

The Divergence Test: If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum a_n$ diverges.

Note: You can only conclude *divergence* using the Divergence Test. If $\lim_{n \rightarrow \infty} a_n = 0$, *you know nothing*.

The Geometric Series Test: This test only applies to geometric series, i.e. series of the form $\sum ar^n$. This series converges if $|r| < 1$ and diverges if $|r| \geq 1$.

Note: If the series converges, then we can compute the value using the formula $\sum ar^n = \frac{\text{first term}}{1-r}$.

The p -Series Test: This test only applies to p -series, i.e. series of the form $\sum \frac{1}{n^p}$. A p -series converges if $p > 1$ and diverges if $p \leq 1$.

The Integral Test: Given $\sum a_n$, if $f(x)$ is a function such that $f(n) = a_n$ for each positive integer n and f is continuous, positive and decreasing, then

$$\sum_{n=k}^{\infty} a_n \text{ converges if and only if } \int_k^{\infty} f(x) dx \text{ converges.}$$

Note: The “if and only if” means that if the improper integral diverges, we can also conclude the series diverges. If the improper integral converges (equals a number), this does not tell you the value of the series!

The Direct Comparison Test (DCT): Suppose $\sum a_n$, $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for each n , then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $b_n \leq a_n$ for each n , then $\sum a_n$ is also divergent.

The Limit Comparison Test (LCT): Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c \quad \text{where } c \text{ is strictly positive and finite (i.e. } 0 < c < \infty),$$

then either both series converge or both diverge.

Note: If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ does not exist, then LCT fails.

The Alternating Series Test (AST): This test applies only to alternating series, i.e. series that can be written as either $\sum (-1)^n b_n$ or $\sum (-1)^{n+1} b_n$ where $b_n > 0$. If

- $\lim_{n \rightarrow \infty} b_n = 0$, and
- $\{b_n\}$ is decreasing,

then the series converges.

Note: If one (or both) of the two conditions above is not true, AST is *inconclusive* (i.e. the test fails). If AST fails, try the Divergence Test.

The Ratio Test: Given a series $\sum a_n$, suppose $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.

- If $L < 1$, the series is absolutely convergent (hence convergent).
- If $L > 1$, the series is divergent.
- If $L = 1$, the test is inconclusive.

Note: If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ does not exist, the Ratio Test fails. However, $L = \infty$ is allowed (we say that $L = \infty > 1$ so the series diverges).