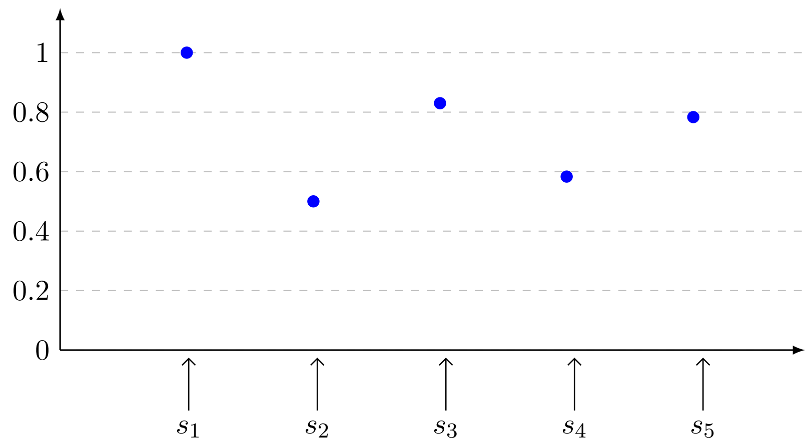
Goal: To estimate the value of a convergent alternating series when we can't find its precise value.

1. We will try to estimate the value of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ using partial sums.

   (a) Compute and plot the first five partial sums on the following graph.



Solution: Recall $s_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N$.

$s_1 = 1$

$s_2 = 1 - \frac{1}{2} = 0.5$

$s_3 = 1 - \frac{1}{2} + \frac{1}{3} \approx .83$

$s_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \approx .5833$

$s_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \approx .7833$

   (b) As you plot these, a pattern should emerge. Make a guess about where the sixth partial sum would fall in relation to the first five, e.g. "greater than all of them," "less than all of them," "between first and second," etc.

Solution: Between the 4th and 5th partial sums

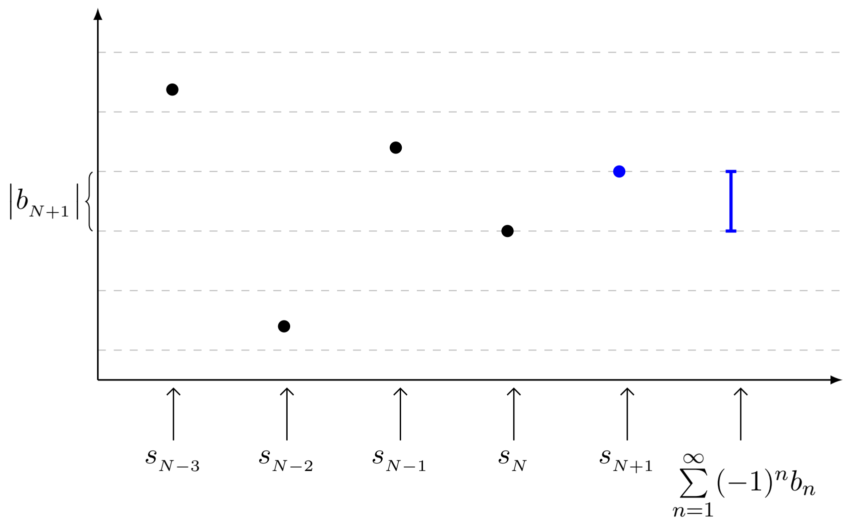
   (c) Where do you think the value of the series would fall in relation to the first five partial sums?

Solution: Also between the 4th and 5th partial sums

2. Now, suppose that $\sum_{n=1}^{\infty}(-1)^n b_n$ is some unknown alternating series with the properties that

- $\lim_{n\to\infty} b_n = 0$
- b_n is decreasing (i.e. $b_{n+1} \leq b_n$).

Some partial sums of this unknown series have been graphed below.



(a) Plot s_{N+1} on the graph. Notice that the distance between the dotted lines on the graph is exactly the size of the next term, $|b_{N+1}|$.

(b) What are the possible values of the series $\sum_{n=1}^{\infty}(-1)^n b_n$? Draw a line segment that represents these possible values on the graph above.

(c) Use your answer in (b) to complete the following statement about the error that occurs when using the partial sums of an alternating series to estimate its value.

Alternating Series Remainder Estimate:

If $\sum_{n=1}^{\infty}(-1)^n b_n$ is an alternating series such that

- $\lim_{n\to\infty} b_n = 0$ and
- b_n is decreasing

then the remainder (i.e. the difference between the Nth partial sum and the value of the series) satisfies

$$|R_N| = |s_N - s| = \left|\sum_{n=1}^{N}(-1)^n b_n - \sum_{n=1}^{\infty}(-1)^n b_n\right| \leq \underline{\quad b_{N+1} \quad}$$

Now put this result to use in estimating the values of alternating series.

3. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

   (a) Show that the series is convergent.

   Solution: We'll use the Alternating Series Test. For this series, $b_n = \frac{1}{n^2}$.

   $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$

   b_n is decreasing since $b_{n+1} = \frac{1}{(n+1)^2} < \frac{1}{n^2} = b_n$ for all n

   The series converges by AST.

   (b) Use the Alternating Series Remainder Estimate to determine how many terms we need to add in order to estimate the value of the series to within 0.01 (i.e. so that $|\text{error}| < 0.01$).

   Solution: By the Alternating Series Remainder Estimate, $|\text{error}| = \left|R_N\right| \leq b_{N+1} = \frac{1}{(N+1)^2}$. To guarantee that the approximation is within 0.01, we set this error estimate (that we can guarantee) less than our desired error of 0.01.

$$\begin{aligned} \frac{1}{(N+1)^2} &< 0.01 \\ \frac{1}{(N+1)^2} &< \frac{1}{100} \\ \frac{1}{N+1} &< \frac{1}{10} \\ 10 &< N+1 \\ 9 &< N \end{aligned}$$

   Since N must be an integer strictly greater than 9, the answer is $\boxed{N = 10.}$

   This means that $s_{10} = \displaystyle\sum_{n=1}^{10} \frac{(-1)^n}{n^2}$ approximates the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ to within 0.01.

4. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n}$.

(a) Show that the series is convergent.

Solution: We'll use the Alternating Series Test. For this series, $b_n = \frac{1}{n5^n}$.

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n5^n} = 0.$

Consider $b_n = \frac{1}{n5^n}$. The denominator of b_n is increasing because both n and 5^n are increasing. Thus b_n is decreasing, since 1 over an increasing sequence must be decreasing.

The series converges by AST.

(b) Use the Alternating Series Remainder Estimate to determine how many terms we need to add in order to estimate the value of the series to within 0.0001 (i.e. so that $|\text{error}| < 0.0001 = 10^{-4}$).

Solution: By the Alternating Series Remainder Estimate, $|\text{error}| = \left|R_N\right| \leq b_{N+1} = \frac{1}{(N+1)5^{N+1}}$. Like before, we set this error estimate less than our desired error of 0.0001.

$$\frac{1}{(N+1)5^{N+1}} < 0.0001$$

However, unlike before, we can't solve this inequality by hand. Instead, we can check the values of b_n using a calculator.

$b_1 = 0.2$
$b_2 = 0.02$
$b_3 = 0.002\overline{6}$
$b_4 = 0.0004$
$b_5 = 0.000064$

Since $b_5 < 0.0001$, we get that $b_{N+1} = b_5$ so $\boxed{N = 4.}$

5. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$.

   (a) Show that the series is convergent.

   Solution: Using AST for this series, $b_n = \dfrac{1}{n!}$.

   $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \dfrac{1}{n!} = 0$.

   b_n is decreasing, since $n!$ is increasing and 1 over an increasing sequence is decreasing.

   The series converges by AST.

   (b) Use the Alternating Series Remainder Estimate to determine how many terms we need to add in order to estimate the value of the series to within 0.005.

   Solution: By the Alternating Series Remainder Estimate, $|\text{error}| = \left|R_N\right| \leq b_{N+1} = \frac{1}{(N+1)!}$. We set this error estimate less than our desired error of 0.005.

$$\frac{1}{(N+1)!} < 0.005$$

   We get an inequality that we can't solve by hand, so instead we check the values of b_n using a calculator.

   $b_1 = \frac{1}{1!} = 1$
   $b_2 = \frac{1}{2!} = \frac{1}{2}$
   $b_3 = \frac{1}{3!} = \frac{1}{6}$
   $b_4 = \frac{1}{4!} = \frac{1}{24}$
   $b_5 = \frac{1}{5!} = \frac{1}{120} = 0.008333\ldots$
   $b_6 = \frac{1}{6!} = \frac{1}{720} = 0.0013888\ldots$

   Since $b_6 < 0.005$, we get that $b_{N+1} = b_6$ so $\boxed{N = 5.}$