Goal: To evaluate integrals of the form $\int \sin^m x \cos^n x \, dx$.

1. $\int \sin x \cos^4 x \, dx$

$\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array}} \Rightarrow (-1)\,du = \sin x \, dx$

$= \int u^4 (-1) \, du$

$= -\frac{1}{5} u^5 + C = -\frac{1}{5}\cos^5 x + C$

2. $\int \sin^3 x \, dx$ (*Hint: Use the identity* $\sin^2 x + \cos^2 x = 1$*, then do a u-substitution.*)

$\hookrightarrow \sin^2 x = 1 - \cos^2 x$

$= \int \sin^2 x \cdot \sin x \, dx$

$= \int (1 - \cos^2 x) \cdot \sin x \, dx$ $\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array}} \Rightarrow (-1)\,du = \sin x \, dx$

$= \int (1 - u^2)(-1) \, du$

$= \int u^2 - 1 \, du = \frac{1}{3} u^3 - u + C = \frac{1}{3}\cos^3 x - \cos x + C$

3. $\int \sin^5 x \cos^2 x \, dx$ (*Hint: write* $\sin^5 x = (\sin^2 x)^2 \sin x$*.*)

$= \int (\sin^2 x)^2 \cdot \sin x \cdot \cos^2 x \, dx$

$= \int (1 - \cos^2 x)^2 \cdot \sin x \cdot \cos^2 x \, dx$ $\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array}}$

$= \int (1 - u^2)^2 \cdot u^2 \cdot (-1) \, du$

$= -\int (1 - 2u^2 + u^4) u^2 \, du = -\int u^2 - 2u^4 + u^6 \, du$

$= -\left(\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7\right) + C = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$

4. Use the same strategy as the previous problem. (The algebra gets hairy, so stop once you do the substitution.)

$$\int \sin^7 x \cos^4 x \, dx = \int (\sin^2 x)^3 \cdot \sin x \cdot \cos^4 x \, dx$$

$$= \int (1-\cos^2 x)^3 \cdot \sin x \cdot \cos^4 x \, dx$$

$$\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x\, dx \end{array}}$$

$$= \int \underbrace{(1-u^2)^3}_{\text{to finish, expand this out}} \cdot u^4 \cdot (-1) \, du$$

5. Describe your strategy to evaluate any integral of the form $\int \sin^m x \cos^n x \, dx$ where m is odd.

Save one copy of $\sin x$ (for du) and convert the remaining even power of sines to cosines. Use $u = \cos x$ (so $(-1)du = \sin x \, dx$.)

6. The same type of trick works if the power on $\cos x$ is odd. What trig identity and u-sub would you use to evaluate the following integral?

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cdot \cos^2 x \cdot \cos x \, dx$$

$$= \int \sin^2 x \cdot (1-\sin^2 x) \cdot \cos x \, dx$$

$$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array}}$$

$$= \int u^2 (1-u^2) \, du$$

Use $\cos^2 x = 1 - \sin^2 x$ and $u = \sin x$.

7. Describe your strategy to evaluate any integral of the form $\int \sin^m x \cos^n x \, dx$ where n is odd.

Save one copy of $\cos x$ (for du) and convert the remaining even power of cosines into sines. Use $u = \sin x$ (so $du = \cos x \, dx$)

If you don't have an odd power of $\sin x$ or $\cos x$, the previous strategies don't work.

8. Evaluate $\int \sin^2 x\, dx$ using the following strategies.

   (a) Use the identity $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$.

$$\int \sin^2 x\, dx = \int \tfrac{1}{2}(1-\cos(2x))\, dx = \tfrac{1}{2}\int 1-\cos(2x)\, dx$$
$$= \tfrac{1}{2}\left(x - \tfrac{1}{2}\sin(2x)\right) + C$$
$$= \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x) + C$$

   (b) Integrate by parts using $\boxed{u = \sin x \text{ and } dv = \sin x\, dx.}$ $\Rightarrow du = \cos x\, dx$ and $v = -\cos x$

$$\int \sin^2 x\, dx = \sin x(-\cos x) - \int -\cos x \cdot \cos x\, dx$$
$$'' \quad = -\sin x \cos x + \int \cos^2 x\, dx$$
$$'' \quad = -\sin x \cos x + \int 1 - \sin^2 x\, dx$$
$$\int \sin^2 x\, dx = -\sin x \cos x + x - \int \sin^2 x\, dx$$
$$+\int \sin^2 x\, dx \qquad\qquad + \int \sin^2 x\, dx$$
$$2\int \sin^2 x\, dx = -\sin x \cos x + x + C$$
$$\Rightarrow \int \sin^2 x\, dx = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x + C$$

   (c) Use a trig identity to show that your answers from part (a) and (b) are the same!

$\sin(2x) = 2\sin x \cos x$ so

$$\tfrac{1}{2}x - \tfrac{1}{4}\sin(2x) + C = \tfrac{1}{2}x - \tfrac{1}{4}(2\sin x\cos x) + C$$
$$= \tfrac{1}{2}x - \tfrac{1}{2}\sin x \cos x + C$$

9. How would you integrate $\int \cos^2 x\, dx$? What about $\int \cos^4 x\, dx$ or $\int \sin^2 x \cos^2 x\, dx$?

Use the "power reducing identities"

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

For example,

$$\int \sin^2 x \cos^2 x\, dx = \int \frac{1}{2}(1-\cos(2x)) \cdot \frac{1}{2}(1+\cos(2x))\, dx$$

$$= \frac{1}{4} \int (1-\cos(2x))(1+\cos(2x))\, dx$$

$$= \frac{1}{4} \int 1 - \cos^2(2x)\, dx$$

use the identity again!

$$= \frac{1}{4} \int 1\, dx - \frac{1}{4} \int \cos^2(2x)\, dx$$

$$= \frac{1}{4} x - \frac{1}{4} \int \frac{1}{2}(1 + \cos(2(2x))\, dx$$

$$= \frac{1}{4} x - \frac{1}{8} \int 1 + \cos(4x)\, dx$$

$$= \frac{1}{4} x - \frac{1}{8}\left(x + \frac{1}{4}\sin(4x)\right) + C$$

$$= \frac{1}{8} x - \frac{1}{32}\sin(4x) + C$$