Definitions

- A set of nonzero vectors $\{v_1, \ldots, v_k\}$ is an **orthogonal set** if each pair is orthogonal. In other words, if $\mathbf{v_i} \neq \mathbf{0}$ for each i and $\mathbf{v_i} \cdot \mathbf{v_j} = 0$ for $i \neq j$.
- An **orthogonal basis** is a basis that is also an orthogonal set.
- An **orthonormal basis** is an orthogonal basis where all the vectors have length 1.

Theorem 15 (The Orthogonal Decomposition Theorem)**.**
If W is a subspace of $\mathbb{R}^n$ with **orthogonal** basis $\mathcal{B} = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}\}$, then

$$\text{proj}_W \mathbf{x} = \left(\frac{\mathbf{x} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}}\right) \mathbf{w_1} + \left(\frac{\mathbf{x} \cdot \mathbf{w_2}}{\mathbf{w_2} \cdot \mathbf{w_2}}\right) \mathbf{w_2} + \cdots + \left(\frac{\mathbf{x} \cdot \mathbf{w_k}}{\mathbf{w_k} \cdot \mathbf{w_k}}\right) \mathbf{w_k}.$$

Furthermore, if $\mathcal{B} = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}\}$ is an **orthonormal** basis of W, then

$$\text{proj}_W \mathbf{x} = (\mathbf{x} \cdot \mathbf{u_1})\mathbf{u_1} + (\mathbf{x} \cdot \mathbf{u_2})\mathbf{u_2} + \cdots + (\mathbf{x} \cdot \mathbf{u_k})\mathbf{u_k}.$$