Volumes (6.2 & 6.3)

For volumes, the overarching idea is that "volume is the integral of area."

- For solids with known cross sections,

$$V = \int_a^b A(x)\,dx \quad \text{where } A(x) \text{ is the cross sectional area.}$$

  The cross sections might be squares, triangles, semicircles, etc. We usually determine $A(x)$ using geometry.

- For solids of revolution, we have the Method of Disks/Washers and the Method of Shells.

  - For Method of Disks/Washers, we slice the solid **perpendicular** to the axis of revolution. This ensures that our cross sections are either disks or washers.

    To compute the volume we use the same idea as solids with known cross sections: "volume is the integral of the cross sectional area."

$$\text{Disks: } dV = \pi(\text{radius})^2\, dx \Big/ dy$$
$$\text{Washers: } dV = \pi\Big((\text{outer radius})^2 - (\text{inner radius})^2\Big)\, dx \Big/ dy$$

    The radius is dictated by the region **and** the axis of revolution given in the problem. The radius is always measured **from the axis of revolution**.

  - For Method of Shells, we slice through our region **parallel** to the axis of revolution. In this method, we add up the volume (dV) of cylindrical shells that are centered about the axis of revolution.

$$dV = 2\pi \cdot \text{ radius} \cdot \text{height} \cdot dx \Big/ dy$$

    The radius here is measured from the axis of revolution to the shell. The height of the shell is dictated by the region.

Work (6.4)

If the force needed to move an object from a to b depends on the position x (so the force is a function, $F(x)$), then we compute the total work, W, by integrating $F(x)$:

$$W = \int_a^b F(x)\,dx.$$

This is the case for **spring problems**, where the forced needed to stretch or compress a spring is given by Hooke's Law:

$$F(x) = kx \quad \text{(assuming equilibrium is at 0).}$$

For **water pumping problems**,

$$dW = \text{the work needed to pump a slice of water.}$$

Remember these slices are always taken *horizontally.*

$$W = \text{total work} = \int_a^b dW$$

The bounds on the integral are determined by the water level. Remember to label your coordinate system when showing your work on these problems!

We determine dW based on the following:

- $dW = Fd$ (where F is the force needed to move the slice and d is the distance traveled by the slice)
- $F =$ weight (a.k.a. force due to gravity) $= m \cdot g$
- $m = \rho \cdot dV$ (where ρ is the density of the liquid and dV is the volume of the *slice*, not the tank!)

Moments and Center of Mass (8.3)

For a lamina with uniform (i.e. constant) density ρ, the x and y moments are

$$M_y = \int_a^b \rho x f(x)\,dx \quad \text{and} \quad M_x = \int_a^b \frac{1}{2}\rho f(x)^2\,dx$$

and the coordinates of the center of mass $(\overline{x}, \overline{y})$ are given by

$$\overline{x} = \frac{M_y}{m} \quad \text{and} \quad \overline{y} = \frac{M_x}{m}$$

where $m = \displaystyle\int_a^b \rho f(x)\,dx$ is the total mass.

Alternatively, if A is the area under the curve f from a to b,

$$\overline{x} = \frac{1}{A}\int_a^b x f(x)\,dx \quad \text{and} \quad \overline{y} = \frac{1}{A}\int_a^b \frac{1}{2} f(x)^2\,dx.$$

Remember, you can use symmetry when determining the center of mass. (If the region is symmetric about a line ℓ, then you know that the center of mass must lie on that line ℓ.)

If you find the value of $\overline{x}$ or $\overline{y}$ using symmetry, you should write "by symmetry" to get credit.

Parametric Curves (10.1 & 10.2)

For a parametric curve $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$, the slope of the tangent line is $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.

Arc Length (6.4)

The length of a curve is

$$L = \int_a^b ds$$

where $ds = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt$ if the curve is given parametrically (i.e. $x = f(t)$ and $y = g(t)$)

$ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx$ if the curve is given by a function of x (i.e. $y = f(x)$)

$ds = \sqrt{1 + \left(\dfrac{dx}{dy}\right)^2}\,dy$ if the curve is given by a function of y (i.e $x = f(y)$)

Recall that the equation for ds when the curve is given parametrically comes from the distance formula:

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} \longrightarrow ds = \sqrt{(dx)^2 + (dy)^2}$$

Polar Curves (10.3 & 10.4)

- **Coordinates:** The connection between polar coordinates (r, θ) and Cartesian coordinates (x, y) is

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

From this, we can get that $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$.

- **Tangents:** To find the slope of the tangent line for a polar curve $r = f(\theta)$, use

$$\begin{aligned} x &= r\cos\theta = f(\theta)\cos\theta \\ y &= r\sin\theta = f(\theta)\sin\theta \end{aligned}$$

and treat this as a parametric curve $\left(\text{so } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}\right)$.

- **Areas:** The area enclosed by a polar curve $r = f(\theta)$ is

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta = \int_a^b \frac{1}{2} (f(\theta))^2 \, d\theta$$

(Remember that this comes from the area of a sector of a circle: $A_s = \pi r^2 \left(\frac{\theta}{2\pi}\right) = \frac{1}{2} r^2 \theta$.)

The area enclosed inside $r = f(\theta)$ and outside $r = g(\theta)$ is

$$A = \int_a^b \frac{1}{2} (f(\theta)^2 - g(\theta)^2) \, d\theta$$

Remember to square f and g separately! Determine the bounds on the integral by finding the intersection points of f and g. To find the intersections points, set the curves equal: $f(\theta) = g(\theta)$.

- **Arc length:** The arc length of a polar curve is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$