

Riemann Problem For A Non-Strictly Hyperbolic System In Chemotaxis

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Our System

System of equations

$$\begin{aligned}u_t + (uv)_x &= 0 \\v_t - u_x &= 0\end{aligned}\tag{1}$$

where $t > 0, x \in \mathbb{R}$ with initial data:

Riemann Problem

$$(u(x, 0), v(x, 0)) = \begin{cases} (u_-, v_-) & \text{for } x < 0 \\ (u_+, v_+) & \text{for } x > 0, \end{cases}\tag{2}$$

where (u_-, v_-) and (u_+, v_+) are constant states.

- ① $u \geq 0$ represents the concentration of bacteria
- ② v represents function of chemical concentration

Chemotaxis and The Keller-Segel Model

- ① What is chemotaxis?
- ② Chemotaxis is the movement of organisms due to chemical response.

Keller-Segel Model

$$\begin{aligned}\rho_t &= D(\rho_x + \alpha \rho \frac{w_x}{w})_x \\ w_t &= \lambda p w - \mu w + \epsilon w_{xx},\end{aligned}\tag{3}$$

where $D > 0, \lambda > 0, \mu \geq 0, \alpha$ constant.

- ③ $\rho \geq 0$: cell density, $w(x, t) \geq 0$: concentration of chemicals
- ④ $\alpha < 0$: 'attractive' chemotaxis system, $\alpha > 0$: 'repulsive' system

Change of Variables

- ① Li et al. showed in [12] that with change of variables,
 $u = \rho, v = \partial_x(\ln(w))$

Equation

$$\begin{aligned}u_x + (uv)_x &= Du_{xx} \\ v_t - (u + \epsilon v^2)_x &= \epsilon v_{xx}\end{aligned}\tag{4}$$

- ② We consider a special case of (4) with $\epsilon = 0, D = 0$, which is (1).

Features

- 1 The system has a strictly hyperbolic and an elliptic region, with a nonstrictly hyperbolic boundary (parabola shaped).
- 2 We solved the Riemann problem **up to the non-strictly hyperbolic boundary** as well as a linearly degenerate region for $u \geq 0$ (physically relevant).
- 3 We will see **shock waves, rarefaction waves, and contact discontinuities** in our solution.

- ① Rascle (1985): Solved problem that was reflected along v -axis[15].
- ② Hillen and Wang (2008): Describe shock wave solutions in parameterized form for our equation [18].
- ③ Li, Wang, and Liu (2016): Derived (1) from the physical point of view via the Keller-Segel model[12].
- ④ Zheng (2001): Helped us find Riemann invariants [20].

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Riemann Problem

- We want to solve the Riemann Problem with n equations.
- Consider $U \in \mathbb{R}^n$.
- $U = (u_1, u_2, \dots, u_n)$, $F(U) = (f_1(u), f_2(u), \dots, f_n(u))$
- The system

$$U_t + F(U)_x = 0 \quad (5)$$

with initial data

$$U(x, 0) = U_0(x) = \begin{cases} U_-, & x < 0 \\ U_+, & x > 0 \end{cases} \quad (6)$$

is the Riemann problem with U_- , U_+ constant vectors.

Genuinely Nonlinear and Linearly Degenerate

- ① If the Jacobian $\nabla F(U)$ has real, distinct eigenvalues, $\lambda_k, k = 1, \dots, n$, then (5) is a system of hyperbolic conservation laws.
- ② The k th family is genuinely nonlinear (*GNL*) when $\nabla \lambda_k \cdot r_k \neq 0$, where r_k is the right eigenvector.
- ③ Linearly degenerate (*LD*) when $\nabla \lambda_k \cdot r_k = 0$
- ④ Shock and rarefaction waves (in region with *GNL* family) and contact discontinuities (in region with *LD* family)

System: Shock Waves

- The Rankine-Hugoniot conditions across a discontinuity for (5)

$$s[U] = [F(U)], \quad (7)$$

where $[U] = U_- - U_+$, $[F(U)] = F(U_-) - F(U_+)$, s is the shock speed.

- The *entropy inequalities* are as follows:

$$\begin{aligned} \lambda_k(U_+) &< s < \lambda_{k+1}(U_+) \\ \lambda_{k-1}(U_-) &< s < \lambda_k(U_-) \end{aligned} \quad (8)$$

for $1 \leq k \leq n$.

Theory - Rarefaction Waves

- A *rarefaction wave* is a continuous solution of the Riemann problem (5) (6) in the form $U = U(x/t)$.

- Let $\xi = x/t$

-

$$-\xi U_\xi + F(U)_\xi = 0$$

$$(\nabla F - \xi I)U_\xi = 0$$

- We require that $U_\xi \neq 0$. Then, U_ξ is an eigenvector of ∇F .

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Preliminary

- ① In our system (1), $U = (u, v)$ and $F(U) = (uv, -u)$.

Jacobian

The Jacobian of F is as follows:

$$A(U) = \begin{pmatrix} v & u \\ -1 & 0 \end{pmatrix}$$

Characteristic Equation

The characteristic equation is

$$\lambda^2 - v\lambda + u = 0 \quad (9)$$

$$\lambda_{1,2} = \frac{v \mp \sqrt{v^2 - 4u}}{2} \quad (10)$$

Regions

Eigenvectors

$$r_{1,2} = \frac{1}{2}(-v \mp \sqrt{v^2 - 4u}, 1)^T \quad (11)$$

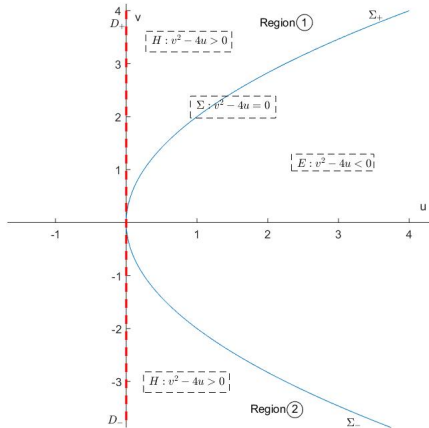
Regions

$$\begin{aligned} H &= \{(u, v) \in \mathbb{R}^2 \mid v^2 - 4u > 0\} \\ E &= \{(u, v) \in \mathbb{R}^2 \mid v^2 - 4u < 0\} \\ \Sigma &= \{(u, v) \in \mathbb{R}^2 \mid v^2 - 4u = 0\} \end{aligned} \quad (12)$$

Axis Regions

$$\begin{aligned} D_- &= \{u = 0, v \leq 0\} \\ D_+ &= \{u = 0, v \geq 0\} \end{aligned} \quad (13)$$

Regions



Non-Strictly Hyperbolic

Feature of System

- ① In H , where $v^2 - 4u > 0$, we have $\lambda_1 < \lambda_2$, (1) is **strictly hyperbolic**.
- ② On Σ , where $v^2 - 4u = 0$, we have $\lambda_1 = \lambda_2$, (1) is **non-strictly hyperbolic**

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Hugoniot Locus

- We now study shock waves.
- The Rankine-Hugoniot conditions (7) applied to (1) yields

$$\begin{aligned} s(u - u_-) &= uv - u_- v_- \\ s(v - v_-) &= -(u - u_-) \end{aligned} \tag{14}$$

- Eliminating the parameter s and applying the entropy inequalities (8), we get

Hugoniot Locus

$$\begin{aligned} S_1(U_-) : u - u_- &= -(v - v_-) \frac{v - \sqrt{v^2 - 4u_-}}{2}, v > v_- \\ S_2(U_-) : u - u_- &= -(v - v_-) \frac{v + \sqrt{v^2 - 4u_-}}{2}, v < v_-. \end{aligned} \tag{15}$$

Rarefaction Waves

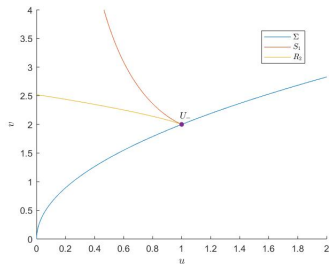
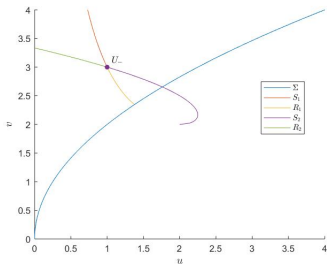
- We now study rarefaction waves.
- Rarefaction waves can be derived by finding the integral curves of the right eigenvectors. For (1), we have:

$$\begin{pmatrix} v - \lambda & u \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Rarefaction

$$\begin{aligned} R_1(U_-) : \frac{du}{dv} &= \frac{-v + \sqrt{v^2 - 4u}}{2}, v < v_- \\ R_2(U_-) : \frac{du}{dv} &= \frac{-v - \sqrt{v^2 - 4u}}{2}, v > v_-. \end{aligned} \tag{16}$$

Shock and Rarefaction Curves



Riemann Invariants I

- Riemann invariants are functions w_i such that $\nabla w_i \cdot r_i = 0$.
- Across rarefaction waves, Riemann invariant does not change.
- $R_i(U_-) = \{U \in \mathbb{R}^2 | w_i(U) = w_i(U_-)\}, i = 1, 2$
- Following Rascle and Zheng [15, 20]

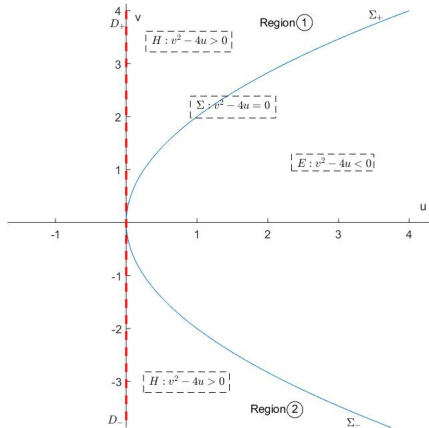
$$\begin{aligned}w_1(U) &= \begin{cases} \sqrt{\lambda_1}(\lambda_1 + 3\lambda_2), & v > 0 \\ \sqrt{-\lambda_1}(\lambda_1 + 3\lambda_2), & v < 0. \end{cases} \\w_2(U) &= \begin{cases} \sqrt{\lambda_2}(3\lambda_1 + \lambda_2), & v > 0 \\ \sqrt{-\lambda_2}(3\lambda_1 + \lambda_2), & v < 0. \end{cases}\end{aligned}\tag{17}$$

Riemann Invariants - II

- Computationally, it was easier to write the Riemann invariants in terms of u and v .

$$\begin{aligned}w_1(U) &= \begin{cases} \frac{\sqrt{v - \sqrt{v^2 - 4u}}(\frac{1}{2}(v - \sqrt{v^2 - 4u}) + \frac{3}{2}(v + \sqrt{v^2 - 4u}))}{\sqrt{2}}, & v > 0 \\ \frac{\sqrt{-v + \sqrt{v^2 - 4u}}(\frac{1}{2}(v - \sqrt{v^2 - 4u}) + \frac{3}{2}(v + \sqrt{v^2 - 4u}))}{\sqrt{2}}, & v < 0. \end{cases} \\w_2(U) &= \begin{cases} \frac{\sqrt{v + \sqrt{v^2 - 4u}}(\frac{3}{2}(v - \sqrt{v^2 - 4u}) + \frac{1}{2}(v + \sqrt{v^2 - 4u}))}{\sqrt{2}}, & v > 0 \\ \frac{\sqrt{-v - \sqrt{v^2 - 4u}}(\frac{3}{2}(v - \sqrt{v^2 - 4u}) + \frac{1}{2}(v + \sqrt{v^2 - 4u}))}{\sqrt{2}}, & v < 0. \end{cases}\end{aligned}\tag{18}$$

Regions



Contact Discontinuity

Lemma

- ① On D_+ , we have $\lambda_2 = v$ is GNL, $\lambda_1 = 0$ is LD
- ② On D_- , we have $\lambda_1 = v$ is GNL, $\lambda_2 = 0$ is LD.

- Let $U_-, U_+ \in D_- \cup D_+$. Then,
- U_- and U_+ can be connected a contact discontinuity of speed $s = 0$.

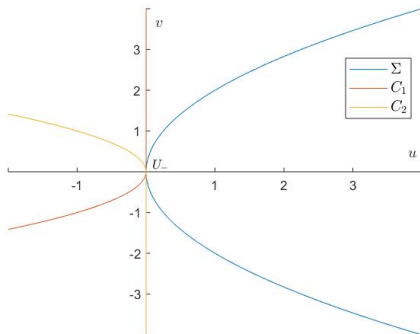


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Theorem

Theorem 4

Suppose U_- and $U_+ \in (H \cup \Sigma) \cap \{u \geq 0\}$. Then the Riemann solutions can be constructed if an intermediate state $\bar{U}$, which connects U_- and U_+ , exists in the same region.

- We will show this via showcases.

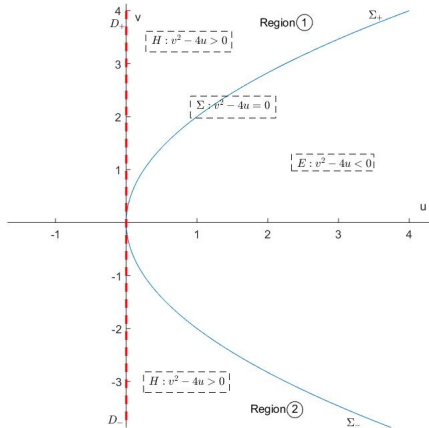
Derivation of Intermediate Step - Main Idea

- 1 Let's consider the case where $U_-, U_+ \in \text{Region 1}$.
- 2 From the Riemann solution, we have $\bar{U} \in S_1(U_-) \cap S_2(U_+)$.

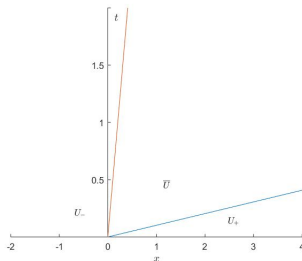
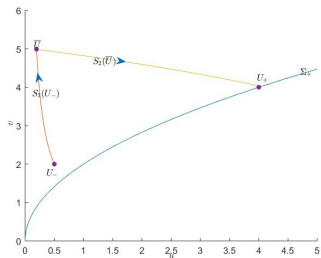
$$\begin{aligned} S_1(u_-, v_-) : u &= u_- - (v - v_-) \frac{v - \sqrt{v^2 - 4u_-}}{2} \\ S_2(u_+, v_+) : u &= u_+ - (v - v_-) \frac{v + \sqrt{v^2 - 4u_+}}{2} \end{aligned} \tag{19}$$

- 3 Equating the equations, we can eliminate u and write an implicit equation in v .
- 4 After solving for v , we can plug v, u_-, u_+ in (19) to solve our u .
- 5 This pair (u, v) are the coordinates for the intermediate state.

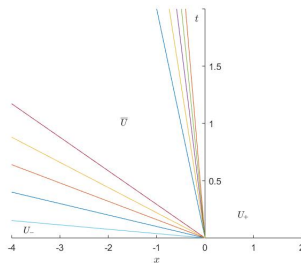
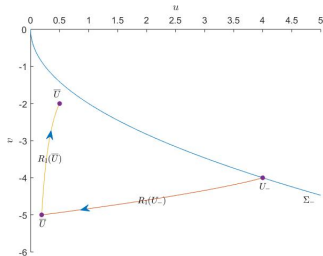
Regions



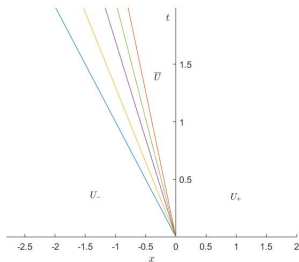
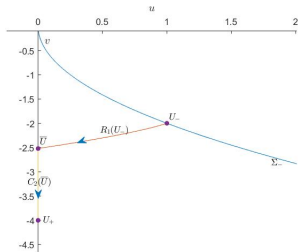
$U_- = (0.5, 2)$, $U_+ = (4, 4) \in \Sigma_+$ and $\bar{U} \approx (0.194, 4.99)$.



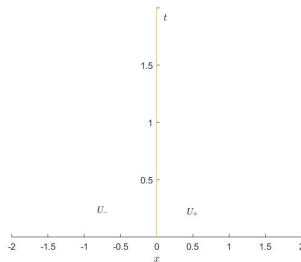
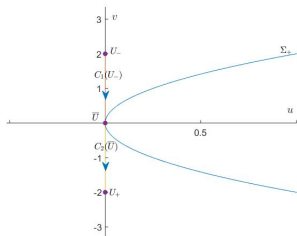
$U_- = (4, -4) \in \Sigma_-$, $U_+ = (0.5, -2)$ and $\bar{U} \approx (0.19, -5)$.



$U_- = (1, -2)$, $U_+ = (0, -4) \in \Sigma_+$ and $\bar{U} \approx (0, -2.52)$.



$U_- = (0, 2), U_+ = (0, -2) \in \Sigma_+$ and $\bar{U} \approx (0, 0)$.



$U_- = (0, 2)$, $U_+ = (1, 2) \in \Sigma_+$ and $\bar{U} \approx (0, -2.52)$.

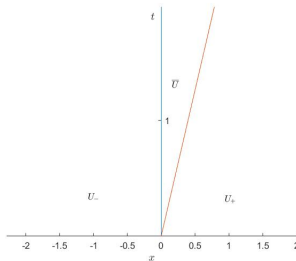
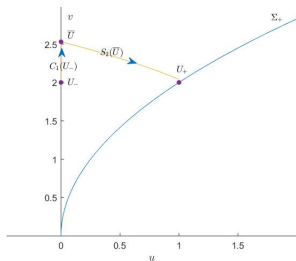


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Conclusion

- We solved the Riemann problem up to the non-strictly hyperbolic boundary for a physically relevant problem in chemotaxis
- Shock waves, rarefaction waves, and contact discontinuities
- Details and further work in our paper [9]

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The End

- Thank You!
- Questions?