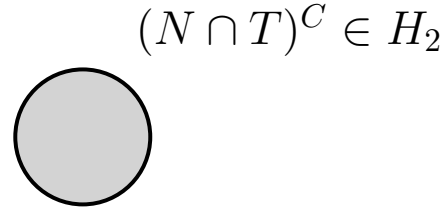



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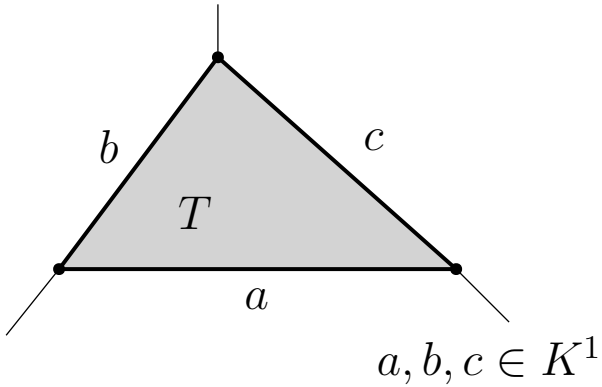


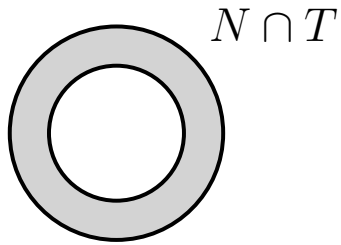



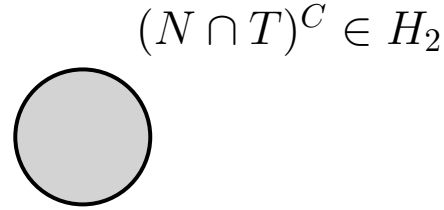




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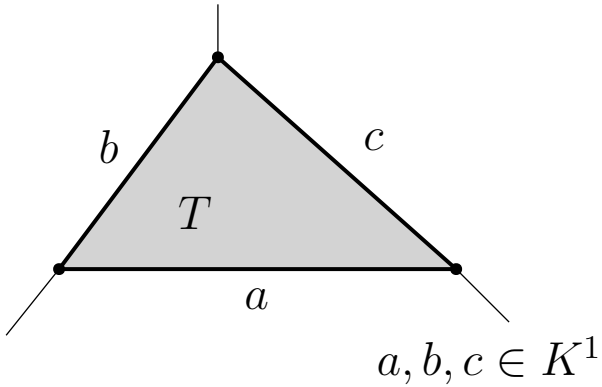


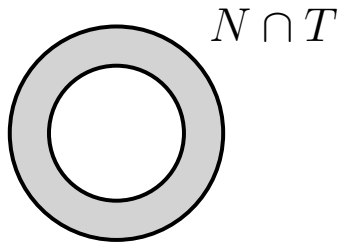



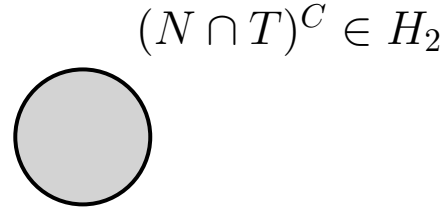




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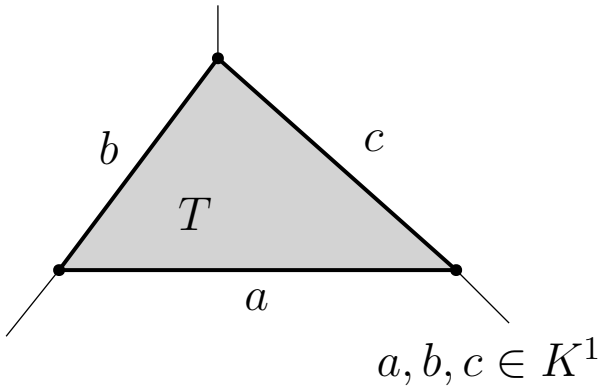



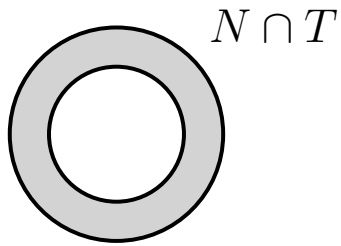



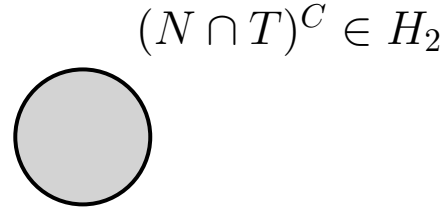

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## Remark

This proof allows us to produce examples.

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## $T^3 = S^1 \times S^1 \times S^1$

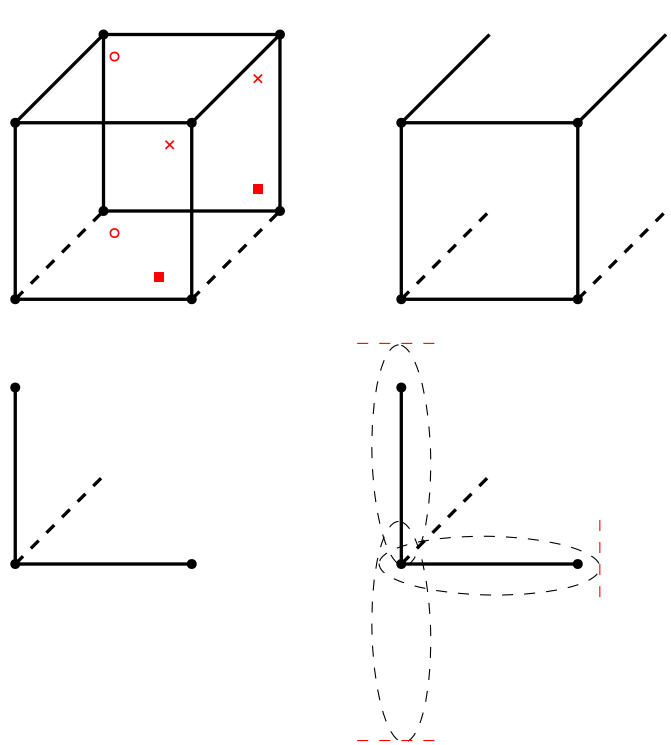
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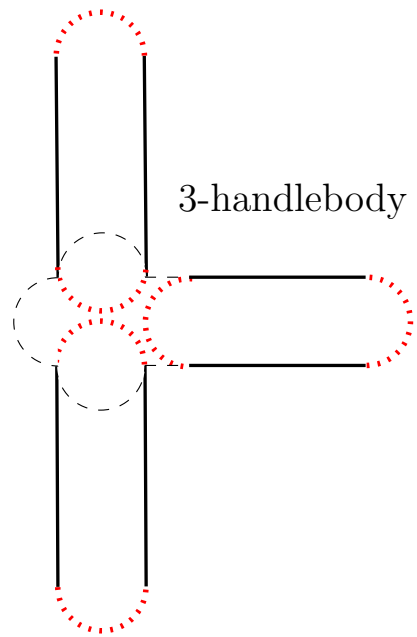
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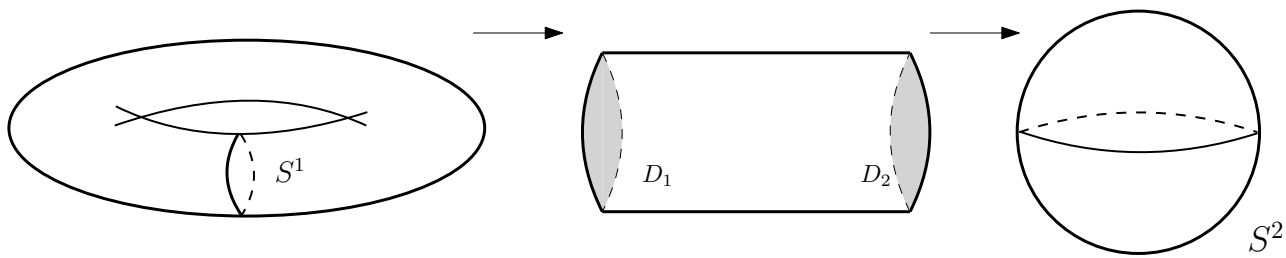
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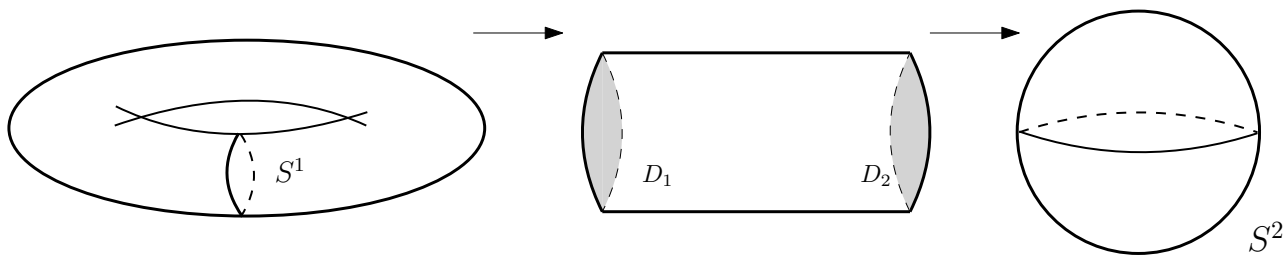
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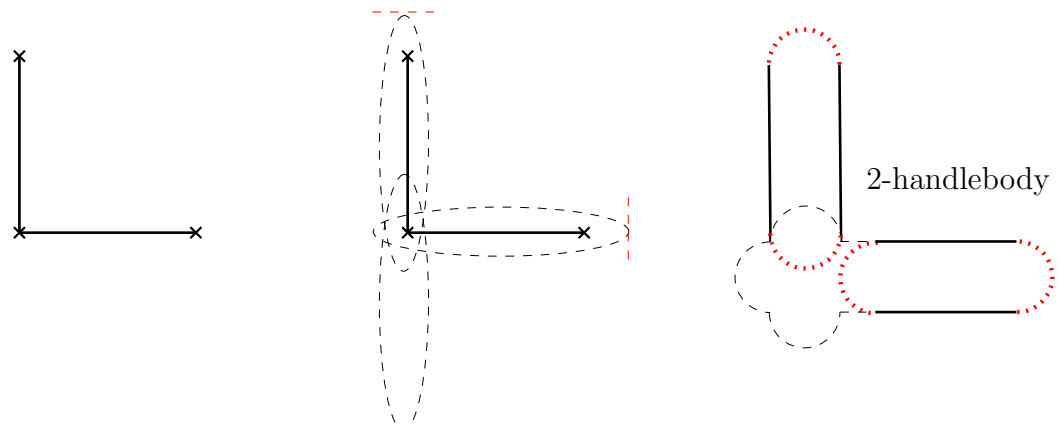
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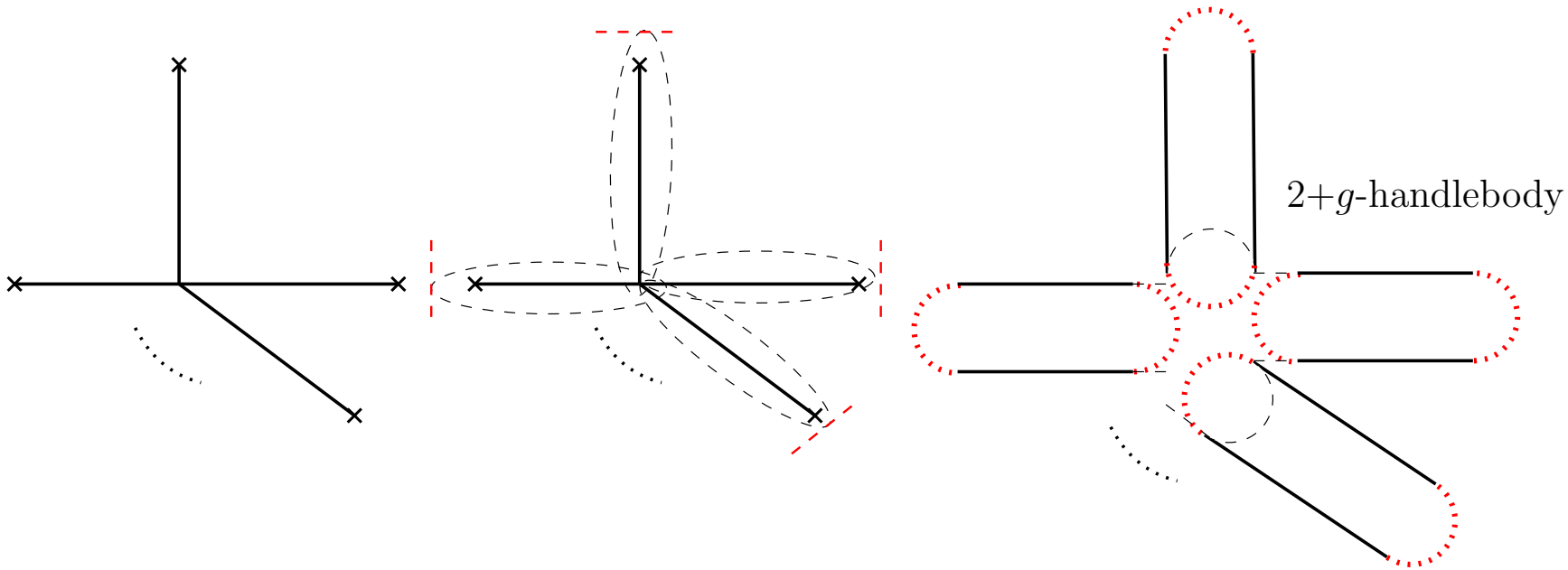



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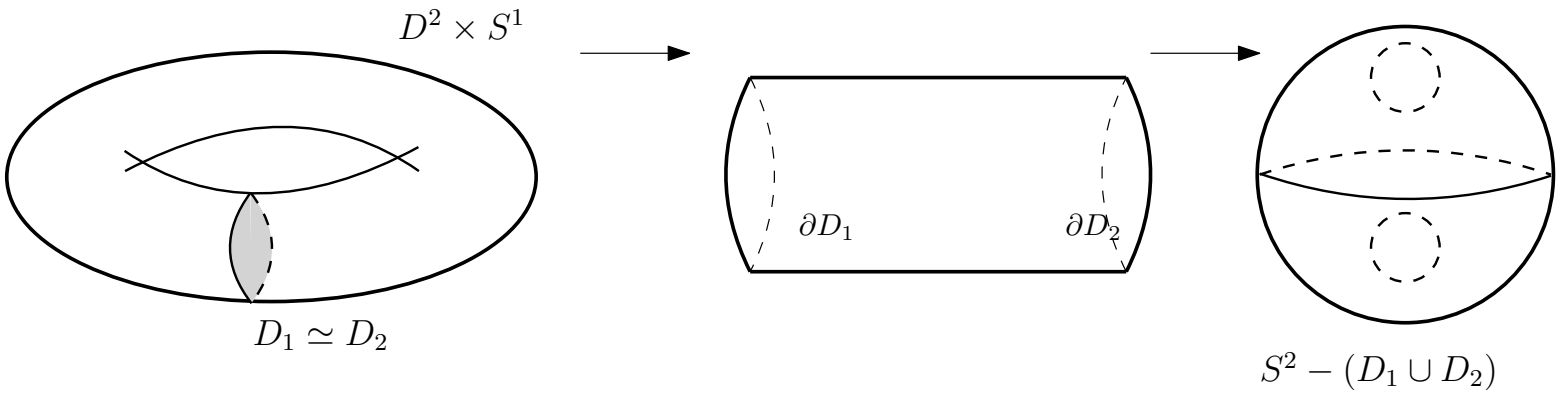
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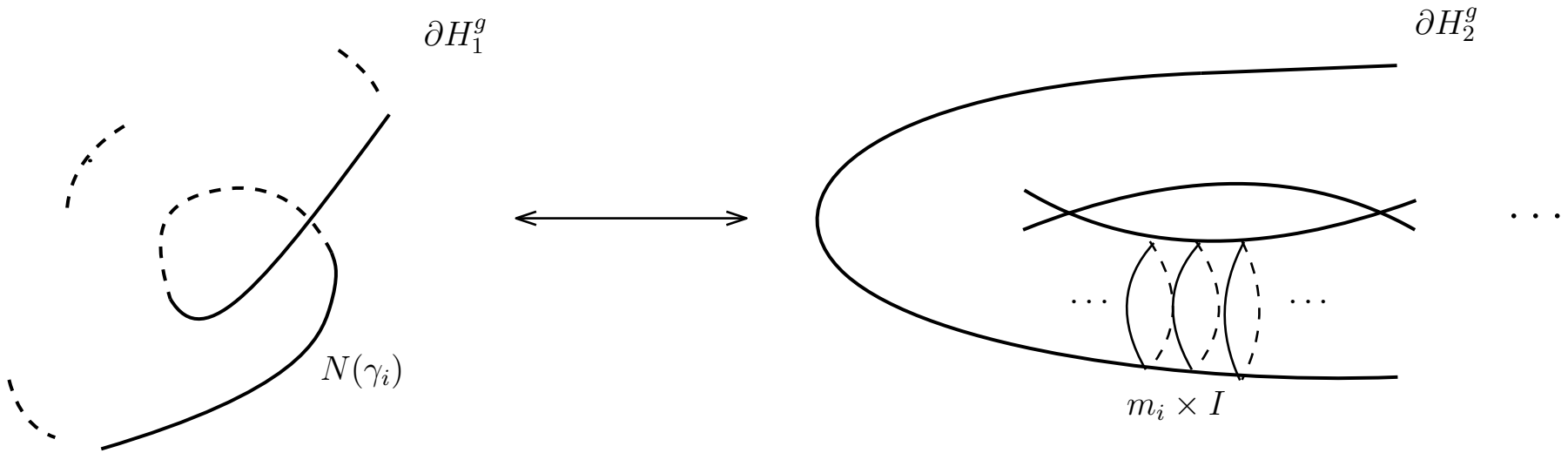
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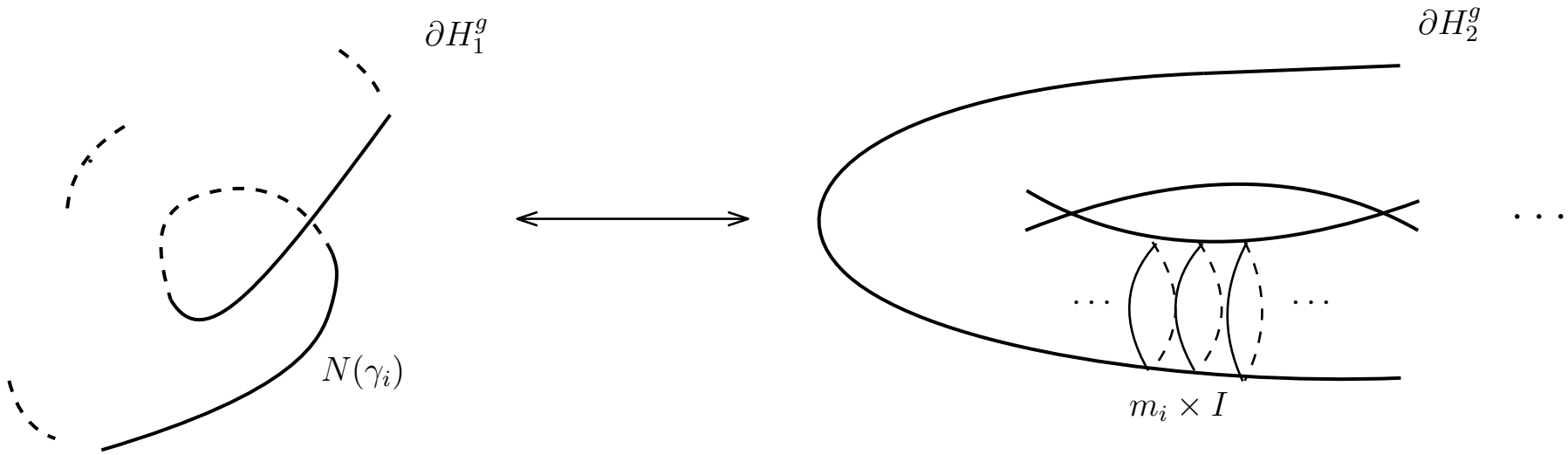
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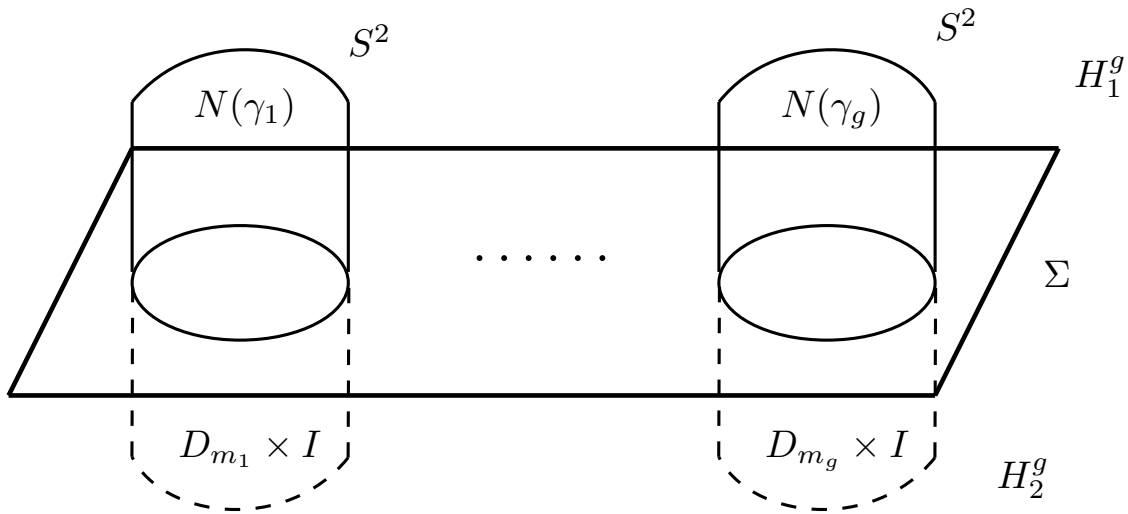
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The resulting manifold $M$ has spheres as boundary components.



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- Depends on the choices of the curves $\gamma_i$.

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- Let $D$ be the disc spanned by $m$. $T_1 = D^2 \times S^1$ and $T_2 = S^1 \times D^2$.
- We may remove a neighborhood of $D^2$ and glue along the boundary.
- There is a homeomorphism $G : D \cup_f T_2 \longrightarrow D \cup_{f'} T_2$.

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- $T_1 \cup_f T_2 = (D \cup_f T_2) \cup_f (T_1 \setminus D)$
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## Conjecture

Let $\{\gamma_i\}_{i=1}^{g}$ be characteristic curves in $H_1^g$ and $h, h' : \partial H_1^g \longrightarrow \partial H_2^g$ homeomorphisms such that $h(\gamma_i) = h'(\gamma_i)$ for each $i$. Then the resulting manifolds glued along $h$ and $h'$ are homeomorphic.

# Lens Spaces

## Definition

*Let $p, q, r, s \in \mathbb{Z}$ with $ps - rq = \pm 1$. The lens space $L^{p,q}_{r,s}$ is the result gluing two solid torus $T_1 = D^2 \times S^1$ and $T_2 = S^1 \times D^2$ along their boundaries with respecto to*

$$\begin{aligned} f : \partial T_1 &\longrightarrow \partial T_2 \\ (z, w) &\longmapsto (z^p w^r, z^q w^s) \end{aligned}$$

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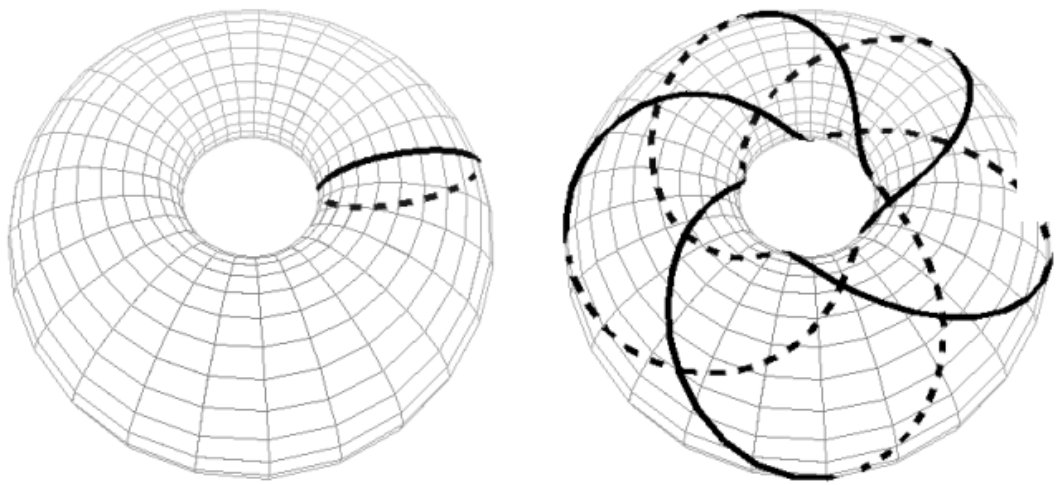


Figure 2.1: $f$ maps a meridian curve to a (5,3) torus curve, when $|a| = 5$, $|b| = 3$.

# The case $p = 0$

## Remark

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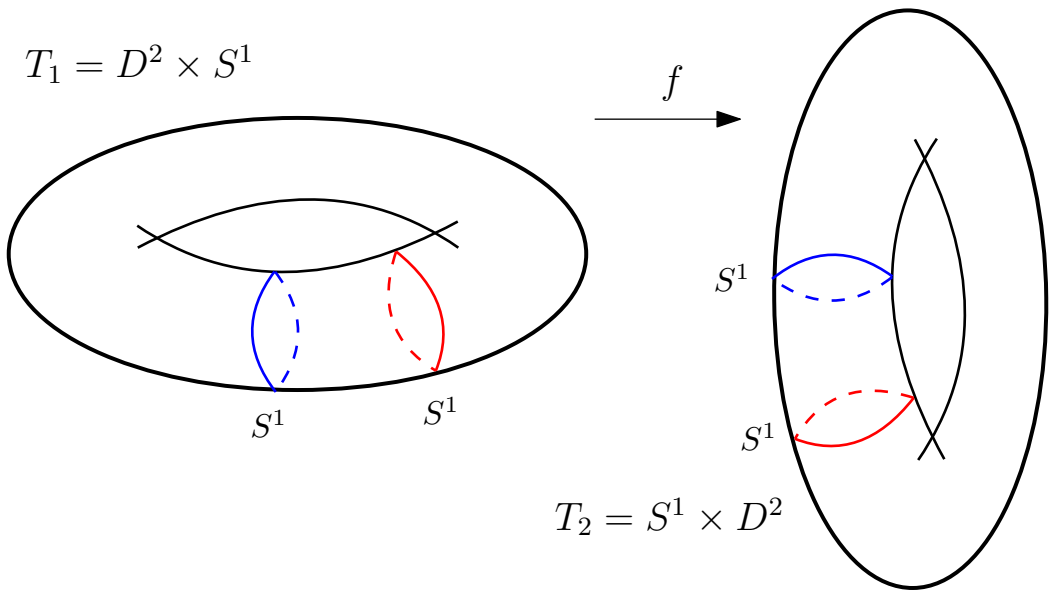
- $p = 0$: $L^{0,q}_{r,s} \simeq L^{1,0}_{0,1} \longrightarrow f(z, w) = (w, z)$.
- Identification of meridian disks along the boundaries.
- $L^{1,0}_{0,1} \simeq (D^2 \cup_f D^2) \times S^1 = S^2 \times S^1$.

# The case $p = 0$

## Remark

The lens space $L^{p,q}_{r,s}$ is uniquely determined (up to homeomorphism) by the numbers $p$ and $q$.

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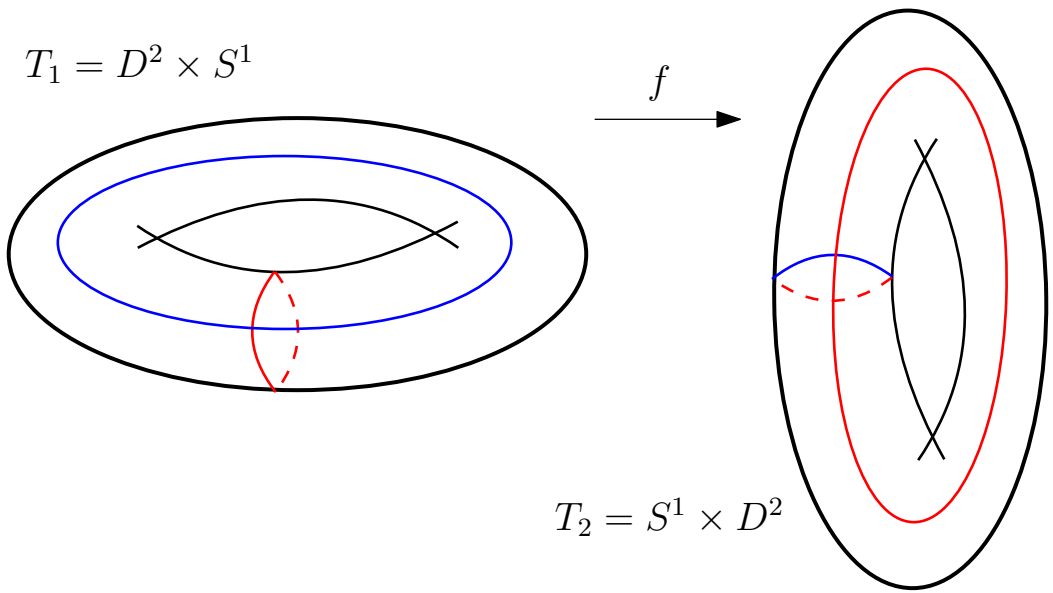


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# Classification

- Let $g : \partial H_1^1 \longrightarrow \partial H_2^1$ a reversing homeomorphism with $g(m) = s$.
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## Proposition

*The Heegaard splittings of genus one are, up to homeomorphism, the lens spaces.*

# More on Lens Spaces

## Definition (Construction from a ball)

*Let* $l, m \in \mathbb{Z}$, $1 \leq l < m$ *and* $(l, m) = 1$. *Divide the equator in* $m$ *segments* $[v_j, v_{j+1}]$, *where*

$$v_j = (\cos(2\pi j/m), \sin(2\pi j/m), 0).$$

*We apply a rotation by* $2\pi l/m$ *in the north hemisphere. After, identify it with the south hemisphere. The triangle* $\Delta_{j+l} = [v_{j+l}, v_{j+l+1}, N]$ *is identified with the triangle* $\Delta_j = [v_j, v_{j+1}, S]$.

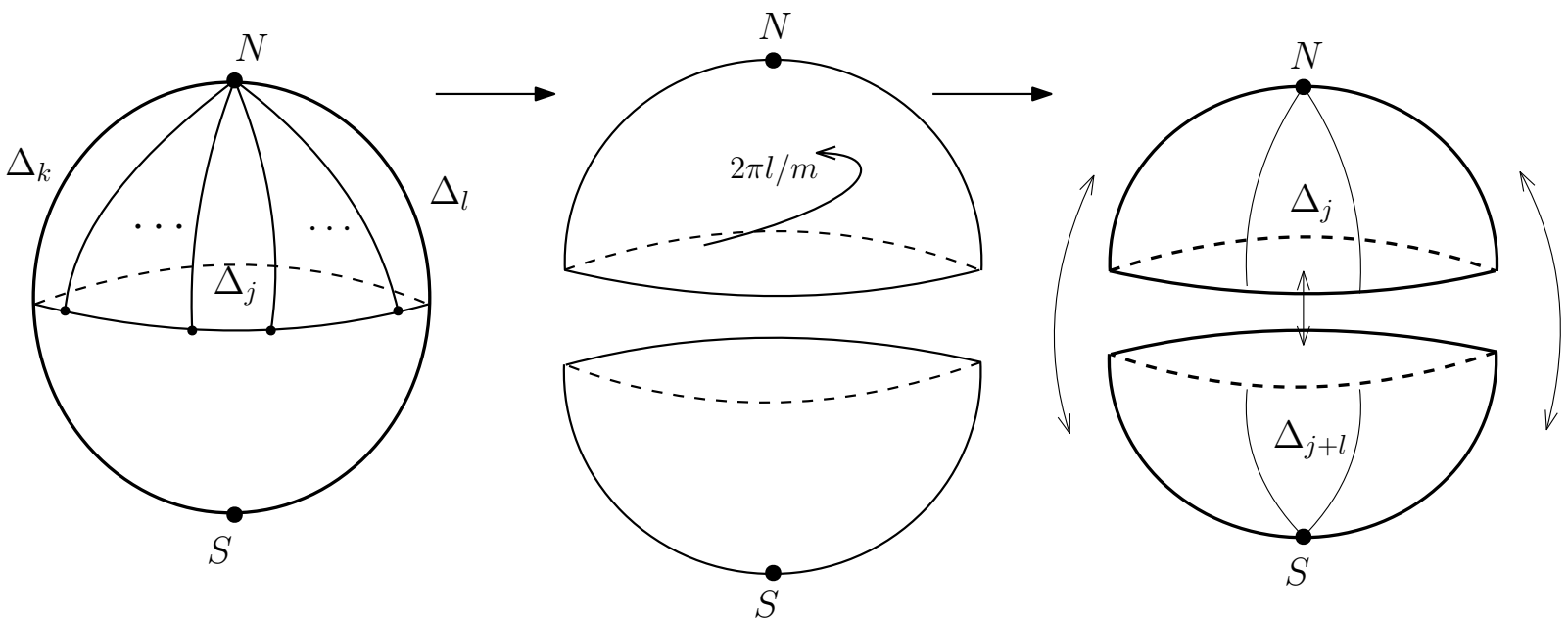
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# Continuation

## Definition

*The lens space* $L(l,m)$ *is the quotient of* $S^3 \subset \mathbb{C}^2$ *by the action of* $\mathbb{Z}_m$ *generated by*

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## Remark

- $\pi_1(L_{l,m}) \simeq \mathbb{Z}_m$
- $L(1,2) = \mathbb{RP}^3$
- *Two torus solid glued with respect to a homeomorphism which takes meridians into a path* $(2,1)$.

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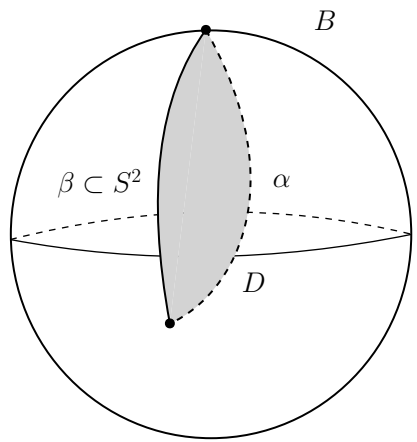
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- More generally, let $\{\alpha_i\}_{i=1}^{g}$ unknotted and unlinked arcs.
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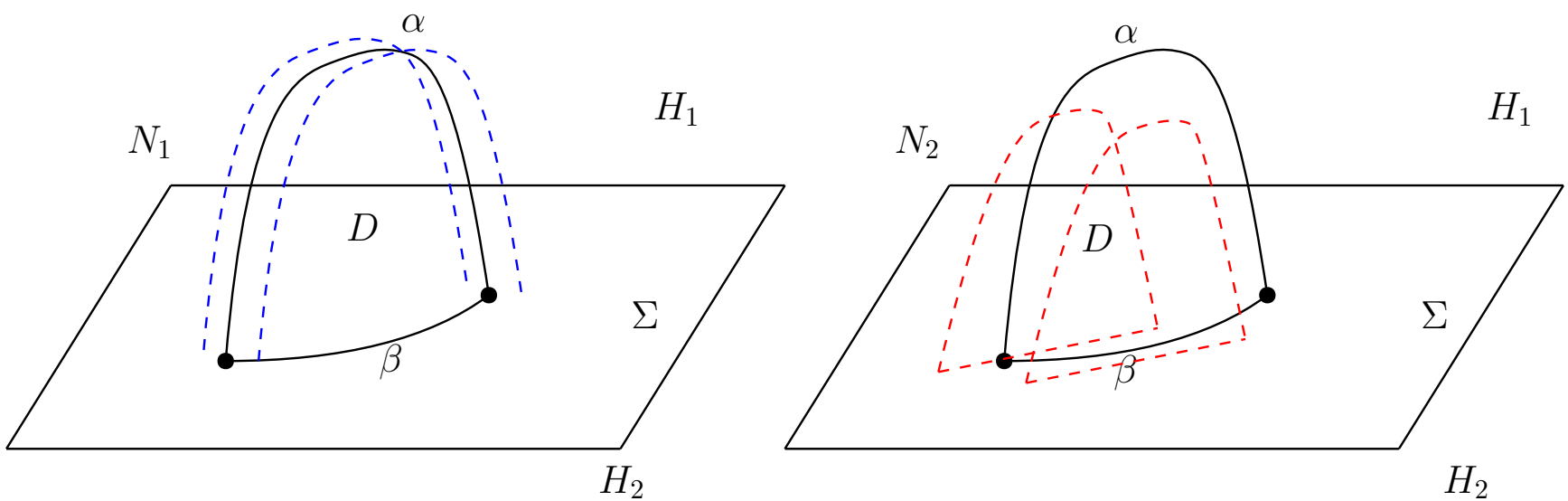
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- $(\Sigma', H_1', H_2')$ , $\Sigma = \partial H_1' \simeq \partial H_2'$, is a H.S.

# Remarks

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- Extension to $h' : \partial H_1' \longrightarrow \partial H_2'$ considering a homeomorphism between $\partial N_1$ and $\partial N_2$.

# A Property

## Theorem

*If* $(\Sigma', H_1', H_2')$ *and* $(\Sigma'', H_1'', H_2'')$ *are stabilizations of the same Heegaard splitting* $(\Sigma, H_1, H_2)$ *and* $\Sigma'$ *and* $\Sigma''$ *have the same genus, then they are isotopic in* $M$*.*

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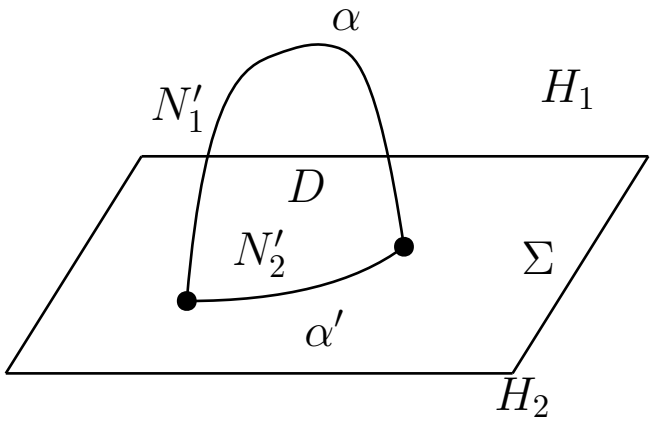
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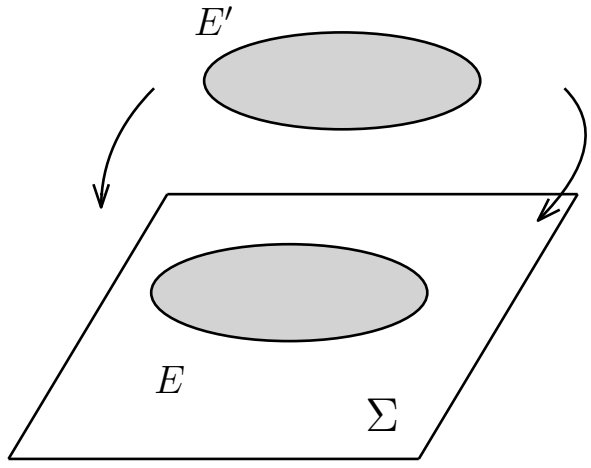
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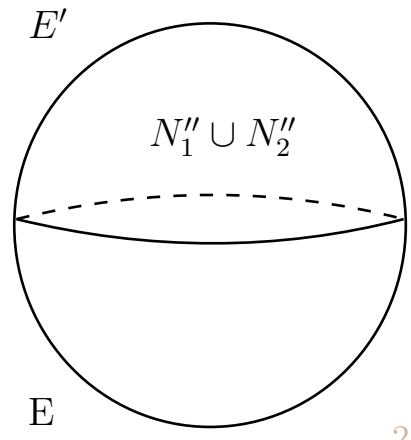
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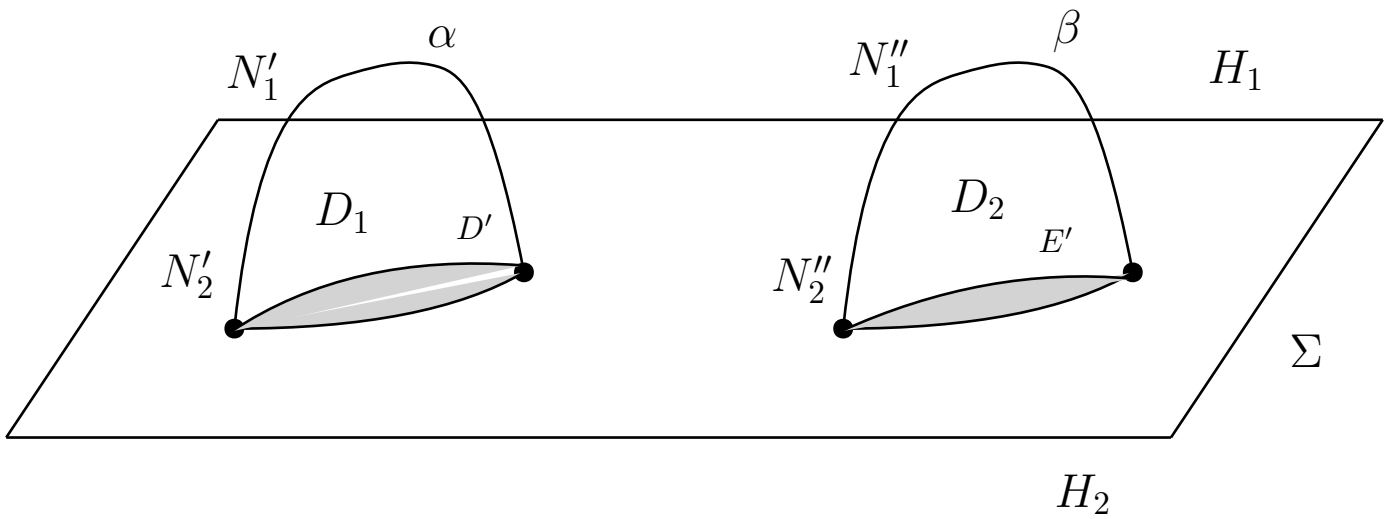
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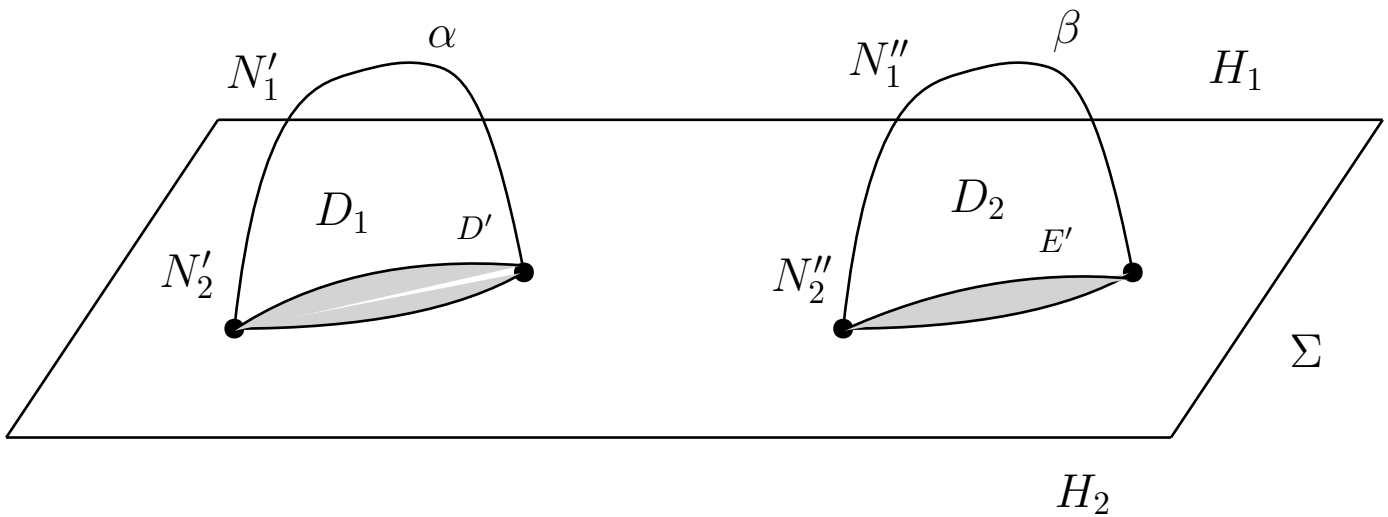
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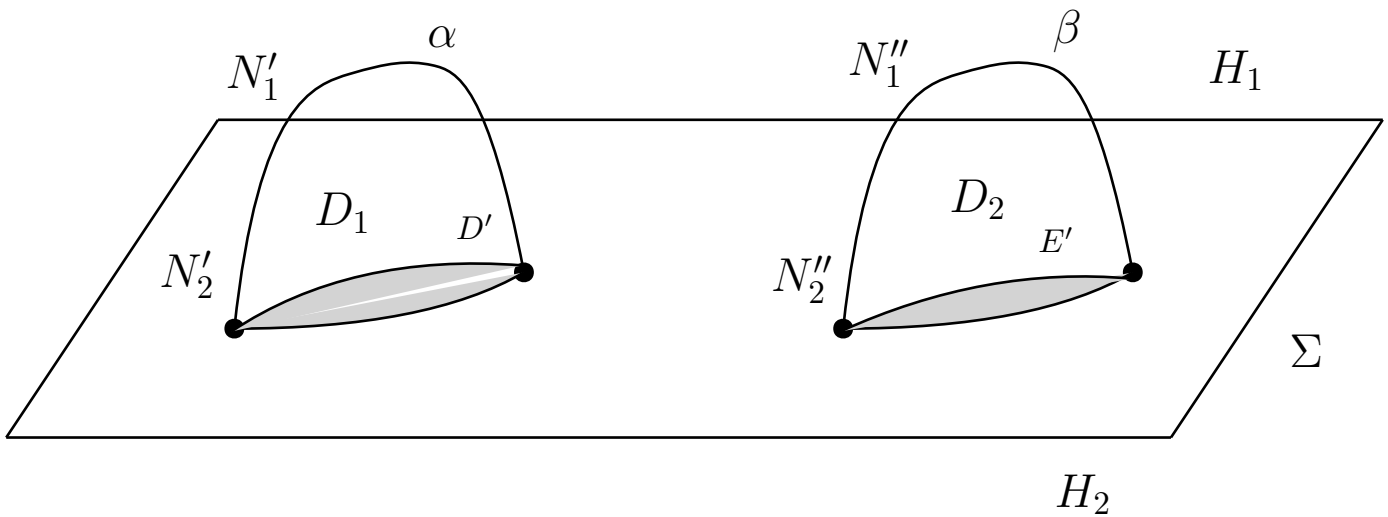



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# Stabilized Splittings

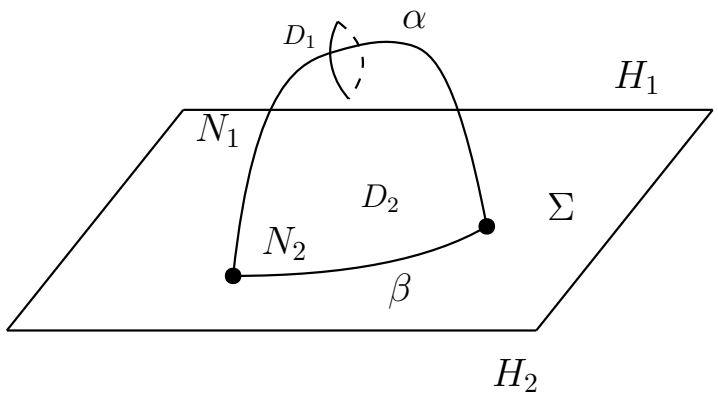
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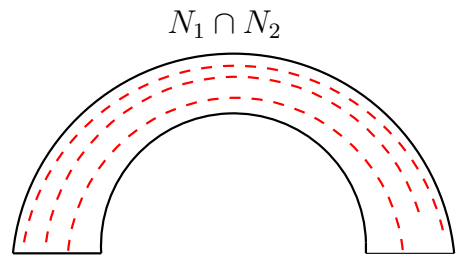
*A Heegaard splitting $(\Sigma', H_1', H_2')$ is stabilized if there is a pair of essential, properly embedded disks $D_1' \subset H_1'$, $D_2' \subset H_2'$ such that $\partial D_1' \cap \partial D_2'$ consists of a single point.*

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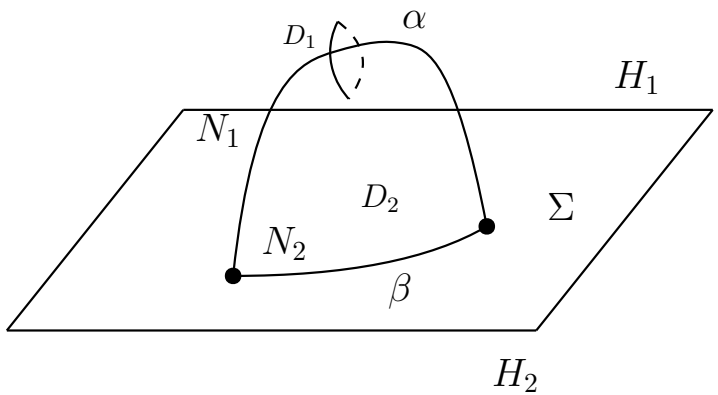


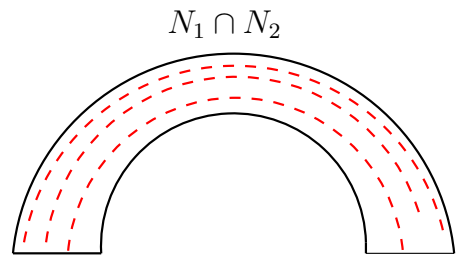



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## Lemma

*A Heegaard splitting is stabilized if and only if it is a stabilization of some Heegard splitting $(\Sigma, H_1, H_2)$.*

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Scheme of the proof:

- Since $S^3$ is connected, for any two points (spines) there exists an isotopy between them.
- By a lemma, the two Heegard surfaces $\Sigma$, $\Sigma'$ of genus 0 are isotopic.

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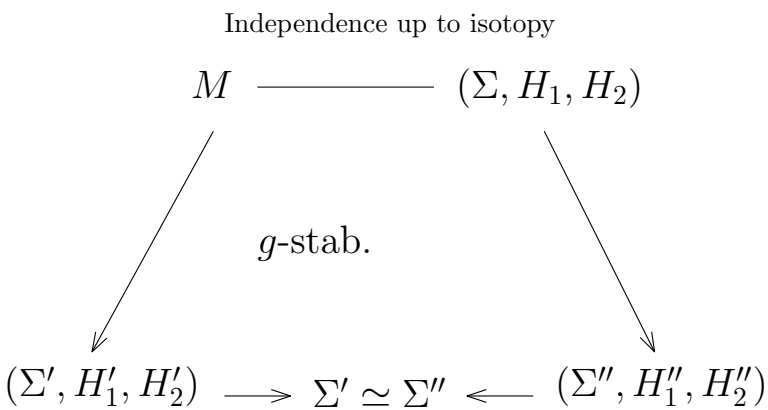
# Further Results

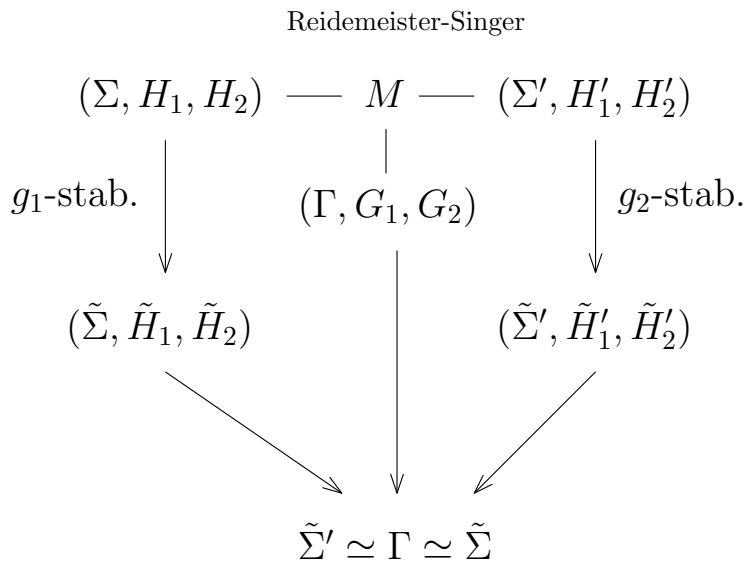
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- Applications in minimal surfaces, symplectic geometry, hyperbolic geometry...

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# THE END