

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN
MATHEMATISCHES INSTITUT

*Lie–Poisson and Euler–Poincaré
Reductions*

BACHELORARBEIT
im Studiengang Mathematik

eingereicht von

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Datum der Einreichung: 01.03.2026

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Introduction

In this thesis, we explore the role of symmetry and the beauty it reveals in geometrical mechanics. One way of describing the geometrical motion of a physical system is the classical Newtonian way $F = m\ddot{x}$, where m is the mass and $\ddot{x}$ is the acceleration and F is the force. When F derives from a potential, this can equivalently be described by the principle of least action, which is derived by setting the first variation of the action, i.e. the time integral of kinetic energy minus potential energy, to zero. That means we have the following equivalence

$$\delta \int (T - V) dt = 0 \quad \Longleftrightarrow \quad m\ddot{x}(t) = -\frac{\partial V}{\partial x}(x(t))$$

where T represent the kinetic energy and V the potential energy. When we move beyond the flat geometry of Euclidean space and want to understand the motion on curved spaces, we need a general framework to work on. So we introduce manifolds, symplectic manifolds, and Poisson manifolds, where geometric structure is integrated directly into the description of dynamics. In this framework, the same variational principle yields the Euler-Lagrange equations given by the equivalence

$$\delta \int L(q, \dot{q}) dt = 0 \quad \Longleftrightarrow \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

where $L : TQ \rightarrow \mathbb{R}$ is a smooth function called Lagrangian defined on the tangent bundle TQ of a smooth manifold Q . Now, let us introduce a map $\mathbb{F}L : TQ \rightarrow T^*Q$ between the tangent bundle TQ and the cotangent bundle T^*Q , called the Legendre transformation and assume that it is a diffeomorphism. Thus, we get an equivalent reformulation of the dynamics in terms of Hamiltonian equations with the Legendre transformation. So, the dynamics are given by a smooth function $H : T^*Q \rightarrow \mathbb{R}$ called Hamiltonian given by $H(q, p) = \langle p, v \rangle - L(q, v)$ where $(q, v) = (\mathbb{F}L)^{-1}(q, p)$, which is usually taken as the total energy of the system. Hence, we get an equivalent description of dynamics in terms of the Hamiltonian equations by the following equivalence

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \quad \Longleftrightarrow \quad \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

Mathematically, symmetry is described by groups. However, in the geometric setting of mechanics, symmetries are continuous and act on manifolds. That requires groups that themselves possess a smooth structure. These are the Lie groups G and their infinitesimal counterparts are Lie algebras $\mathfrak{g}$.

To motivate the use of symmetry in mechanics, let us consider an example: the Euler equations for the rigid body, which are given by

$$I_1 \dot{\Omega}_1 = (I_2 - I_3)\Omega_2\Omega_3, \quad I_2 \dot{\Omega}_2 = (I_3 - I_1)\Omega_3\Omega_1, \quad I_3 \dot{\Omega}_3 = (I_1 - I_2)\Omega_1\Omega_2$$

where Ω_i are the components of the angular velocity and I_i are the moments of inertia. Nevertheless, these equations can not be solved in the sense of the Hamiltonian equations, since there is odd number of equations which can not be described by even number of canonical coordinates. One can use the Euler angles to describe the dynamics, but then we get 6 equations instead of 3, which are more complicated to solve. But, instead we can introduce the Lagrangian as

$$L = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)$$

Then the "reduced" Euler-Lagrange equations, the Euler-Poincaré equations, are given by

$$\frac{d}{dt} \frac{\partial L}{\partial \Omega} = \frac{\partial L}{\partial \Omega} \times \Omega$$

Hence the dynamics of the rigid body can be described by the Euler-Poincaré equations with only 3 equations instead of 6. This is the main motivation for using symmetry in mechanics, which allows us to reduce the number of equations and to get a simpler description of the dynamics.

Having now the essential tools and after seeing the concrete example of the rigid body, we can now proceed to the reduction process in the more general setting of Lie groups and Lie algebras. First, we assume that the Lagrangian L is invariant under the action of a Lie group G . This means the dynamics of the system are symmetric with respect to the transformations in G . Then, we define a reduced Lagrangian $l : \mathfrak{g} \rightarrow \mathbb{R}$ by $l(\xi) = L(e, \xi)$. Hence, by setting the first variation of the action to zero and by using the variations of the form $\delta\xi = \dot{\eta} + [\xi, \eta]$, we get the variational principle for the Euler-Poincaré reduction on the Lie algebra $\mathfrak{g}$ as

$$\delta \int l(\xi(t))dt = 0 \quad \Longleftrightarrow \quad \frac{d}{dt} \frac{\delta l}{\delta \xi} = \text{ad}_\xi^* \frac{\delta l}{\delta \xi}$$

Furthermore, by applying the reduced Legendre transformation $\mathfrak{f}l : \mathfrak{g} \rightarrow \mathfrak{g}^*$ to the reduced Lagrangian l , and by assuming it to be a diffeomorphism, we get the reduced Hamiltonian $h : \mathfrak{g}^* \rightarrow \mathbb{R}$ defined by $h(\mu) = \langle \mu, \xi \rangle - l(\xi)$ where $\xi = (\mathfrak{f}l)^{-1}(\mu)$ and $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Hence, this gives an equivalent formulation

of the dynamics in terms of the Lie–Poisson reduction as

$$\frac{d}{dt} \frac{\delta l}{\delta \xi} = \text{ad}_\xi^* \frac{\delta l}{\delta \xi} \iff \dot{\mu} = \text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu$$

The aim of this thesis is to understand the reduction processes, Lie–Poisson and Euler–Poincaré reduction, in the general setting of Lie groups and Lie algebras.

The thesis is organised as follows: The first chapter introduces the Lagrangian and Hamiltonian formalism of the dynamics, followed by the Symplectic and Poisson structures in the second chapter. In the third chapter, we will present momentum maps, which are essential to prove the Lie–Poisson reduction theorem, and will explicitly compute the Poisson bracket on the quotient space called the Lie–Poisson bracket by identifying the dual of the Lie algebra $\mathfrak{g}^*$ with the quotient space T^*G/G . Then, we will introduce the Euler–Poincaré reduction, which is the Lagrangian counterpart of Lie–Poisson reduction. In the fourth chapter, we will describe the dynamics of the rigid body and apply these reduction theorems to it as an example. As an appendix, we also provide an introduction to Lie groups and Lie algebras to make the thesis self contained. We will mainly follow the book *Introduction to Mechanics and Symmetry* by Marsden and Ratiu [MR99] and use additional sources to complement the material, which are listed in the bibliography.

1 Lagrangian Mechanics and Variational Principles

In this section we first present the Newtonian mechanics in the setting of N particles in Euclidean space and show how the equations of motion can be derived from the principle of least action. Then we generalize it to the smooth manifolds and give an equivalent formulation in terms of the Lagrangian on the tangent bundle TQ . Then we will introduce the Legendre transformation, which allows us to reformulate the dynamics in terms of the Hamiltonian on the cotangent bundle T^*Q . Lastly, we will show that for a hyperregular Lagrangian, Hamilton's equations are equivalent to the Euler-Lagrange equations and reformulate the variational principle for the Hamiltonian.

1.1 Newton's Equation of Motion and Principle of Least Action

Newton's second law states that the motion of a particle of mass m under a force field $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is described by the second order differential equation

$$F(x(t)) = m \frac{d^2 x}{dt^2}(t)$$

where $t \mapsto x(t)$ is the trajectory of the particle. The work of a path $\gamma : [a, b] \rightarrow \mathbb{R}^n$ with $\gamma(a) = q_0$ and $\gamma(b) = q_1$ is defined by

$$W_\gamma = \int_a^b F(\gamma(t)) \frac{d\gamma(t)}{dt} dt$$

By assuming F is conservative, i.e. $F = -\nabla V$ for some potential function V , and fixing a base point $q_0 \in \mathbb{R}^n$, we may define the potential energy as

$$V : \mathbb{R}^n \rightarrow \mathbb{R}, \quad V(q_1) := W_\gamma$$

Thus, we can rewrite Newton's second law as

$$m \frac{d^2 x}{dt^2}(t) = -\frac{\partial V}{\partial x}(x(t)) \tag{1}$$

Define for now the action as the kinetic energy $T = \frac{1}{2}m\dot{x}(t)^2$ minus the potential energy $V(x)$ integrated over time as

$$A_x = \int_a^b T - V dt = \int_a^b \left(\frac{1}{2}m\dot{x}(t)^2 - V(x(t)) \right) dt$$

Principle of least Action: The physical path $x(t)$ joining q_0 to q_1 is the path for which the action A_x is minimal.

So by the principle of least action, the actual physical path $x(t)$ is locally an extremum of A . In other words, infinitesimal variations of $x(t)$, keeping endpoints fixed, do not change A to first order. So by varying the time dependent coordinate by small $\varepsilon(t)$ with $\varepsilon(a) = 0 = \varepsilon(b)$ we get

$$A_{x+\varepsilon} - A_x = \int_a^b \left[\frac{1}{2}m(\dot{x} + \dot{\varepsilon})^2 - V(x + \varepsilon) \right] dt - \int_a^b \left[\frac{1}{2}m\dot{x}^2 - V(x) \right] dt$$

By expanding $V(x + \varepsilon)$ in Taylor series around x we get $V(x + \varepsilon) = V(x) + \frac{\partial V}{\partial x}(x)\varepsilon + O(\varepsilon^2)$, where $O(\varepsilon^2)$ denotes the second order terms and higher. Hence

$$\begin{aligned} A_{x+\varepsilon} - A_x &= \int_a^b \left[\frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{\varepsilon} + \dot{\varepsilon}^2) - V(x) - \frac{\partial V}{\partial x}(x)\varepsilon + O(\varepsilon^2) \right] dt - \int_a^b \left[\frac{1}{2}m\dot{x}^2 - V(x) \right] dt \\ &= \int_a^b \left[m\dot{x}\dot{\varepsilon} - \frac{\partial V}{\partial x}(x)\varepsilon + O(\varepsilon^2) \right] dt \end{aligned}$$

Integrating by parts

$$\int_a^b m\dot{x}\dot{\varepsilon} dt = \left[m\dot{x}\varepsilon \right]_a^b - m \int_a^b \ddot{x}\varepsilon dt = -m \int_a^b \ddot{x}\varepsilon dt$$

and discarding $O(\varepsilon^2)$ terms we get the first variation of the action as

$$\delta A = - \int_a^b \left[m\ddot{x}(t) + \frac{\partial V}{\partial x}(x(t)) \right] \varepsilon(t) dt$$

By the principle of least action one has $\delta A = 0$ for any arbitrary $\varepsilon(t)$. Thus

$$m\ddot{x}(t) + \frac{\partial V}{\partial x}(x(t)) = 0$$

That is exactly Newton's law (1) we derived earlier!

1.2 Lagrangian - Hamiltonian Mechanics

As we have seen above, where we derived Newton's second law from the principle of least action by taking the first variation of the action, we would like to do a similar approach in the manifold setting. For this, we start by defining the action.

Definition 1.1. Let Q be a n dimensional smooth manifold and $L : TQ \rightarrow \mathbb{R}$ be a smooth function called **Lagrangian**. The **action** of a curve $\gamma : [a, b] \rightarrow Q$ is defined as

$$A_\gamma = \int_a^b (\tilde{\gamma}^* L)(t) dt = \int_a^b L(\gamma(t), \frac{d\gamma}{dt}(t)) dt$$

where $\tilde{\gamma} : [a, b] \mapsto TQ$, $t \mapsto (\gamma(t), \frac{d\gamma}{dt}(t))$ is called the lift of γ to TQ .

For fixed $q_0, q_1 \in Q$ let $\mathcal{P}(a, b, q_0, q_1) := \{\gamma \in C^\infty([a, b], Q) \mid \gamma(a) = q_0, \gamma(b) = q_1\}$.

Proposition 1.2. *The tangent space $T_\gamma \mathcal{P}(a, b, q_0, q_1)$ at a given curve $\gamma \in \mathcal{P}(a, b, q_0, q_1)$ consists of all smooth vector fields $v : [a, b] \rightarrow TQ$ along γ satisfying $\tau_Q \circ v = \gamma$ with vanishing end points $v(a) = 0$ and $v(b) = 0$, where $\tau_Q : TQ \rightarrow Q$ is the tangent bundle projection.*

Proof. Let $\gamma \in \mathcal{P}(a, b, q_0, q_1)$ and $\eta : [a, b] \times (-\epsilon, \epsilon) \rightarrow Q$ be a smooth map s.t. $\eta(t, 0) = \gamma(t)$ for all $t \in [a, b]$ and $\eta(a, \epsilon) = q_0$ and $\eta(b, \epsilon) = q_1$ for all $\epsilon \in (-\epsilon, \epsilon)$. Let us set $\gamma_\epsilon = \eta(\cdot, \epsilon) \in \mathcal{P}(a, b, q_0, q_1)$ a variation of the curve γ . Then the tangent vector to the curve $\epsilon \mapsto \gamma_\epsilon$ in $\mathcal{P}(a, b, q_0, q_1)$ at $s = 0$ is given by the vector field $v : [a, b] \rightarrow TQ$ along $\gamma_0 = \gamma$ defined by

$$v(t) = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \gamma_\epsilon$$

By construction, $\tau_Q(v(t)) = \gamma(t)$ for all $t \in [a, b]$, since $v(t) \in T_{c(t)}Q$. Moreover

$$v(a) = \left. \frac{\partial}{\partial s} \right|_{s=0} \gamma_\epsilon(a) = \left. \frac{\partial}{\partial s} \right|_{s=0} q_1 = 0$$

and similarly $v(b) = 0$. □

Such a vector field v is called an **infinitesimal variation** of the curve γ that preserves the endpoints. v is denoted by

$$v = \delta\gamma$$

Having now defined both the action functional and the infinitesimal variation of a curve on a manifold, we want to proceed in the same sense as in section 1.1. So we set the first variation of the action to zero. Then this condition yields the Euler-Lagrange equations, which provides an equivalent description of the dynamics of a particle moving on the manifold.

Theorem 1.3. *Let $(U, q_1, \dots, q_n)$ be a coordinate chart on Q . Let $\gamma : [a, b] \rightarrow TU$, $t \mapsto (\gamma_1(t), \dots, \gamma_n(t))$ with variations $\delta\gamma_i(a) = 0 = \delta\gamma_i(b)$ for $i = 1, \dots, n$ and let $L : TQ \rightarrow \mathbb{R}$ be a Lagrangian. Then the following are equivalent*

1) γ satisfies the **Euler-Lagrange equations**:

$$\frac{\partial L}{\partial q_i}(\gamma, \dot{\gamma}) = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}(\gamma, \dot{\gamma})$$

where $(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) \in TU$ associated with a trivialization of TU by $\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_n}$.

2) $\delta \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt = 0$

Proof. Suppose $\gamma : [a, b] \rightarrow TU$, $t \mapsto (\gamma_1(t), \dots, \gamma_n(t))$ and $\delta\gamma_1, \dots, \delta\gamma_n \in C^\infty([a, b])$ with $\delta\gamma_i(a) = \delta\gamma_i(b) = 0$ for $i = 1, \dots, n$ and define

$$\gamma_\varepsilon(t) = (\gamma_1(t) + \varepsilon\delta\gamma_1(t), \dots, \gamma_n(t) + \varepsilon\delta\gamma_n(t)), \quad t \in [a, b]$$

for small ε . Now compute the first variation of the action as follows

$$\begin{aligned} \delta \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt &= \left. \frac{dA_{\gamma_\varepsilon}}{d\varepsilon} \right|_{\varepsilon=0} \\ &= \int_a^b \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} L(\gamma_1 + \varepsilon\delta\gamma_1, \dots, \gamma_n + \varepsilon\delta\gamma_n, \dot{\gamma}_1 + \varepsilon\delta\dot{\gamma}_1, \dots, \dot{\gamma}_n + \varepsilon\delta\dot{\gamma}_n) dt \\ &= \int_a^b \sum_{i=1}^n \left[\frac{\partial L}{\partial q_i}(\gamma, \dot{\gamma}) \frac{\partial}{\partial \varepsilon} (\gamma_i + \varepsilon\delta\gamma_i) \Big|_{\varepsilon=0} + \frac{\partial L}{\partial \dot{q}_i}(\gamma, \dot{\gamma}) \frac{\partial}{\partial \varepsilon} (\dot{\gamma}_i + \varepsilon\delta\dot{\gamma}_i) \Big|_{\varepsilon=0} \right] dt \\ &= \int_a^b \sum_{i=1}^n \left[\frac{\partial L}{\partial q_i}(\gamma, \dot{\gamma}) \delta\gamma_i + \frac{\partial L}{\partial \dot{q}_i}(\gamma, \dot{\gamma}) \delta\dot{\gamma}_i \right] dt \end{aligned}$$

Integrate the second term by parts with $\delta\gamma_i(a) = \delta\gamma_i(b) = 0$

$$\int_a^b \frac{\partial L}{\partial \dot{q}_i}(\gamma, \dot{\gamma}) \delta\dot{\gamma}_i dt = \left[\frac{\partial L}{\partial \dot{q}_i}(\gamma, \dot{\gamma}) \delta\gamma_i \right]_a^b - \int_a^b \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}(\gamma, \dot{\gamma}) \delta\gamma_i(t) dt$$

$$= - \int_a^b \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}(\gamma, \dot{\gamma}) \delta \gamma_i dt$$

Thus

$$\begin{aligned} \left. \frac{dA_{\gamma_\varepsilon}}{d\varepsilon} \right|_{\varepsilon=0} &= \int_a^b \sum_{i=1}^n \left[\frac{\partial L}{\partial q_i}(\gamma, \dot{\gamma}) \delta \gamma_i - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}(\gamma, \dot{\gamma}) \delta \gamma_i \right] dt \\ &= \int_a^b \sum_{i=1}^n \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) (\gamma(t), \dot{\gamma}(t)) \delta \gamma_i(t) dt \end{aligned}$$

Since this is satisfied for all smooth $\delta \gamma_i$ with vanishing endpoints $\delta \gamma_i(a) = \delta \gamma_i(b) = 0$. This implies

$$\delta \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt = 0 \quad \Leftrightarrow \quad \frac{\partial L}{\partial q_i}(\gamma(t), \dot{\gamma}(t)) = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}(\gamma(t), \dot{\gamma}(t))$$

□

Remark. As a consequence of this theorem the Euler-Lagrange equations are invariant under any coordinate change.

So the Euler-Lagrange equations give us a description of the dynamics of a physical system on the tangent bundle TQ , where the Lagrangian is defined as a smooth function $L : TQ \rightarrow \mathbb{R}$. We now define the Legendre transformation to obtain an equivalent formulation of the dynamics but this time on the cotangent bundle T^*Q . In this setting, the equations of motion become Hamilton's equations, which are expressed in terms of a smooth function $H : T^*Q \rightarrow \mathbb{R}$ called Hamiltonian. The Hamiltonian in mechanics is usually taken as the total energy of the system, which is the sum of kinetic and potential energy.

Definition 1.4. Let $L : TQ \rightarrow \mathbb{R}$ be a Lagrangian. Define the **Legendre transformation** $\mathbb{F}L : TQ \rightarrow T^*Q$ as

$$\langle \mathbb{F}L(q, v), w \rangle := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} L(q, v + \varepsilon w)$$

where $v, w \in T_q Q$ and $\langle \cdot, \cdot \rangle$ denotes the natural duality pairing between covectors and vectors.

Remark. Notice that, the Legendre transformation is just defined as the fiber derivative of the Lagrangian L with respect to the velocity variable v .

Now assume $\mathbb{F}L : TQ \rightarrow T^*Q$ to be a diffeomorphism (i.e. L to be hyperregular), then for each $\alpha_q \in T_q^*Q$, there exists a unique velocity $v = (\mathbb{F}L)^{-1}(\alpha_q) \in T_qQ$ allowing us to define the dynamics by a Hamiltonian $H : T^*Q \rightarrow \mathbb{R}$ as follows.

Definition 1.5. For a hyperregular Lagrangian $L : TQ \rightarrow \mathbb{R}$ the **Hamiltonian** is defined by

$$H := E \circ (\mathbb{F}L)^{-1} : T^*Q \rightarrow \mathbb{R}$$

with the energy $E(q, v) = \langle \mathbb{F}L(q, v), v \rangle - L(q, v)$. Equivalently

$$H(q, \alpha_q) = \langle \alpha_q, v \rangle - L(q, v), \quad v = (\mathbb{F}L)^{-1}(\alpha_q)$$

Let $(U, q_1, \dots, q_n)$ be any local chart around $q \in Q$ that induce local coordinates $(q_i, \dot{q}_i)$ on TU we get for any $w \in T_qQ$

$$\begin{aligned} \langle \mathbb{F}L(q, \dot{q}), w \rangle &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} L(q_1, \dots, q_n, \dot{q}_1 + \varepsilon w_1, \dots, \dot{q}_n + \varepsilon w_n) \\ &= \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}) w_i \end{aligned}$$

Thus

$$\mathbb{F}L(q_i, \dot{q}_i) = (q_i, p_i), \quad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

The Hamiltonian in these coordinates is therefore

$$H(q, p) = \sum_{i=1}^n p_i \dot{q}_i - L(q, \dot{q}), \quad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Now using the chain rule, we compute the partial derivatives of H

$$\begin{aligned} \frac{\partial H}{\partial q_j}(q, p) &= \sum_i p_i \frac{\partial \dot{q}_i}{\partial q_j} - \frac{\partial L}{\partial q_j} - \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial q_j} = -\frac{\partial L}{\partial q_j} \\ \frac{\partial H}{\partial p_j}(q, p) &= \dot{q}_j + \sum_i p_i \frac{\partial \dot{q}_i}{\partial p_j} - \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial p_j} = \dot{q}_j \end{aligned}$$

Recall the Euler–Lagrange equations $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$ and consequently

$$\dot{p}_i = \frac{\partial L}{\partial q_i}$$

Thus, substituting the expressions derived above, we obtain the **Hamilton's equa-**

tions

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

for $i = 1, \dots, n$. Hence, we have shown that Hamilton's equations on T^*Q are equivalent to the Euler–Lagrange equations on TQ for a hyperregular Lagrangian. Thus, for a hyperregular Lagrangian, we can reformulate the variational principle directly in terms of the Hamiltonian.

Theorem 1.6. *Let $(U, q_1, \dots, q_n)$ be a coordinate chart on Q . Let $\gamma : [a, b] \rightarrow T^*U$, $t \mapsto (q_1(t), \dots, q_n(t), p_1(t), \dots, p_n(t))$ with variations $\delta q_i(a) = 0 = \delta q_i(b)$ for $i = 1, \dots, n$ and let $H : T^*Q \rightarrow \mathbb{R}$ be a Hamiltonian. Then the following are equivalent:*

1) γ satisfies the Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

2) $\delta \int_a^b (\sum_{i=1}^n p_i \dot{q}_i - H(q, p)) dt = 0$

Proof. It is a direct result of previous calculations, but to be precise, we consider an arbitrary variation of the curve γ in T^*U of the form

$$q_i(t) \rightarrow q_i(t) + \varepsilon \delta q_i(t), \quad p_i(t) \rightarrow p_i(t) + \varepsilon \delta p_i(t)$$

where ε is sufficiently small and $\delta q_i, \delta p_i$ are smooth functions with $\delta q_i(a) = 0 = \delta q_i(b)$ for all $i = 1, \dots, n$. Then

$$\begin{aligned} \delta \int_a^b (\sum_{i=1}^n p_i \dot{q}_i - H(q, p)) dt &= \sum_{i=1}^n \int_a^b \delta p_i \dot{q}_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i dt \\ &= \sum_{i=1}^n [p_i \delta q_i \Big|_a^b + \int_a^b \delta p_i \dot{q}_i - \dot{p}_i \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i dt] \\ &= \sum_{i=1}^n \int_a^b (\dot{q}_i - \frac{\partial H}{\partial p_i}) \delta p_i - (\dot{p}_i + \frac{\partial H}{\partial q_i}) \delta q_i dt \end{aligned}$$

Since this is satisfied for all smooth δp_i and δq_i , this implies

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \Leftrightarrow \quad \delta \int_a^b (\sum_{i=1}^n p_i \dot{q}_i - H(q, p)) dt = 0$$

□

2 Symplectic Geometry and Poisson Geometry

We first introduce the notion of symplectic manifold. Then we define Poisson manifolds, which are a more general setting and are particularly useful in the context of reduction by symmetries.

2.1 Symplectic Manifolds

Let M be a smooth manifold endowed with a 2-form $\omega \in \Omega^2(M)$ defined on it

Definition 2.1. ω is called a **symplectic form**, if $d\omega = 0$ and ω_x are non-degenerate for all $x \in M$, i.e.

$$\begin{aligned} T_x M &\rightarrow T_x^* M \\ v &\mapsto i_v \omega_x = \omega_x(v, \cdot) \end{aligned}$$

is an isomorphism.

A smooth manifold equipped with a symplectic form is called a *symplectic manifold*.

Definition 2.2. A **symplectic manifold** is a pair (M, ω) where M is a manifold and ω is a symplectic form on M .

Remark. Since ω_x is a nondegenerate skew symmetric bilinear form on $T_x M$ for each $x \in M$, $\dim T_x M = \dim M$ is even. Thus finite dimensional symplectic manifolds are always even dimensional.

Definition 2.3. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds and let $\varphi: M_1 \rightarrow M_2$ be a smooth map. Then φ is a **symplectic** if

$$\varphi^* \omega_2 = \omega_1$$

i.e.

$$(\omega_2)_{\varphi(x)}(d\varphi_x(v_1), d\varphi_x(v_2)) = (\omega_1)_x(v_1, v_2)$$

for all $x \in M_1$ and all $v_1, v_2 \in T_x M_1$. If φ is a diffeomorphism, then it is called a **symplectomorphism**.

Now we would like to classify symplectic manifolds up to symplectomorphism. Locally, this classification is given by the Darboux theorem below, and the only local invariant is the dimension of the manifold.

Theorem 2.4 (Darboux). *Let (M, ω) be a $2n$ -dimensional symplectic Manifold and let $x \in M$. Then, there is local coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ in a neighborhood of x such that*

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i$$

*The local coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ are called **canonical coordinates** and the 2-form ω is called the **canonical symplectic form**.*

Proof. See [MR99, Corollary 5.1.3.] or [CdS01, Theorem 8.1]. □

Remark. Let $\pi : T^*Q \rightarrow Q$ be the canonical projection. Then we can define the **Liouville 1-form** Θ on T^*Q by

$$\Theta_\alpha(w) = \langle \alpha, T\pi(w) \rangle, \quad \forall \alpha \in T^*Q, w \in T_\alpha T^*Q$$

We can now define the **canonical symplectic form** ω on T^*Q with the Liouville 1-form Θ by

$$\omega = -d\Theta$$

Take any local coordinates $(q_1, \dots, q_n)$ on Q and lift them to the local coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ on T^*Q . Then the Liouville 1-form Θ in these coordinates is given by

$$\Theta := \sum_{i=1}^n p_i dq_i$$

and the canonical symplectic form ω is given by

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i$$

Note that the definition is independent of the choice of local coordinates.

The non-degeneracy of the symplectic form ω guarantees that for every Hamiltonian function $H \in C^\infty(M)$ there exists a unique vector field $X_H \in \mathfrak{X}(M)$, called the **Hamiltonian vector field** of the Hamiltonian H , satisfying

$$i_{X_H} \omega = \omega(X_H, \cdot) = dH$$

In lifted local coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ on T^*Q , the Hamiltonian vector

field X_H is then given by

$$X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right)$$

2.2 Poisson Manifolds

Definition 2.5. A **Poisson manifold** is any manifold P endowed with a $\mathbb{R}$ -bilinear map, called **Poisson bracket**

$$\{\cdot, \cdot\} : C^\infty(P) \times C^\infty(P) \rightarrow C^\infty(P), \quad (f, g) \mapsto \{f, g\}$$

satisfying

$$1) \quad \{f, g\} = -\{g, f\} \quad (\text{Skew-Symmetry})$$

$$2) \quad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \quad (\text{Jacobi identity})$$

$$3) \quad \{f, g \cdot h\} = g \cdot \{f, h\} + \{f, g\} \cdot h \quad (\text{Leibniz rule})$$

for all $f, g, h \in C^\infty(P)$. So the Poisson bracket $\{\cdot, \cdot\}$ is a Lie bracket on $C^\infty(P)$ satisfying the Leibniz rule.

Remark. In a local chart $(U, x_1, \dots, x_n)$ a Poisson bracket $\{\cdot, \cdot\}$ takes the form

$$\{f, g\}|_U = \sum_{i,j=1}^n \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

where $\pi_{ij} = \{x_i, x_j\}|_U \in C^\infty(U)$. Thus a Poisson bracket is completely determined by the functions π_{ij} . Note also that every symplectic manifold is a Poisson manifold with the Poisson bracket defined by $\{f, g\} = \omega(X_f, X_g)$. However, not every Poisson manifold is symplectic. For instance, any manifold M with the trivial Poisson bracket $\{f, g\} = 0$ for all $f, g \in C^\infty(P)$ is a Poisson manifold but it is not symplectic, since the Poisson bracket is degenerate unless P is trivial.

Now we can extend the notion of Hamiltonian vector fields to Poisson manifolds. In fact, the Leibniz rule implies that the map $f \mapsto \{H, f\}$ is a derivation on $C^\infty(P)$ for all $H \in C^\infty(P)$. Thus, by the identification of vector fields with derivations on $C^\infty(P)$, we can associate a **Hamiltonian vector field** X_H to each Hamiltonian function H by

$$\{H, f\} = X_f(H) = -X_H(f), \quad \forall f \in C^\infty(P)$$

Denote now the set of all Hamiltonian vector fields by $\mathfrak{X}^{\text{Ham}}(P)$. Then we have the following result.

Proposition 2.6. *For a Poisson manifold $(P, \{\cdot, \cdot\})$, the map*

$$\begin{aligned} C^\infty(P) &\rightarrow \mathfrak{X}^{\text{Ham}}(P) \\ f &\mapsto X_f \end{aligned}$$

is a Lie algebra anti homomorphism, i.e. for any $f, g \in C^\infty(P)$ we have

$$[X_f, X_g] = -X_{\{f, g\}}$$

Proof. Let $f, g \in C^\infty(P)$, then

$$\begin{aligned} [X_f, X_g](h) &= X_f(X_g(h)) - X_g(X_f(h)) \\ &= X_f(\{h, g\}) - X_g(\{h, f\}) \\ &= \{\{h, g\}, f\} - \{\{h, f\}, g\} \\ &= \{f, \{g, h\}\} + \{g, \{h, f\}\} \quad \text{skew-symmetry} \\ &= -\{h, \{f, g\}\} \quad \text{Jacobi identity} \\ &= -X_{\{f, g\}}(h) \end{aligned}$$

□

Remark. Note that the Jacobi identity of Poisson bracket is equivalent to the condition $[X_f, X_g] = -X_{\{f, g\}}$ for all $f, g \in C^\infty(P)$. So we could have defined a Poisson manifold with that condition replacing the Jacobi identity.

Proposition 2.7. *Let $(P, \{\cdot, \cdot\})$ be a Poisson manifold and let φ_t be a flow on P . Furthermore, assume $H \in C^\infty(P)$ to be a Hamiltonian and $U \subset P$ to be an open set. Then the following are equivalent*

- 1) φ_t is the flow of X_H
- 2) $\frac{d}{dt}(F \circ \varphi_t) = \{F \circ \varphi_t, H\} = \{F, H\} \circ \varphi_t$ for all $F \in C^\infty(U)$

We denote the 2nd condition shortly by $\dot{F} = \{F, H\}$ along all integral curves of X_H .

Proof. See [MR99, Proposition 10.2.3]

□

As an example, we reformulate Hamilton's equations in terms of the canonical Poisson bracket on the cotangent bundle T^*Q .

Example 2.8. Take the lifted local coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ on T^*Q then we get the Poisson bracket $\{\cdot, \cdot\}$ on T^*Q by

$$\{f, g\} = \omega(X_f, X_g) = \sum_{i=1}^n dq_i \wedge dp_i(X_f, X_g) = \sum_{i=1}^n \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}$$

Let $(q(t), p(t))$ be a integral curve of X_H , then

$$\begin{aligned} \dot{F} &= \frac{d}{dt} F(q(t), p(t)) = \sum_i \frac{\partial F}{\partial q^i} \dot{q}^i + \frac{\partial F}{\partial p_i} \dot{p}_i \\ &\stackrel{!}{=} \sum_i \frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q^i} \\ &= \{F, H\} \end{aligned}$$

This is true if and only if $\frac{\partial H}{\partial p_i} = \dot{q}^i$ and $\frac{\partial H}{\partial q^i} = -\dot{p}_i$, which are exactly the Hamilton's equations.

Definition 2.9. A **Poisson homomorphism** between Poisson manifolds $(P_1, \{\cdot, \cdot\}_1)$ and $(P_2, \{\cdot, \cdot\}_2)$ is a smooth map $\phi : P_1 \rightarrow P_2$ such that

$$\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi$$

for all $f, g \in C^\infty(P_2)$.

Remark. If G acts on a Poisson manifold P by Poisson homomorphisms, then we say that the action is **canonical** (or we say G acts **canonically** on P).

As we are interested in studying symmetries, we want to consider quotient spaces obtained by factoring out symmetry group actions. But these quotients are not necessarily smooth manifolds. So we need to impose some conditions on the group action to ensure that the quotient is a smooth manifold. Assuming that the group action is free and proper¹, we can guarantee that the quotient is a smooth manifold and the projection map is a submersion. The following theorem shows that, these quotients inherit a Poisson structure under the assumption that the group action is canonical, free and proper. We will later apply this theorem to cotangent bundles with symmetry, where the reduced space can be identified with the dual of a Lie algebra $\mathfrak{g}^*$ and explicitly compute the resulting Poisson bracket called the Lie-Poisson bracket.

Theorem 2.10 (Poisson Reduction Theorem). *Let G be a Lie group acting canonically on the Poisson Manifold P . Suppose also that the action is free and proper.*

¹See the discussion before the Definition 0.25 and the Proposition 0.26 for more details.

Then there is a unique reduced Poisson structure on P/G denoted $\{\cdot, \cdot\}_{P/G}$ such that the projection $\pi : P \rightarrow P/G$ is a Poisson map, i.e.

$$\{f, g\}_{P/G} \circ \pi = \{f \circ \pi, g \circ \pi\}, \quad \forall f, g \in C^\infty(P/G)$$

Proof. See [MR99, Theorem 10.5.1]. □

3 Symmetry and Reduction

In this section we first introduce the concept of momentum maps, which helps us to prove the Lie–Poisson reduction 3.11. Then we will show that the reduced space T^*G/G obtained by factoring out the symmetry group action can be identified with the dual of a Lie algebra $\mathfrak{g}^*$ and we will explicitly compute the resulting Poisson bracket on $\mathfrak{g}^*$, which is called the Lie–Poisson bracket (6). That then gives us the Lie–Poisson equations (7) describing the dynamics on $\mathfrak{g}^*$. Then we will introduce the Euler–Poincaré reduction 3.13, which is a Lagrangian analogue of the Lie–Poisson reduction. That is, we will reformulate the variational principle on $\mathfrak{g}$ and obtain the Euler–Poincaré equations. Lastly, we will show that the resulting Euler–Poincaré equations are equivalent to the Lie–Poisson equations on $\mathfrak{g}^*$.

3.1 Momentum maps

Assume that the Lie group G acts on the Poisson manifold P canonically on the left by $\phi : G \rightarrow \text{Diff}(P)$ and denote the action as $\phi_g : P \mapsto P$, $\phi_g(p) = g \cdot p$ for $g \in G$. To get the induced canonical infinitesimal action of this, differentiate $\phi_g^*\{f, h\} = \{\phi_g^*f, \phi_g^*h\}$ with respect to $g \in G$ in the direction $\xi \in \mathfrak{g}$. On left hand side we have

$$\left. \frac{d}{dt} \right|_{t=0} \phi_{\exp(t\xi)}^* \{f, h\} = \xi^P(\{f, h\})$$

where ξ^P is the infinitesimal generator of the action corresponding to $\xi \in \mathfrak{g}$ ², and on the right hand side

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \{\phi_{\exp(t\xi)}^* f, \phi_{\exp(t\xi)}^* h\} &= \left\{ \left. \frac{d}{dt} \right|_{t=0} \phi_{\exp(t\xi)}^* f, \phi_e^* h \right\} + \{\phi_e^* f, \left. \frac{d}{dt} \right|_{t=0} \phi_{\exp(t\xi)}^* h\} \\ &= \{\xi^P(f), h\} + \{f, \xi^P(h)\} \end{aligned}$$

²See the definition 0.27 for the definition of the infinitesimal generator of the action.

Thus we obtain the condition

$$\xi^P(\{f, h\}) = \{\xi^P(f), h\} + \{f, \xi^P(h)\} \quad (2)$$

When this condition holds for every $\xi \in \mathfrak{g}$, we say that the Lie algebra $\mathfrak{g}$ acts canonically on P . Now let us define the moment map of the action of $\mathfrak{g}$ on P .

Definition 3.1. Let the Lie algebra $\mathfrak{g}$ act canonically on the Poisson manifold P by $\xi \mapsto \xi^P$ ³ and let $\mu : \mathfrak{g} \rightarrow C^\infty(P)$ be the linear map such that

$$X_{\mu(\xi)} = \xi^P, \quad \forall \xi \in \mathfrak{g}$$

i.e. $\mu(\xi)$ is the hamiltonian function of the vector field ξ^P . Then the map $J : P \rightarrow \mathfrak{g}^*$ defined by

$$\langle J(p), \xi \rangle = \mu(\xi)(p), \quad \forall \xi \in \mathfrak{g}, \forall p \in P$$

is called **momentum map** of the action.

Example 3.2. Let $G \subset GL_n(\mathbb{R})$ be a matrix Lie group. Let us consider the action of G on $\mathbb{R}^n$ given by matrix multiplication $\phi_A(q) = Aq$ for $A \in G$ and $q \in \mathbb{R}^n$. So first we are going to lift the action to the cotangent bundle. For that first differentiate ϕ_A with respect to $q \in \mathbb{R}^n$ in the direction $v \in T_q\mathbb{R}^n$ to get the tangent lift of the action

$$T_q\phi_A(v) = \left. \frac{d}{dt} \right|_{t=0} \phi_A(q + tv) = \left. \frac{d}{dt} \right|_{t=0} A(q + tv) = Av, \quad \forall v \in T_q\mathbb{R}^n \simeq \mathbb{R}^n$$

Thus $T\phi_A : T\mathbb{R}^n \rightarrow T\mathbb{R}^n$, $(q, v) \mapsto (Aq, Av)$. Then the cotangent lift of this is given by the usual duality pairing

$$\langle T^*\phi_A(\alpha_s), v \rangle = \langle \alpha_s, T\phi_A(v) \rangle = \langle \alpha_s, A \cdot v \rangle = \langle A^T \alpha_s, \cdot v \rangle$$

where $\alpha_s \in T_s^*\mathbb{R}^n$, $v \in T_q\mathbb{R}^n$ and $s = Aq$. To obtain a proper lift, we rebase the cotangent map to the original point q , which gives the lifted action on $T^*\mathbb{R}^n$ as

$$\begin{aligned} T_{Aq}^*\phi_{A^{-1}} : T_q^*\mathbb{R}^n &\rightarrow T_{Aq}^*\mathbb{R}^n \\ (q, p) &\mapsto A \cdot (q, p) = (Aq, (A^T)^{-1}p) \end{aligned} \quad (3)$$

To compute the infinitesimal generator corresponding to $\xi \in \mathfrak{g}$, we differentiate (3) with respect to $A \in G$ in direction ξ . That is, we consider $A(t) = \exp(t\xi)$ and

³See the definition 0.31 for the definition of the Lie algebra action.

differentiate it at $t = 0$

$$\xi^{T^*\mathbb{R}^n}(q, p) = \left. \frac{d}{dt} \right|_{t=0} (\exp(t\xi)q, (\exp(t\xi)^T)^{-1}p) = (\xi q, -\xi^T p)$$

Furthermore, by the definition of the canonical symplectic structure on $T^*\mathbb{R}^n$ we have

$$X_{\mu(\xi)}(q, p) = \left(\frac{\partial \mu(\xi)}{\partial p_i}, -\frac{\partial \mu(\xi)}{\partial q_i} \right)$$

Then, $X_{\mu(\xi)}(q, p) = \xi^{T^*\mathbb{R}^n}(q, p)$ implies that

$$\frac{\partial \mu(\xi)}{\partial p} = \xi q, \quad \frac{\partial \mu(\xi)}{\partial q} = \xi^T p$$

Then a solution is given by $\mu(\xi)(q, p) = (\xi q) \cdot p$. Hence $\langle J(p), \xi \rangle = (\xi q) \cdot p$. For example, for the case of the special orthogonal group $G = SO(3)$ acting on $\mathbb{R}^3$ we obtain $J(q, p) = q \times p$ for all $(q, p) \in T^*\mathbb{R}^3$.

Definition 3.3. We denote by $\mathcal{P} : \mathfrak{X}(Q) \rightarrow C^\infty(T^*Q)$ the **momentum function** defined for any $X \in \mathfrak{X}(Q)$ and $\alpha_q \in T_q^*Q$ by

$$\mathcal{P}(X)(\alpha_q) = \langle \alpha_q, X_q \rangle$$

In local coordinates $\alpha_q = (q, p)$ one has $\mathcal{P}(X)(q_i, p_i) = \sum_j X^j(q_i) p_j$.

Let $\mathcal{L}(T^*Q)$ denote the space of fiberwise-linear functions on T^*Q , i.e.

$$\mathcal{L}(T^*Q) = \{F \in C^\infty(T^*Q) \mid F(\alpha_q) \text{ is linear in } \alpha_q \in T_q^*Q \text{ for each } q \in Q\}$$

Since $\mathcal{L}(T^*Q)$ is closed under the canonical Poisson bracket on T^*Q , it forms a Lie subalgebra of $C^\infty(T^*Q)$ with the Poisson bracket as the Lie bracket.

Lemma 3.4. *Let $\mathcal{P} : \mathfrak{X}(Q) \rightarrow C^\infty(T^*Q)$, $X \mapsto \mathcal{P}(X)$ be the momentum map with $\mathcal{P}(X)(\alpha_q) = \langle \alpha_q, X_q \rangle$. Then it follows that*

- 1) *The momentum map $\mathcal{P}$ is a Lie algebra anti-isomorphism between the Lie algebra of vector fields $(\mathfrak{X}(Q), [\cdot, \cdot])$ and the Lie Algebra $(\mathcal{L}(T^*Q), \{\cdot, \cdot\})$*

$$(\mathfrak{X}(Q), [\cdot, \cdot]) \cong (\mathcal{L}(T^*Q), \{\cdot, \cdot\})$$

- 2) *The map $F \mapsto X_F$ is a Lie algebra anti-isomorphism between the Lie algebra $(\mathcal{L}(T^*Q), \{\cdot, \cdot\})$ and the Lie algebra of Hamiltonian vector fields generated by*

fiberwise-linear functions, i.e.

$$(\mathcal{L}(T^*Q), \{\cdot, \cdot\}) \cong (\{X_F \in \mathfrak{X}(T^*Q) \mid F \in \mathcal{L}(T^*Q)\}, [\cdot, \cdot])$$

Proof. (1) By construction $P(X) : T^*Q \rightarrow \mathbb{R}$ is smooth and linear on each fiber T_q^*Q for fixed $q \in Q$. Moreover, to show that $P(X)$ is injective, let $P(X) = 0$, then $\langle \alpha_q, X_q \rangle = \alpha_q(X_q) = 0$ for each $q \in Q$ and each $\alpha_q \in T_q^*Q$. Thus $X_q = 0$ for every $q \in Q$ and therefore $X = 0$. Furthermore for each $F \in \mathcal{L}(T^*Q)$, one can define a vector field $X^F \in \mathfrak{X}(Q)$ by

$$F(\alpha_q) = \langle \alpha_q, X_q^F \rangle, \quad \forall \alpha_q \in T_q^*Q$$

Thus $P(X^F) = F$ by

$$P(X^F)(\alpha_q) = \langle \alpha_q, X_q^F \rangle = F(\alpha_q), \quad \forall \alpha_q \in T_q^*Q$$

showing the surjectivity. Furthermore, in local coordinates $\alpha_q = (q, p)$ on T_q^*Q one has

$$\mathcal{P}(X)(q, p) = \sum_i X^i(q) p_i, \quad \mathcal{P}(Y)(q, p) = \sum_i Y^i(q) p_i$$

Hence the Poisson bracket is

$$\begin{aligned} \{\mathcal{P}(X), \mathcal{P}(Y)\}(q, p) &= \sum_k \frac{\partial \mathcal{P}(X)}{\partial q^k} \frac{\partial \mathcal{P}(Y)}{\partial p_k} - \frac{\partial \mathcal{P}(X)}{\partial p_k} \frac{\partial \mathcal{P}(Y)}{\partial q^k} \\ &= \sum_k \left(\sum_i p_i \frac{\partial X^i}{\partial q^k} \right) Y^k - \sum_k X^k \left(\sum_i p_i \frac{\partial Y^i}{\partial q^k} \right) \\ &= \sum_{i,k} p_i \frac{\partial X^i}{\partial q^k} Y^k - \sum_{i,k} p_i X^k \frac{\partial Y^i}{\partial q^k} \\ &= \sum_i p_i \sum_k \left(Y^k \frac{\partial X^i}{\partial q^k} - X^k \frac{\partial Y^i}{\partial q^k} \right) \\ &= \sum_i p_i [Y, X]^i(q) \\ &= -\mathcal{P}([X, Y])(q, p) \end{aligned}$$

Thus $\{\mathcal{P}(X), \mathcal{P}(Y)\} = -\mathcal{P}([X, Y])$

(2) Recall that for any two functions $F, G \in C^\infty(T^*Q)$, we have $[X_F, X_G] = -X_{\{F, G\}}$. Furthermore, the map $F \mapsto X_F$ is surjective by definition, since every Hamiltonian vector field X_F is generated by some function F . To see that it is also

injective, suppose $X_F = 0$. Then, for any $G \in \mathcal{L}(T^*Q)$, we have

$$\{F, G\} = X_G(F) = 0$$

Thus F must be constant on each fiber. But F is linear on each fiber and the only linear function that is constant is zero. Hence $F = 0$ and $F \mapsto X_F$ is injective. $\square$

Proposition 3.5. *The flow of the Hamiltonian vector field $X_{\mathcal{P}(X)} \in \mathfrak{X}(T^*Q)$ associated to the momentum function $\mathcal{P}(X) \in C^\infty(T^*Q)$ is given by the cotangent lift of the flow of $X \in \mathfrak{X}(Q)$, i.e. if ϕ_t is the flow of the vector field X , then the flow of the Hamiltonian vector field $X_{\mathcal{P}(X)}$ is given by $T^*\phi_{-t} : T_q^*Q \rightarrow T_{\phi_t(q)}^*Q$.*

Proof. Let $\pi_Q : T^*Q \rightarrow Q$ be the projection and let $\phi_t : Q \rightarrow Q$ be the flow of the vector field $X \in \mathfrak{X}(Q)$. Then the cotangent lift of ϕ_t is given by $T^*\phi_{-t} : T_q^*Q \rightarrow T_{\phi_t(q)}^*Q$. Thus

$$\pi_Q \circ T^*\phi_{-t}(\alpha_q) = \phi_t(q) = \phi_t \circ \pi_Q(\alpha_q)$$

for all $\alpha_q \in T_q^*Q$. Therefore

$$\pi_Q \circ T^*\phi_{-t} = \phi_t \circ \pi_Q$$

Differentiating at $t = 0$ gives

$$T\pi_Q \circ Y = X \circ \pi_Q$$

where

$$Y(\alpha) = \left. \frac{d}{dt} \right|_{t=0} T^*\phi_{-t}(\alpha) \in T_\alpha(T^*Q)$$

Now use the fact that for any diffeomorphism ψ on Q , the cotangent lift preserves the Liouville one-form on T^*Q , i.e. $(T^*\psi)^*\Theta = \Theta$. Now apply this to $\psi = \phi_{-t}$ and differentiate at $t = 0$

$$\mathfrak{L}_Y \Theta = \left. \frac{d}{dt} \right|_{t=0} (T^*\phi_{-t})^*(\Theta) = \left. \frac{d}{dt} \right|_{t=0} \Theta = 0$$

Using Cartan's formula and $\omega = -d\Theta$ we get

$$\begin{aligned} 0 &= \mathfrak{L}_Y \Theta \\ &= d(i_Y \Theta) + i_Y d\Theta \\ &= d(i_Y \Theta) - i_Y \omega \\ &\Rightarrow i_Y \omega = d(i_Y \Theta) \end{aligned}$$

Thus, by definition of the Liouville 1-form Θ , it follows that for any $\alpha_q \in T_q^*Q$

$$\begin{aligned}
 (i_Y \Theta)(\alpha_q) &= \langle \Theta(\alpha_q), Y(\alpha_q) \rangle \\
 &= \langle \alpha_q, T\pi_Q(Y(\alpha_q)) \rangle && \text{Definition of } \Theta \\
 &= \langle \alpha_q, X(\pi_Q(\alpha_q)) \rangle && \text{Using the relation we derived above} \\
 &= \mathcal{P}(X)(\alpha_q) && \text{Definition of } \mathcal{P}(X)
 \end{aligned}$$

Hence $i_Y \omega = d\mathcal{P}(X)$, so that $Y = X_{\mathcal{P}(X)}$. $\square$

$X_{\mathcal{P}(X)} \in \mathfrak{X}(T^*Q)$ is called the cotangent lift of $X \in \mathfrak{X}(Q)$ denoted by $X' := X_{\mathcal{P}(X)}$. Furthermore

$$\begin{aligned}
 [X', Y'] &= [X_{\mathcal{P}(X)}, X_{\mathcal{P}(Y)}] \\
 &= -X_{\{\mathcal{P}(X), \mathcal{P}(Y)\}} \\
 &= -X_{-\mathcal{P}[X, Y]} \\
 &= [X, Y]'
 \end{aligned}$$

Thus, the cotangent lift is a Lie algebra homomorphism $\mathfrak{X}(Q) \rightarrow \mathfrak{X}(T^*Q)$, $X \mapsto X'$.

Definition 3.6. Momentum maps $J : P \rightarrow \mathfrak{g}^*$ are called **infinitesimally equivariant**, if one of the following equivalent conditions are satisfied

- 1) $T_z J \cdot \eta^P(z) = -\text{ad}_\eta^* J(z)$ for all $\eta \in \mathfrak{g}$ and $z \in P$
- 2) The map $\mu : \mathfrak{g} \rightarrow C^\infty(P)$ associated to J satisfies $\mu([\xi, \eta]) = \{\mu(\xi), \mu(\eta)\}$ for all $\xi, \eta \in \mathfrak{g}$

Furthermore, Lie algebra actions admitting infinitesimally equivariant momentum maps are called Hamiltonian actions. Moreover, assume that the Lie group G acts canonically on P . Then its Lie algebra $\mathfrak{g}$ acts canonically on P by $\xi \mapsto \xi^P$ for any $\xi \in \mathfrak{g}$. The momentum map $J : P \rightarrow \mathfrak{g}^*$ corresponding to this induced canonical Lie algebra action is called **equivariant** if

$$Ad_{g^{-1}}^* \circ J = J \circ \phi_g, \quad \forall g \in G$$

To compute the momentum map in an easier and straightforward way for cotangent lifted actions, we have the following theorem.

Theorem 3.7 (Momentum Maps for Lifted Actions). *Let $P = T^*Q$ be the cotangent bundle of a manifold Q equipped with the canonical symplectic structure. Suppose*

that the Lie algebra $\mathfrak{g}$ acts on the manifold Q by vector fields $\xi \mapsto \xi^Q$ for $\xi \in \mathfrak{g}$. Then $\mathfrak{g}$ acts on P on the left by the cotangent lifted action $\xi \mapsto \xi^P = (\xi^Q)'$. This lifted action on P is Hamiltonian with infinitesimally equivariant momentum map $J : P \rightarrow \mathfrak{g}^*$ given by

$$\langle J(\alpha_q), \xi \rangle = \langle \alpha_q, \xi^Q(q) \rangle = \mathcal{P}(\xi^Q)(\alpha_q) \quad (4)$$

Moreover, if $\mathfrak{g}$ is the Lie algebra of a Lie group G that acts on Q , then the cotangent lifted action of G on P is Hamiltonian with equivariant momentum map $J : P \rightarrow \mathfrak{g}^*$ given by the same formula (4) above.

Proof. We first show that $\mathfrak{g} \rightarrow T^*Q$, $\xi \mapsto \xi^P$ is indeed a canonical Lie algebra action. Let $\xi, \eta \in \mathfrak{g}$

$$\begin{aligned} [\xi, \eta]^P &= ([\xi, \eta]^Q)' \\ &= -[\xi^Q, \eta^Q]' \\ &= -[(\xi^Q)', (\eta^Q)'] \\ &= -[\xi^P, \eta^P] \end{aligned}$$

Furthermore, for any $F, H \in C^\infty(T^*Q)$ we have

$$\begin{aligned} \xi^P(\{F, H\}) &= X_{\mathcal{P}(\xi^Q)}(\{F, H\}) \\ &= \{X_{\mathcal{P}(\xi^Q)}(F), H\} + \{F, X_{\mathcal{P}(\xi^Q)}(H)\} \\ &= \{\xi^P(F), H\} + \{F, \xi^P(H)\} \end{aligned}$$

Thus $\xi \mapsto \xi^P$ is a canonical Lie algebra action. By using the Proposition 3.5, if ϕ_t is the flow of ξ^Q , the flow of $\xi^P = (\xi^Q)' = X_{\mathcal{P}(\xi^Q)}$ is $T^*\phi_{-t}$. Thus $\mu(\xi) = \mathcal{P}(\xi^Q)$ is a induced "comoment map". Furthermore, using the fact that $\mathcal{P}$ is a Lie algebra anti-isomorphism, we get

$$\begin{aligned} \{\mu(\xi), \mu(\eta)\} &= \{\mathcal{P}(\xi^Q), \mathcal{P}(\eta^Q)\} \\ &= -\mathcal{P}([\xi^Q, \eta^Q]) \\ &= \mathcal{P}([\xi, \eta]^Q) \\ &= \mu([\xi, \eta]) \end{aligned}$$

Hence J is the infinitesimally equivariant momentum map. Now assume that $\mathfrak{g}$ is

the Lie algebra of the Lie group G that acts on Q . Then

$$\begin{aligned}\langle J(g \cdot \alpha_q), \xi \rangle &= \langle g \cdot \alpha_q, \xi^Q(g \cdot q) \rangle \\ &= \langle T_{gq}^* \phi_g^{-1}(\alpha_q), \xi^Q(\phi_g(q)) \rangle \\ &= \langle \alpha_q, T_{gq} \phi_g^{-1}(\xi^Q(\phi_g(q))) \rangle \\ &= \langle \alpha_q, \phi_g^* \xi^Q(q) \rangle\end{aligned}$$

Using the relation $\phi_g^* \xi^Q(q) = (\text{Ad}_{g^{-1}} \xi)^Q(q)$ proven in Proposition 0.30, we get

$$\begin{aligned}\langle J(g \cdot \alpha_q), \xi \rangle &= \langle \alpha_q, (\text{Ad}_{g^{-1}} \xi)^Q(q) \rangle \\ &= \langle J(\alpha_q), \text{Ad}_{g^{-1}} \xi \rangle \\ &= \langle \text{Ad}_{g^{-1}}^* J(\alpha_q), \xi \rangle\end{aligned}$$

Thus $J \circ \Phi_g = \text{Ad}_{g^{-1}}^* \circ J$. □

Example 3.8. Let $G \subset GL_n(\mathbb{R})$ be a matrix Lie group. Let us again consider the action of G on $\mathbb{R}^n$ given by matrix multiplication $\phi_A(q) = Aq$ for $A \in G$ and $q \in \mathbb{R}^n$. As in the example 3.2, the infinitesimal generator of this action is then given by

$$\xi_q^{\mathbb{R}^n} = \xi q$$

for $\xi \in \mathfrak{g}$. By the above theorem, the cotangent lift of this action to $T^*\mathbb{R}^n$ has an equivariant momentum map $J: T^*\mathbb{R}^n \rightarrow \mathfrak{g}^*$ given by

$$\langle J(q, p), \xi \rangle = p \cdot (\xi q)$$

As we can see, that was easier compared to Example 3.2

Example 3.9 (Momentum map of the left action). Let G be a Lie group and recall the left translation map

$$\begin{aligned}L_g : G &\rightarrow G \\ h &\mapsto gh\end{aligned}$$

Its tangent lift is given by

$$\begin{aligned}TL_g : TG &\rightarrow TG \\ v_h &\mapsto (gh, T_h L_g(v_h))\end{aligned}$$

Hence its infinitesimal Lie algebra action is given by

$$\xi^G(g) = \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi)g = \xi g = T_e R_g(\xi)$$

for $\xi \in \mathfrak{g}$. Then the momentum map for the cotangent lift of the left translation action is equal to

$$\langle J_L(\alpha_g), \xi \rangle = \langle \alpha_g, \xi^G(g) \rangle = \langle \alpha_g, T_e R_g(\xi) \rangle = \langle T_e^* R_g(\alpha_g), \xi \rangle$$

that is

$$J_L(\alpha_g) = T_e^* R_g(\alpha_g)$$

Notice that J_L is right invariant and is a equivariant momentum map, so they are Poisson maps according the following theorem

Theorem 3.10. *If $J : P \rightarrow \mathfrak{g}^*$ is an infinitesimally equivariant momentum map for a left Hamiltonian action of the Lie algebra $\mathfrak{g}$ on any Poisson manifold P , then J is a Poisson map, i.e. for all $F_1, F_2 \in C^\infty(\mathfrak{g}^*)$ we have*

$$J^*\{F_1, F_2\}_{LP} = \{J^*F_1, J^*F_2\}$$

where $\{\cdot, \cdot\}_{LP}$ denotes the Lie–Poisson bracket given in the Theorem 3.11.

Proof. See [MR99, Theorem 12.4.1] □

3.2 Lie–Poisson Reduction

Let G be a Lie group and consider the cotangent bundle T^*G . We now want to compute the Poisson bracket on the reduced space T^*G/G obtained by factoring out the left translation action of G on T^*G . Recall that, a smooth function $F_L : T^*G \rightarrow \mathbb{R}$ is called *left-invariant* if for every $g \in G$

$$F_L \circ T^*L_g = F_L$$

Let $F \in C^\infty(\mathfrak{g}^*)$, then we can extend it to a left invariant smooth function $F_L : T^*G \rightarrow \mathbb{R}$ by

$$F_L(\alpha_g) = F(T_e^* L_g \cdot \alpha_g) = (F \circ J_R)(\alpha_g)$$

where, for all $\alpha_g \in T^*G$, the momentum map $J_R : T^*G \rightarrow \mathfrak{g}^*$ is defined by

$$J_R(\alpha_g) = T_e^* L_g \cdot \alpha_g$$

Furthermore, note that the trivialization of T^*G by left translations given by

$$\begin{aligned} T_g^*G &\rightarrow G \times \mathfrak{g}^* \\ \alpha_g &\mapsto (g, T_e^*L_g \cdot \alpha_g) = (g, J_R(\alpha_g)) \end{aligned}$$

So this trivialization identifies the cotangent lift of left translation action on T^*G with the left action on $G \times \mathfrak{g}^*$ given by

$$g \cdot (h, \mu) = (gh, \mu), \quad \forall g, h \in G \text{ and } \mu \in \mathfrak{g}^*$$

This means that G acts trivially on the $\mathfrak{g}^*$ -component, so that the quotient T^*G/G is diffeomorphic to $\mathfrak{g}^*$. Thus, we can regard $J_R : T^*G \rightarrow \mathfrak{g}^*$ as the canonical projection $T^*G \rightarrow T^*G/G$ and endow $\mathfrak{g}^*$ with the unique Poisson structure that makes J_R a Poisson map by the Poisson reduction theorem. Thus, $\mathfrak{g}^*$ inherits a Poisson structure $\{\cdot, \cdot\}_{LP}$ uniquely given by

$$\{F, H\}_{LP} \circ J_R = \{F \circ J_R, H \circ J_R\} = \{F_L, H_L\} \quad (5)$$

Now we would like to compute this Poisson structure called the **Lie-Poisson structure**.

Theorem 3.11 (Lie-Poisson Reduction). *Let $F, H \in C^\infty(\mathfrak{g}^*)$ and extend them to left invariant functions $F_L : T^*G \rightarrow \mathbb{R}$ and $H_L : T^*G \rightarrow \mathbb{R}$ respectively. Then the restriction of the canonical Poisson bracket $\{F_L, H_L\}$ on T^*G to $\mathfrak{g}^*$ endows $\mathfrak{g}^*$ with a Poisson structure $\{\cdot, \cdot\}_{LP}$ given by*

$$\{F, H\}_{LP}(\mu) = - \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle \quad (6)$$

for all $\mu \in \mathfrak{g}^*$. Here, $\frac{\delta F}{\delta \mu} \in \mathfrak{g}$ is defined by

$$\left\langle \delta \mu, \frac{\delta F}{\delta \mu} \right\rangle = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} F(\mu + \epsilon \delta \mu), \quad \forall \delta \mu \in \mathfrak{g}^*$$

So $(\mathfrak{g}^*, \{\cdot, \cdot\}_{LP})$ is a Poisson manifold and the bracket $\{\cdot, \cdot\}_{LP}$ is called the **Lie-Poisson bracket**.

Proof. Let $C_L^\infty(T^*G)$ be the space of left-invariant functions on T^*G , so it is isomorphic to $C^\infty(\mathfrak{g}^*)$ by the isomorphism

$$C^\infty(\mathfrak{g}^*) \rightarrow C_L^\infty(T^*G), \quad F \mapsto F_L$$

where $F_L(\alpha_g) = F(J_R(\alpha_g)) = F(T_e^*L_g \cdot \alpha_g)$. Indeed, the inverse is just the restriction to the fiber over the identity

$$C_L^\infty(T^*G) \rightarrow C^\infty(\mathfrak{g}^*), F \mapsto F|_{T_e^*G}$$

Moreover, T^*L_g is a symplectic map for each $g \in G$, thus the space $C_L^\infty(T^*G)$ is closed under the Poisson bracket on T^*G . Therefore $C^\infty(\mathfrak{g}^*)$ gets endowed with a unique Poisson structure by the Poisson reduction theorem. Furthermore, the Poisson bracket $\{F, H\}$ for $F, H \in C^\infty(\mathfrak{g}^*)$ depends only on the differentials of F and H at each point. Thus, to determine the bracket $\{\cdot, \cdot\}_{LP}$ on $\mathfrak{g}^*$ we can assume that F and H are linear functions on $\mathfrak{g}^*$ of a particular point. So let $F \in C^\infty(\mathfrak{g}^*)$ be linear, we can then write $F(\mu) = \langle \mu, \frac{\delta F}{\delta \mu} \rangle$, where $\frac{\delta F}{\delta \mu}$ is a constant in $\mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. Now let $\mu = T_e^*L_g \cdot \alpha_g$, then

$$\begin{aligned} F_L(\alpha_g) &= F(T_e^*L_g \cdot \alpha_g) \\ &= \left\langle T_e^*L_g \cdot \alpha_g, \frac{\delta F}{\delta \mu} \right\rangle \\ &= \left\langle \alpha_g, T_e L_g \cdot \frac{\delta F}{\delta \mu} \right\rangle \\ &= \left\langle \alpha_g, \left(\frac{\delta F}{\delta \mu} \right)_L (g) \right\rangle \\ &= \mathcal{P} \left(\left(\frac{\delta F}{\delta \mu} \right)_L \right) (\alpha_g) \end{aligned}$$

where $\left(\frac{\delta F}{\delta \mu} \right)_L (g) = T_e L_g \cdot \frac{\delta F}{\delta \mu}$ is the left-invariant vector field on G generated by $\frac{\delta F}{\delta \mu}$. Similarly, we can write $H_L(\alpha_g) = \mathcal{P} \left(\left(\frac{\delta H}{\delta \mu} \right)_L \right) (\alpha_g)$ for some $\frac{\delta H}{\delta \mu} \in \mathfrak{g}$. Moreover, recall that $\{\mathcal{P}(X), \mathcal{P}(Y)\} = -\mathcal{P}([X, Y])$ for any vector fields X, Y . Using this and the fact that $\left(\frac{\delta F}{\delta \mu} \right)_L$ and $\left(\frac{\delta H}{\delta \mu} \right)_L$ are left-invariant vector fields⁴, we get

$$\begin{aligned} \{F_L, H_L\}(\alpha_g) &= \left\{ \mathcal{P} \left(\left(\frac{\delta F}{\delta \mu} \right)_L \right), \mathcal{P} \left(\left(\frac{\delta H}{\delta \mu} \right)_L \right) \right\} (\alpha_g) \\ &= -\mathcal{P} \left(\left[\left(\frac{\delta F}{\delta \mu} \right)_L, \left(\frac{\delta H}{\delta \mu} \right)_L \right] \right) (\alpha_g) \\ &= -\mathcal{P} \left(\left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right]_L \right) (\alpha_g) \\ &= -\left\langle \alpha_g, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right]_L (g) \right\rangle \end{aligned}$$

⁴Left invariant vector fields are closed under the Lie bracket, see Definition 0.11 and the remark right after that

$$\begin{aligned}
&= - \left\langle \alpha_g, T_e L_g \cdot \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle \\
&= - \left\langle T_e^* L_g \cdot \alpha_g, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle \\
&= - \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle
\end{aligned}$$

Finally, using the fact that $F \circ J_R = F_L$ and $H \circ J_R = H_L$ and the characterization of the Lie-Poisson bracket given by the equality (5) we have

$$\begin{aligned}
\{F, H\}_{LP}(\mu) &= \{F, H\}_{LP} \circ J_R(\alpha_g) \\
&= \{F \circ J_R, H \circ J_R\}(\alpha_g) \\
&= \{F_L, H_L\}(\alpha_g) \\
&= - \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle
\end{aligned}$$

which proves the Equation (6). $\square$

Example 3.12. Let us apply the above theorem for the Lie group $G = SO(3)$ and its Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$. Using the identification $\mathfrak{so}(3) \cong \mathbb{R}^3$ derived in example 0.15 and identifying $\mathbb{R}^3 \cong (\mathbb{R}^3)^*$ via the standard inner product, the Lie-Poisson bracket on $\mathbb{R}^3$ is given by

$$\{F, H\}(\Pi) = -\Pi \cdot (\nabla F(\Pi) \times \nabla H(\Pi))$$

for $F, H \in C^\infty(\mathbb{R}^3)$ and $\Pi \in \mathbb{R}^3$. We call this bracket the rigid body bracket.

Remark. Instead of extending F and H to left invariant functions, we can restrict the functions to a coadjoint orbit $\mathcal{O}_\mu$ of $\mu \in \mathfrak{g}^*$, which is given by

$$\mathcal{O}_\mu = \{\text{Ad}_{g^{-1}}^* \mu : g \in G\} := G \cdot \mu$$

Then we get an induced symplectic structure ω on $\mathcal{O}_\mu$ given by

$$\omega(\mu)(\xi^{g^*}(\mu), \eta^{g^*}(\mu)) = -\langle \mu, [\xi, \eta] \rangle, \quad \forall \mu \in \mathcal{O}_\mu, \quad \xi, \eta \in \mathfrak{g}$$

This symplectic structure induces also a Poisson structure $\{\cdot, \cdot\}$ on $\mathcal{O}_\mu$ given by

$$\{F, H\}(\mu) = - \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle, \quad \forall F, H \in C^\infty(\mathcal{O}_\mu), \quad \mu \in \mathcal{O}_\mu$$

which is exactly the Lie-Poisson structure on $\mathcal{O}_\mu$ as a submanifold of $\mathfrak{g}^*$. For more

details, see [[MR99], Section 14].

Now having this Lie-Poisson structure on $\mathfrak{g}^*$ in our hands, we can write the Hamilton's equations $\dot{F} = \{F, H\}_{LP}$ for a Hamiltonian $H \in C^\infty(\mathfrak{g}^*)$ and any $F \in C^\infty(\mathfrak{g}^*)$. Again by assuming that F is linear, we can write $F(\mu) = \langle \mu, \frac{\delta F}{\delta \mu} \rangle$ for some $\frac{\delta F}{\delta \mu} \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. Then

$$\begin{aligned} \left\langle \dot{\mu}, \frac{\delta F}{\delta \mu} \right\rangle &= \dot{F}(\mu) = \{F, H\}_{LP}(\mu) \\ &= - \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle \\ &= \left\langle \mu, \text{ad}_{\frac{\delta H}{\delta \mu}}^* \frac{\delta F}{\delta \mu} \right\rangle \\ &= \left\langle \text{ad}_{\frac{\delta H}{\delta \mu}}^* \mu, \frac{\delta F}{\delta \mu} \right\rangle \end{aligned}$$

Since this holds for all $\frac{\delta F}{\delta \mu} \in \mathfrak{g}$, we get

$$\dot{\mu} = \text{ad}_{\frac{\delta H}{\delta \mu}}^* \mu \tag{7}$$

which is called the **Lie-Poisson equation**.

3.3 Euler–Poincaré Reduction

Let G be a Lie group and consider its tangent bundle TG . When a Lagrangian $L : TG \rightarrow \mathbb{R}$ is invariant under left translations, then this symmetry naturally leads to reduced equations of motion on the Lie algebra $\mathfrak{g}$. Recall that a Lagrangian $L : TG \rightarrow \mathbb{R}$ is called *left-invariant* if for every $g \in G$ we have

$$L \circ TL_g = L$$

that is

$$L(gh, T_h L_g \dot{h}) = L(h, \dot{h})$$

Left-invariance of L implies that L is determined by its value at the identity, i.e.

$$L(g, \dot{g}) = L(g^{-1}g, g^{-1}\dot{g}) = L(e, \xi)$$

where $\xi = g^{-1}\dot{g} = T_g L_{g^{-1}}\dot{g} \in \mathfrak{g}$. Thus the restriction of L in $\mathfrak{g}$ defines a **reduced Lagrangian** $l : \mathfrak{g} \rightarrow \mathbb{R}$ such that

$$l(\xi) = L(e, \xi) = L(g, \dot{g}), \quad \forall g \in G, \quad \xi = g^{-1}\dot{g} \in \mathfrak{g}$$

The following theorem shows that the Euler–Lagrange equations for L on G are equivalent to the Euler–Poincaré equations for l on $\mathfrak{g}$ with their respective variational principles.

Theorem 3.13 (Euler–Poincaré Reduction). *Let G be a Lie group and let $L : TG \rightarrow \mathbb{R}$ be a left-invariant Lagrangian. Let $g : [a, b] \rightarrow G$ be any smooth curve with variations $\delta g(a) = \delta g(b) = 0$ and set $\xi(t) = g(t)^{-1}\dot{g}(t)$. Let $l : \mathfrak{g} \rightarrow \mathbb{R}$ be the reduced Lagrangian defined by $l(\xi) = L(e, \xi)$. Then the following are equivalent*

- 1) *The curve $g(t)$ satisfies the Euler–Lagrange equations for L on G*
- 2) *The variational principle with fixed endpoints holds*

$$\delta \int_a^b L(g(t), \dot{g}(t)) dt = 0$$

- 3) *The **Euler–Poincaré equations** hold along the curve $\xi(t)$*

$$\frac{d}{dt} \frac{\delta l}{\delta \xi} = \text{ad}_\xi^* \frac{\delta l}{\delta \xi}$$

- 4) *The variational principle on $\mathfrak{g}$ holds*

$$\delta \int_a^b l(\xi(t)) dt = 0$$

using variations of the form

$$\delta \xi = \dot{\eta} + [\xi, \eta]$$

where η vanishes at the endpoints, $\eta(a) = \eta(b) = 0$.

Proof. We have already shown the equivalence (1) $\Leftrightarrow$ (2) in the Theorem 1.3. Now let us show that (2) $\Leftrightarrow$ (4). For that we assume $G \subset GL_n \mathbb{R}$.⁵ Since l uniquely determines L , we just need to prove that all variations $\delta g(t) \in T_{g(t)}G$ of $g(t)$ with $\delta g(a) = \delta g(b) = 0$ are exactly induced by variations $\delta \xi(t)$ of $\xi(t)$ of the form $\delta \xi = \dot{\eta} +$

⁵For the general proof see [BKMM96]

$[\xi, \eta]$, where $\eta(t)$ vanishes at the endpoints, $\eta(a) = \eta(b) = 0$. Let $g_\lambda \in \mathcal{P}(a, b, q_1, q_2)$ be a variation of the curve g s.t. $\delta g = \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} g_\lambda$. Then

$$\begin{aligned} \delta \xi &= \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} (g_\lambda^{-1} \dot{g}_\lambda) \\ &= \left(\frac{\partial}{\partial \lambda} g_\lambda^{-1} \right) \dot{g}_\lambda + g_\lambda^{-1} \left(\frac{\partial}{\partial \lambda} \dot{g}_\lambda \right) \Big|_{\lambda=0} \end{aligned}$$

Since $g_\lambda g_\lambda^{-1} = e$, one has $\frac{d}{d\lambda}(g_\lambda)g_\lambda^{-1} + g_\lambda \frac{d}{d\lambda}(g_\lambda^{-1}) = 0$. Therefore $\frac{d}{d\lambda}(g_\lambda^{-1}) = -g_\lambda^{-1} \frac{d}{d\lambda}(g_\lambda)g_\lambda^{-1}$. Thus

$$\begin{aligned} \delta \xi &= -g_\lambda^{-1} \frac{d}{d\lambda}(g_\lambda)g_\lambda^{-1} \dot{g}_\lambda + g_\lambda^{-1} \left(\frac{\partial}{\partial \lambda} \dot{g}_\lambda \right) \Big|_{\lambda=0} \\ &= (-g^{-1} \delta g g^{-1}) \dot{g} + g^{-1} \left(\frac{\partial}{\partial \lambda} \right)_{\lambda=0} \dot{g}_\lambda \end{aligned}$$

Now let $\eta = g^{-1} \delta g$. Thus

$$\delta \xi = \eta \xi + g^{-1} \left(\frac{\partial}{\partial \lambda} \right)_{\lambda=0} \dot{g}_\lambda$$

Furthermore

$$\begin{aligned} \dot{\eta} &= \frac{d}{dt}(g^{-1} \delta g) \\ &= (-g^{-1} \dot{g} g^{-1}) \delta g + g^{-1} \frac{d}{dt} \left(\frac{\partial}{\partial \lambda} \right)_{\lambda=0} g_\lambda \\ &= -\xi \eta + g^{-1} \left(\frac{\partial}{\partial \lambda} \right)_{\lambda=0} \dot{g}_\lambda \end{aligned}$$

Hence $\delta \xi - \dot{\eta} = [\xi, \eta]$. Conversely, given $\delta \xi$ of the form $\delta \xi = \dot{\eta} + [\xi, \eta]$ where η vanishes at the endpoints, we can define $\delta g = g\eta$. Then δg is a variation of g with fixed endpoints and it induces the given $\delta \xi$. This shows the equivalence (2) $\Leftrightarrow$ (4).

Let us now show (3) $\Leftrightarrow$ (4) by using the variational principle. We have

$$\begin{aligned} \delta \int_a^b l(\xi) dt &= \int_a^b \left\langle \frac{\delta l}{\delta \xi}, \delta \xi \right\rangle dt \\ &= \int_a^b \left\langle \frac{\delta l}{\delta \xi}, \dot{\eta} + ad_\xi \eta \right\rangle dt \\ &= \left[\left\langle \frac{d}{dt} \frac{\delta l}{\delta \xi}, \eta \right\rangle \right]_a^b - \int_a^b \left\langle \frac{d}{dt} \frac{\delta l}{\delta \xi}, \eta \right\rangle dt + \int_a^b \left\langle ad_\xi^* \frac{\delta l}{\delta \xi}, \eta \right\rangle dt \end{aligned}$$

$$\begin{aligned}
&= - \int_a^b \left\langle \frac{d}{dt} \frac{\delta l}{\delta \xi}, \eta \right\rangle dt + \int_a^b \left\langle ad_\xi^* \frac{\delta l}{\delta \xi}, \eta \right\rangle dt \\
&= \int_a^b \left\langle -\frac{d}{dt} \frac{\delta l}{\delta \xi} + ad_\xi^* \frac{\delta l}{\delta \xi}, \eta \right\rangle dt
\end{aligned}$$

for all variations η vanishing at the endpoints. Since they are arbitrary, we get

$$\delta \int_a^b l(\xi) dt = 0 \quad \Leftrightarrow \quad -\frac{d}{dt} \frac{\delta l}{\delta \xi} + ad_\xi^* \frac{\delta l}{\delta \xi} = 0$$

□

The Legendre Transformation. Since in the hyperregular case (i.e. $\mathbb{F}L$ is a diffeomorphism) the Euler-Lagrange and Hamilton equations on TQ and T^*Q are equivalent, it follows that the Lie–Poisson and Euler–Poincaré equations are also equivalent. To see this directly, we define the reduced Legendre transform

Definition 3.14. Let $l : \mathfrak{g} \rightarrow \mathbb{R}$ be the reduced Lagrangian defined by $l(\xi) = L(e, \xi)$. Assuming L is hyperregular, define the **reduced Legendre transform** as

$$\mathfrak{f}l : \mathfrak{g} \rightarrow \mathfrak{g}^*, \quad \langle \mathfrak{f}l(\xi), \eta \rangle := \left. \frac{d}{ds} \right|_{s=0} l(\xi + s\eta) = \left\langle \frac{\delta l}{\delta \xi}, \eta \right\rangle$$

for all $\xi, \eta \in \mathfrak{g}$. Thus, by definition, we have $\mathfrak{f}l(\xi) = \frac{\delta l}{\delta \xi}$. Note that, since

$$\mathbb{F}L(e, \xi) = (e, \mathfrak{f}l(\xi))$$

the map $\mathfrak{f}l$ is the restriction of $\mathbb{F}L$ to $\mathfrak{g}$. Thus, if L is hyperregular, then $\mathfrak{f}l$ is a diffeomorphism.

Definition 3.15. For a hyperregular Lagrangian $L : TQ \rightarrow \mathbb{R}$ and its reduced Lagrangian $l(\xi) = L(e, \xi)$, the **reduced Hamiltonian** $h : \mathfrak{g}^* \rightarrow \mathbb{R}$ is defined by

$$h := \mathfrak{e} \circ (\mathfrak{f}l)^{-1}$$

with the energy $\mathfrak{e}(\xi) = \langle \mathfrak{f}l(\xi), \xi \rangle - l(\xi)$. Equivalently

$$h(\mu) = \langle \mu, \xi \rangle - l(\xi), \quad \xi = (\mathfrak{f}l)^{-1}(\mu)$$

Note that

$$\frac{\delta h}{\delta \mu} = \xi + \left\langle \mu, \frac{\delta \xi}{\delta \mu} \right\rangle - \left\langle \frac{\delta l}{\delta \xi}, \frac{\delta \xi}{\delta \mu} \right\rangle = \xi + \left\langle \mu, \frac{\delta \xi}{\delta \mu} \right\rangle - \left\langle \mu, \frac{\delta \xi}{\delta \mu} \right\rangle = \xi$$

Since we can write the Euler-Poincaré equations in the new variable μ , this proves the following proposition

Proposition 3.16. *Euler-Poincaré Equations are equivalent to the Lie-Poisson equations, i.e. the following two equations are equivalent*

$$1) \quad \frac{d}{dt} \frac{\delta l}{\delta \xi} = \text{ad}_\xi^* \frac{\delta l}{\delta \xi}$$

$$2) \quad \dot{\mu} = \text{ad}_{\delta h / \delta \mu}^* \mu$$

with μ , h and l as defined above.

4 The Free Rigid Body

A rigid body is a mechanical system that is constrained to move in such a way that the distance between any two points of the body remains constant during the motion. We assume that the body rotates about its center of mass taken as the origin of the coordinate system in $\mathbb{R}^3$. We fix a reference configuration of the body $\mathcal{B}$, which is the closure of an open set in $\mathbb{R}^3$. Then the position of each particle is given by $X = (X_1, X_2, X_3) \in \mathcal{B}$ in the reference configuration. Furthermore, assume that there are no external forces or torques acting on the body, meaning that it is a free rigid body. A configuration of the body is then given by a map $\mathcal{B} \rightarrow \mathbb{R}^3$ that is C^1 and invertible on its image. Moreover, the configuration space of this system is the Lie group $SO(3)$, since it describes all possible rotations about the origin in $\mathbb{R}^3$. Thus, the velocity phase space is the tangent bundle $TSO(3)$, and the momentum phase space is the cotangent bundle $T^*SO(3)$.

The motion of the free rigid body can be described by time dependent family of configurations given by a curve $x(X, t)$ in $\mathbb{R}^3$, which is related to the reference configuration X by a rotation matrix $\mathbf{R}(t) \in SO(3)$, i.e.

$$x(X, t) = \mathbf{R}(t)X$$

The velocity $V(X, t)$ in the reference configuration is then obtained by differentiating the above relation with respect to time, which gives

$$V(X, t) = \dot{x}(X, t) = \dot{\mathbf{R}}(t)X$$

Then the Lagrangian $L : TSO(3) \rightarrow \mathbb{R}$ is given by the kinetic energy of the system. But, if we had a system of N particles with masses m_i and positions x_i , then the

kinetic energy would have been given by $\sum_{i=1}^N m_i \|\dot{x}_i\|^2/2$. Nevertheless, in the case of a rigid body, we have infinitely many particles, so the mass of each particle at the position X is given by a mass density function $\rho(X)$ and the mass of the body is given by the integral of the mass density function over the reference configuration, that is $\int_{\mathcal{B}} \rho(X) d^3X$. Thus, the Lagrangian $L : TSO(3) \rightarrow \mathbb{R}$ is given by

$$\begin{aligned}
 L(\mathbf{R}, \dot{\mathbf{R}}) &= \frac{1}{2} \int_{\mathcal{B}} \rho(X) \|\dot{x}\|^2 d^3X \\
 &= \frac{1}{2} \int_{\mathcal{B}} \rho(X) \|\dot{\mathbf{R}}X\|^2 d^3X \\
 &= \frac{1}{2} \int_{\mathcal{B}} \rho(X) \text{tr} \left((\dot{\mathbf{R}}X)(\dot{\mathbf{R}}X)^T \right) d^3X \\
 &= \frac{1}{2} \text{tr} \left(\int_{\mathcal{B}} \rho(X) (\dot{\mathbf{R}}X)(X^T \dot{\mathbf{R}}^T) d^3X \right) \\
 &= \frac{1}{2} \text{tr} \left(\dot{\mathbf{R}} \left(\int_{\mathcal{B}} \rho(X) X X^T d^3X \right) \dot{\mathbf{R}}^T \right) \\
 &= \frac{1}{2} \text{tr}(\dot{\mathbf{R}} \mathbb{I} \dot{\mathbf{R}}^T)
 \end{aligned}$$

where the inertia tensor $\mathbb{I}$ of the body, with respect to the origin, is given by

$$\mathbb{I} = \int_{\mathcal{B}} \rho(X) X X^T d^3X$$

which is symmetric and constant. Now, let us show that the Lagrangian L is left invariant. For that, let $\mathbf{Q} \in SO(3)$, then

$$\begin{aligned}
 L(\mathbf{Q}\mathbf{R}, \mathbf{Q}\dot{\mathbf{R}}) &= \frac{1}{2} \text{tr}((\mathbf{Q}\dot{\mathbf{R}})(\mathbf{Q}\dot{\mathbf{R}})^T) \\
 &= \frac{1}{2} \text{tr}(\mathbf{Q}\dot{\mathbf{R}}\dot{\mathbf{R}}^T\mathbf{Q}^T) \\
 &= \frac{1}{2} \text{tr}(\dot{\mathbf{R}}\dot{\mathbf{R}}^T) \\
 &= L(\mathbf{R}, \dot{\mathbf{R}})
 \end{aligned}$$

Now, by identifying $T_{\mathbf{R}}SO(3)$ with the Lie algebra $\mathfrak{so}(3)$ via left or right translation, the differentials of left and right multiplication by $\mathbf{R}^{-1}$ gives

$$TL_{\mathbf{R}^{-1}}(\mathbf{R}, \dot{\mathbf{R}}) = (I, \mathbf{R}^{-1}\dot{\mathbf{R}}), \quad TR_{\mathbf{R}^{-1}}(\mathbf{R}, \dot{\mathbf{R}}) = (I, \dot{\mathbf{R}}\mathbf{R}^{-1})$$

for any $(\mathbf{R}, \dot{\mathbf{R}}) \in TSO(3)$. Therefore, $\mathbf{R}^{-1}\dot{\mathbf{R}}, \dot{\mathbf{R}}\mathbf{R}^{-1} \in \mathfrak{so}(3)$, thus they both are skew symmetric. By using the hat map $\hat{\cdot} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ defined in the Example 0.15

define the spatial angular velocity $\omega \in \mathbb{R}^3$ by

$$\hat{\omega} = \dot{\mathbf{R}}\mathbf{R}^{-1}$$

Note that this implies

$$\dot{x} = \hat{\omega}x = \omega \times x$$

The body angular velocity $\Omega \in \mathbb{R}^3$ is then given by

$$\Omega = \mathbf{R}^{-1}\omega$$

Thus, we get

$$\hat{\Omega}X = \Omega \times X = \mathbf{R}^{-1}\omega \times \mathbf{R}^{-1}x = \mathbf{R}^{-1}(\omega \times x) = \mathbf{R}^{-1}\dot{\mathbf{R}}\mathbf{R}^{-1}x = \mathbf{R}^{-1}\dot{\mathbf{R}}X$$

Hence, $\hat{\Omega} = \mathbf{R}^{-1}\dot{\mathbf{R}}$, thus the reduced Lagrangian $l : \mathfrak{so}(3) \rightarrow \mathbb{R}$ is given by

$$l(\hat{\Omega}) = L(e, \hat{\Omega}) = \frac{1}{2}\text{tr}(\hat{\Omega}\mathbb{I}\hat{\Omega}^T) = \frac{1}{2}\text{tr}((\mathbf{R}^{-1}\dot{\mathbf{R}})\mathbb{I}(\mathbf{R}^{-1}\dot{\mathbf{R}})^T) = \frac{1}{2}\text{tr}(\dot{\mathbf{R}}\mathbb{I}\dot{\mathbf{R}}^T) = L(\mathbf{R}, \dot{\mathbf{R}})$$

Furthermore, notice that

$$\begin{aligned} l(\hat{\Omega}) &= L(\mathbf{R}, \dot{\mathbf{R}}) = \frac{1}{2} \int_{\mathcal{B}} \rho(X) \|\dot{x}\|^2 d^3X \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(X) \|\omega \times x\|^2 d^3X \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(X) \|\mathbf{R}^{-1}(\omega \times x)\|^2 d^3X \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(X) \|\Omega \times X\|^2 d^3X \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(X) \|X \times \Omega\|^2 d^3X \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(X) \|\hat{X}\Omega\|^2 d^3X \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(X) \Omega^T \hat{X}^T \hat{X} \Omega d^3X \\ &= \frac{1}{2} \Omega^T \left(\int_{\mathcal{B}} \rho(X) \hat{X}^T \hat{X} d^3X \right) \Omega \end{aligned}$$

Consider

$$\hat{X}^T \hat{X} = -\hat{X}^2 = -(X \times (X \times \cdot)) = \|X\|^2 I - XX^T$$

where we used that $\hat{X}^T = -\hat{X}$ and the vector triple product identity $X \times (X \times y) =$

$X(X \cdot y) - \|X\|^2 y$. Thus

$$l(\hat{\Omega}) = \frac{1}{2} \Omega^T \int_{\mathcal{B}} \rho(X) \hat{X}^T \hat{X} d^3 X \Omega = \frac{1}{2} \Omega^T \int_{\mathcal{B}} \rho(X) (\|X\|^2 I - X X^T) d^3 X \Omega = \frac{1}{2} \Omega^T \mathbb{J} \Omega$$

where the moment of inertia tensor $\mathbb{J}$ is given by ⁶

$$\mathbb{J} = \int_{\mathcal{B}} \rho(X) (\|X\|^2 I - X X^T) d^3 X$$

Using the identification ⁷ $\mathfrak{so}(3) \simeq \mathbb{R}^3 \simeq \mathfrak{so}(3)^*$, we regard the reduced Lagrangian l as a function of Ω . Then, for any variation $\delta\Omega \in \mathbb{R}^3$, the directional derivative is given by

$$\begin{aligned} \left\langle \frac{\delta l}{\delta \Omega}, \delta \Omega \right\rangle &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} l(\Omega + \varepsilon \delta \Omega) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \frac{1}{2} (\Omega + \varepsilon \delta \Omega)^T \mathbb{J} (\Omega + \varepsilon \delta \Omega) \\ &= \frac{1}{2} (\delta \Omega^T \mathbb{J} \Omega + \Omega^T \mathbb{J} \delta \Omega) \\ &= \frac{1}{2} \Omega^T (\mathbb{J} + \mathbb{J}^T) \delta \Omega \\ &= \Omega^T \mathbb{J}^T \delta \Omega \quad \text{since } \mathbb{J} \text{ is symmetric} \\ &= \langle \mathbb{J} \Omega, \delta \Omega \rangle \end{aligned}$$

Since this holds for all $\delta\Omega \in \mathbb{R}^3$, we get

$$\frac{\delta l}{\delta \Omega} = \mathbb{J} \Omega$$

Hence, on the left hand side of the Euler-Poincaré equations we get $\frac{d}{dt} \frac{\delta l}{\delta \Omega} = \mathbb{J} \dot{\Omega}$ and on the right hand side

$$\begin{aligned} \langle \text{ad}_\Omega^* \left(\frac{\delta l}{\delta \Omega} \right), \eta \rangle &= \left\langle \frac{\delta l}{\delta \Omega}, \text{ad}_\Omega(\eta) \right\rangle \\ &= \langle \mathbb{J} \Omega, [\Omega, \eta] \rangle \\ &= \langle \mathbb{J} \Omega, \Omega \times \eta \rangle \\ &= (\mathbb{J} \Omega) \cdot (\Omega \times \eta) \\ &= ((\mathbb{J} \Omega) \times \Omega) \cdot \eta \\ &= \langle (\mathbb{J} \Omega) \times \Omega, \eta \rangle \end{aligned}$$

for any $\eta \in \mathbb{R}^3$. Hence, $\text{ad}_\Omega^* \left(\frac{\delta l}{\delta \Omega} \right) = (\mathbb{J} \Omega) \times \Omega$. Thus, the Euler-Poincaré equation for

⁶The inertia's are related by the equality $\text{tr}(\hat{\xi} \mathbb{J} \hat{\eta}^T) = \xi^T \mathbb{J} \eta$ for any $\xi, \eta \in \mathfrak{so}(3)$

⁷See Example 0.15 and 0.16

the free rigid body is given by

$$\mathbb{J}\dot{\Omega} = \mathbb{J}\Omega \times \Omega$$

which is exactly the Euler's equation for the rigid body. The reduced Legendre transform $\mathfrak{fl} : \mathfrak{so}(3) \rightarrow \mathfrak{so}^*(3)$, $\Omega \mapsto \Pi$ is then given under the identification $\mathfrak{so}(3) \simeq \mathbb{R}^3 \simeq \mathfrak{so}(3)^*$ by

$$\Pi = \mathfrak{fl}(\Omega) = \frac{\delta l}{\delta \Omega} = \mathbb{J}\Omega$$

with inverse $\mathfrak{fl}^{-1}(\Pi) = \mathbb{J}^{-1}\Pi$. The reduced energy function $\mathfrak{e} : \mathfrak{so}(3) \rightarrow \mathbb{R}$ is

$$\mathfrak{e}(\Pi) = \langle \Pi, \Omega \rangle - \frac{1}{2} \langle \Omega, \mathbb{J}\Omega \rangle = \langle \Pi, \Omega \rangle - \frac{1}{2} \langle \mathbb{J}\Omega, \Omega \rangle = \frac{1}{2} \langle \Pi, \Omega \rangle$$

Thus, the reduced Hamiltonian is

$$h : \mathfrak{so}(3)^* \rightarrow \mathbb{R}, \quad h(\Pi) = \mathfrak{e} \circ \mathfrak{fl}^{-1}(\Pi) = \frac{1}{2} \langle \mathbb{J}^{-1}\Pi, \Pi \rangle$$

and the Lie–Poisson equations are

$$\dot{\Pi} = \text{ad}_{\delta h / \delta \Pi}^* \Pi = \Pi \times \mathbb{J}^{-1}\Pi.$$

Appendix: Lie Groups

Lie groups are the mathematical concept that describes the continuously varying symmetries of a geometrical system. In this section we give the definition of Lie groups, Lie algebras and exponential map and show how they are related to each other. Then we introduce the Lie group actions. Finally, we illustrate their relevance through the example of the Lie group $SO(3)$, particularly useful for the study of rigid body dynamics.

Lie Groups and Lie Algebras

Definition 0.1. A Lie group is a smooth manifold G with a group structure such that the group product and the inverse map

$$G \times G \rightarrow G, (g, h) \mapsto gh \quad \text{and} \quad G \rightarrow G, g \mapsto g^{-1}$$

are smooth maps.

Definition 0.2. A Lie group homomorphism is a group homomorphism that is smooth.

Example 0.3. Consider the general linear group $GL_n\mathbb{C}$, which is the set of all invertible $n \times n$ matrices with complex entries. The group operation is given by matrix multiplication and the inverse map is given by taking the matrix inverse. Since both operations are smooth maps, $GL_n\mathbb{C}$ is a Lie group.

Definition 0.4. A matrix Lie group is a closed subgroup of $GL_n\mathbb{C}$.

Example 0.5. Some examples of matrix Lie groups are

$$\begin{aligned} O(n) &= \{A \in GL_n\mathbb{R} : A^T A = I\} \\ SO(n) &= \{A \in O(n) : \det(A) = 1\} \\ U(n) &= \{A \in GL_n\mathbb{C} : A^* A = I\} \\ SU(n) &= \{A \in U(n) : \det(A) = 1\} \\ SL(n) &= \{A \in GL_n\mathbb{R} : \det(A) = 1\} \\ O(n, k) &= \{A \in GL_{n+k}\mathbb{R} : A^T I_{n,k} A = I_{n,k}\} \\ SO(n, k) &= \{A \in O(n, k) : \det(A) = 1\} \end{aligned}$$

where I is the identity matrix and $I_{n,k}$ is the diagonal matrix with n entries of 1 and k entries of -1 on the diagonal.

Later we are going to look more closely into $SO(n)$ since it describes rotations in n -dimensional space. For $n = 3$, it contains all possible rotations about the origin in $\mathbb{R}^3$, thus it is the configuration space of the rigid body dynamics. Now let us give the definition of a Lie algebra.

Definition 0.6. A Lie algebra is a vector space $\mathfrak{g}$ equipped with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ called the Lie bracket satisfying

$$1) [x, y] = -[y, x] \quad (\text{Skew-Symmetry})$$

$$2) [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \quad (\text{Jacobi Identity})$$

for all $x, y, z \in \mathfrak{g}$.

Having defined a Lie algebra, now we would like to find a finite dimensional Lie algebra $\mathfrak{g}$ associated to any Lie group G . For that reason define now the following mappings

Definition 0.7. Let G be a Lie group. For any $g \in G$ define the *left* and *right translations*

$$L_g : G \rightarrow G, \quad h \mapsto gh, \quad R_g : G \rightarrow G, \quad h \mapsto hg$$

It is easy to show that these mappings have the following properties

Property 1. Let $g, h \in G$

$$1) L_g \circ L_h = L_{gh} \text{ and } R_g \circ R_h = R_{hg}$$

$$2) (L_g)^{-1} = L_{g^{-1}} \text{ and } (R_g)^{-1} = R_{g^{-1}}$$

$$3) L_g \circ R_h = R_h \circ L_g$$

It follows from 2 that L_g and R_g are diffeomorphisms on G . Now we can use the left translation L_g to define left-invariant vector fields on G .⁸

Definition 0.8. A vector field $X \in \mathfrak{X}(G)$ is called left invariant if for every $g \in G$ we have $L_g^*(X) = X$, i.e. $(T_h L_g)X_h = X_{gh}$ for every $h \in G$

⁸Analogously, one can define right-invariant vector fields by pulling back via right translation instead of left translation and proceed in the same way to define the Lie algebra of a Lie group. Since the results obtained using right-invariant vector fields are analogous to those for left-invariant vector fields, we fix a convention and work only with the left-invariant ones.

Definition 0.9. A Lie subalgebra of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ closed under the bracket, i.e. $X, Y \in \mathfrak{g} \Rightarrow [X, Y] \in \mathfrak{g}$. Thus any Lie subalgebra is itself a Lie algebra with the same brackets.

Now we can use the Lie bracket of vector fields to define a Lie algebra structure on the set of left-invariant vector fields. But first we need to show that the set of left-invariant vector fields is closed under the Lie bracket so that it becomes a Lie subalgebra of the Lie algebra of all vector fields on G .

Corollary 0.10. *The set of left-invariant vector fields $\mathfrak{X}_L(G) = \{X \in \mathfrak{X}(G) : X \text{ is left invariant}\}$ is closed under the Lie bracket $[\cdot, \cdot]$ and thus is a Lie subalgebra of $\mathfrak{X}(G)$.*

Proof. Since L_g is a diffeomorphism, we have

$$L_g^*[X, Y] \underset{L_g \text{ is diffeomorphism}}{=} [L_g^*X, L_g^*Y] \underset{X, Y \text{ left invariant}}{=} [X, Y]$$

Thus $[X, Y]$ is left invariant, which implies that $\mathfrak{X}_L(G)$ is closed under the Lie bracket and hence is a Lie subalgebra of $\mathfrak{X}(G)$. $\square$

Furthermore, notice that every vector field $X \in \mathfrak{X}_L(G)$ is uniquely determined through the vector $X_e \in T_eG$, since for any vector $\xi \in T_eG$ we can define a left-invariant vector field $X^\xi \in \mathfrak{X}_L(G)$ by

$$X_g^\xi := T_eL_g(\xi), \quad \forall g \in G$$

To show that X^ξ is left invariant, let $g, h \in G$. Then we have

$$X_{gh}^\xi = T_eL_{gh}(\xi) = T_e(L_g \circ L_h)(\xi) = T_hL_g \circ T_eL_h(\xi) = T_hL_g(X_h^\xi)$$

showing that X^ξ is left invariant. Furthermore, define the maps

$$\begin{aligned} \phi_1 : \mathfrak{X}_L(G) &\rightarrow T_eG, \quad X \rightarrow X_e \\ \phi_2 : T_eG &\rightarrow \mathfrak{X}_L(G), \quad \xi \rightarrow X^\xi \end{aligned}$$

Notice that $\phi_1\phi_2(\xi) = \phi_1(X^\xi) = X_e^\xi = \xi$. Thus $\phi_1\phi_2 = id_{T_eG}$ and similarly $\phi_2\phi_1 = id_{\mathfrak{X}_L(G)}$. Thus we get an isomorphism between vector spaces $\mathfrak{X}_L(G)$ and T_eG . This now enables us to define the Lie algebra of G by passing to the tangent space at the identity T_eG . But we need a meaningful bracket on it. So we equip the tangent

space $T_e G$ with a Lie bracket, which we obtain by using the isomorphism introduced above in the following way.

Definition 0.11. The tangent vector space $T_e G$ of a Lie group G with the Lie bracket defined by

$$[\xi, \eta] := [X^\xi, X^\eta]_e, \quad \forall \xi, \eta \in T_e G$$

is called the Lie algebra of G and is denoted by $\mathfrak{g} = T_e G$.

Remark. By construction, the Lie bracket of two left invariant vector fields X^ξ and X^η is again a left invariant vector field and is given by ⁹

$$[X^\xi, X^\eta] = X^{[\xi, \eta]}, \quad \forall \xi, \eta \in T_e G$$

Example 0.12. Consider the Lie group $GL_n \mathbb{R}$. The tangent space at the identity $T_I GL_n \mathbb{R}$ is then given by the set of all $n \times n$ matrices $M_n \mathbb{R}$ and the Lie bracket on $M_n \mathbb{R}$ is given by the commutator

$$[A, B] = AB - BA, \quad A, B \in M_n \mathbb{R}$$

Thus $\mathfrak{g} := \mathfrak{gl}_n \mathbb{R} = M_n \mathbb{R}$ with the commutator as Lie bracket.

Example 0.13. Consider now the Lie group $SO(3)$. Then the tangent space at the identity $T_I SO(3)$ is given by the set of all 3×3 skew-symmetric matrices, i.e. $T_I SO(3) = \mathfrak{so}(3) = \{A \in M_3 \mathbb{R} : A^T = -A\}$. To show this, consider the curve $A : (-\varepsilon, \varepsilon) \rightarrow SO(3)$ with $A(0) = I$. Then we have

$$\left. \frac{d}{dt} (A(t)^T A(t)) \right|_{t=0} = \dot{A}(0)^T I + I^T \dot{A}(0) = \dot{A}(0)^T + \dot{A}(0) = 0$$

Thus $\dot{A}(0)$ is skew-symmetric, implying $T_I SO(3) \subset \{A \in M_3 \mathbb{R} : A^T = -A\}$.

Conversely, take any matrix $A \in M_3 \mathbb{R}$ with $A^T = -A$ and consider the curve $A(t) = \exp(tA)$, where $\exp$ is the matrix exponential. Then we have $A(0) = I$, $A'(0) = A$ and

$$(A(t)^T A(t)) = \exp(tA)^T \exp(tA) = \exp(-tA) \exp(tA) = I$$

Furthermore, $A^T = -A$ implies $\text{tr}(A) = 0$. Thus, using the property $\det(\exp A) =$

⁹In the proof of Lie Poisson reduction 3.11, we denote the left invariant vector field associated to $\xi \in T_e G$ by ξ_L instead of X^ξ and use the fact that $[\xi_L, \eta_L] = [\xi, \eta]_L$ for all $\xi, \eta \in T_e G$. This is exactly the same as the relation stated in the remark, but with different notations.

$\exp(\text{tr}(A))$, we get

$$\det(A(t)) = \det(\exp(tA)) = \exp(\text{tr}(tA)) = \exp(0) = 1$$

Thus $A(t) \in SO(3)$ for all t , implying $A'(0) = A \in T_I SO(3)$. Hence $\{A \in M_3\mathbb{R} : A^T = -A\} \subset T_I SO(3)$.

Definition 0.14. Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ be Lie algebras. A Lie algebra homomorphism is a linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ such that $\phi([g_1, g_2]_{\mathfrak{g}}) = [\phi(g_1), \phi(g_2)]_{\mathfrak{h}}$ for all $g_1, g_2 \in \mathfrak{g}$. A Lie algebra isomorphism is a bijective Lie algebra homomorphism.

Example 0.15. The hat map $\hat{\cdot} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ defined by

$$\hat{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \quad \text{for } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$

is a Lie algebra isomorphism between $(\mathbb{R}^3, \times)$ and $(\mathfrak{so}(3), [\cdot, \cdot])$, where $\times$ is the cross product in $\mathbb{R}^3$ and $[\cdot, \cdot]$ is the matrix commutator defined by $[A, B] = AB - BA$ for $A, B \in \mathfrak{so}(3)$. First it is clear that the map is linear and bijective, since any skew symmetric matrix is of that given form¹⁰. Let us show that this is also a Lie algebra homomorphism. First note that

$$\hat{x}y = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -x_3y_2 + x_2y_3 \\ x_3y_1 - x_1y_3 \\ -x_2y_1 + x_1y_2 \end{bmatrix} = x \times y$$

for any $x, y \in \mathbb{R}^3$. Thus, for any $x, y, z \in \mathbb{R}^3$ we have

$$[\hat{x}, \hat{y}]z = \hat{x}(\hat{y}z) - \hat{y}(\hat{x}z) = x \times (y \times z) - y \times (x \times z) \underset{\text{Jacobi}}{=} (x \times y) \times z = \widehat{x \times y} z$$

That implies $[\hat{x}, \hat{y}] = \widehat{x \times y}$. Hence, the hat map is a Lie algebra isomorphism.

Example 0.16. Now define the breve map $\breve{\cdot} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)^*$ by

$$\langle \breve{\Pi}, \hat{x} \rangle = \Pi^T \cdot x = \Pi_1 x_1 + \Pi_2 x_2 + \Pi_3 x_3, \quad \text{for } \Pi = \begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{bmatrix} \in \mathbb{R}^3 \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$

¹⁰In Example (0.13) it is shown that $\mathfrak{so}(3) = \{A \in M_3\mathbb{R} : A^T = -A\}$

One can analogously show that the breve map is also a Lie algebra isomorphism. Furthermore, the hat and breve maps are related by the following identity

$$\begin{aligned}
 \text{tr}(\hat{\Pi}\hat{x}^T) &= \text{tr}\left(\begin{bmatrix} 0 & -\Pi_3 & \Pi_2 \\ \Pi_3 & 0 & -\Pi_1 \\ -\Pi_2 & \Pi_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}\right) \\
 &= \text{tr}\left(\begin{bmatrix} \Pi_3x_3 + \Pi_2x_2 & * & * \\ * & \Pi_3x_3 + \Pi_1x_1 & * \\ * & * & \Pi_2x_2 + \Pi_1x_1 \end{bmatrix}\right) \\
 &= 2(\Pi_1x_1 + \Pi_2x_2 + \Pi_3x_3) = 2\langle \breve{\Pi}, \hat{x} \rangle
 \end{aligned}$$

Thus, we have $\langle \breve{\Pi}, \hat{x} \rangle = \frac{1}{2}\text{tr}(\hat{\Pi}\hat{x}^T)$ for any $\Pi, x \in \mathbb{R}^3$. Hence, $\breve{\Pi}$ corresponds to $\hat{\Pi}$ under the identification $\mathfrak{so}(3)^* \simeq \mathfrak{so}(3)$ given by the trace pairing.

The Exponential Map

Definition 0.17. Let G be a Lie group. A smooth function $\phi : \mathbb{R} \rightarrow G$ is called a one-parameter subgroup of G if

- 1) $\phi(0) = e$
- 2) $\phi(s+t) = \phi(s)\phi(t)$

Proposition 0.18. Let $\gamma_\xi : \mathbb{R} \rightarrow G$ be the integral curve of the left-invariant vector field $X^\xi \in \mathfrak{X}_L(G)$ generated by $\xi \in \mathfrak{g}$ satisfying

$$\gamma_\xi(0) = e, \quad \frac{d}{dt}\gamma_\xi(t) = X^\xi(\gamma_\xi(t))$$

Then γ_ξ is a one-parameter subgroup generated by X^ξ .

Proof. Let $s, t \in \mathbb{R}$ and differentiate the curve $\sigma(t) = \gamma_\xi(s)\gamma_\xi(t)$

$$\begin{aligned}
 \frac{d}{dt}\sigma(t) &= \frac{d}{dt}(\gamma_\xi(s)\gamma_\xi(t)) \\
 &= T_{\gamma_\xi(t)}L_{\gamma_\xi(s)}\left(\frac{d}{dt}\gamma_\xi(t)\right) \\
 &= T_{\gamma_\xi(t)}L_{\gamma_\xi(s)}(X^\xi(\gamma_\xi(t))) && \gamma_\xi \text{ integral curve of } X^\xi \\
 &= X^\xi(\gamma_\xi(s)\gamma_\xi(t)) && X^\xi \text{ left invariant} \\
 &= X^\xi(\sigma(t))
 \end{aligned}$$

now we fix s and differentiate the curve $\Sigma(t) = \gamma_\xi(s + t)$

$$\frac{d}{dt}\Sigma(t) = \frac{d}{dt}\gamma_\xi(s + t) = X^\xi(\gamma_\xi(s + t)) = X^\xi(\Sigma(t))$$

Let $t = 0$, then

$$\sigma(0) = \gamma_\xi(0)\gamma_\xi(s) = e\gamma_\xi(s) = \gamma_\xi(s) = \gamma_\xi(0 + s) = \Sigma(0)$$

Thus, both $\sigma(t)$ and $\Sigma(t)$ are integral curves of X^ξ with the same initial condition $\gamma_\xi(s)$. Hence, by uniqueness of integral curves we get $\gamma_\xi(s + t) = \gamma_\xi(s)\gamma_\xi(t)$. $\square$

Definition 0.19. The exponential map $\exp : \mathfrak{g} \rightarrow G$ is defined by

$$\exp(\xi) = \gamma_\xi(1)$$

Example 0.20. As we have seen above, for any matrix Lie group G the exponential map is given by the matrix exponential

$$\gamma_\xi(1) = \exp(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{k!}$$

for any $\xi \in \mathfrak{g}$.

Proposition 0.21. *The following statements hold*

- 1) *The exponential map maps the line $\{t\xi : t \in \mathbb{R}\} \subset \mathfrak{g}$ onto the one-parameter subgroup $\{\gamma_\xi(t) : t \in \mathbb{R}\} \subset G$.*
- 2) *For any one-parameter subgroup $\gamma : \mathbb{R} \rightarrow G$ there exists a unique $\xi \in \mathfrak{g}$ such that $\gamma(t) = \{\exp(t\xi), t \in \mathbb{R}\}$.*
- 3) *The exponential map $\exp : \mathfrak{g} \rightarrow G$ is a local diffeomorphism (étale) from a neighborhood of $0 \in \mathfrak{g}$ onto a neighborhood of $e \in G$.*

Proof. 1. Let $t \in \mathbb{R}$, then by definition of the exponential map we have $\exp(t\xi) = \gamma_{t\xi}(1)$. Now consider the curve $\sigma(s) = \gamma_{t\xi}(s)$, then for a fixed t we have

$$\begin{aligned} \frac{d}{ds}\sigma(s) &= \frac{d}{ds}\gamma_{t\xi}(s) \\ &= X^{t\xi}(\gamma_{t\xi}(s)) && \gamma_{t\xi} \text{ integral curve of } X^{t\xi} \\ &= T_e L_{\gamma_{t\xi}(s)}(t\xi) && X^{t\xi} \text{ left invariant} \\ &= tT_e L_{\gamma_{t\xi}(s)}(\xi) && T_e L_g \text{ linear} \end{aligned}$$

$$= tX^\xi(\sigma(s)) \quad X^{t\xi} \text{ left invariant}$$

and $\sigma(0) = \gamma_\xi(0) = e$. Thus $\sigma(s)$ is the integral curve of $X^{t\xi}$ with initial condition e . Now consider the curve $\Sigma(s) = \gamma_\xi(ts)$, then

$$\begin{aligned} \frac{d}{ds}\Sigma(s) &= \frac{d}{ds}(\gamma_\xi(ts)) \\ &= t \frac{d}{d(ts)}\gamma_\xi(ts) && \text{Chain rule} \\ &= tX^\xi(\Sigma(s)) && \gamma_\xi \text{ integral curve of } X^\xi \end{aligned}$$

and $\Sigma(0) = \gamma_\xi(0) = e$. Thus $\Sigma(s)$ is also an integral curve of $X^{t\xi}$ with initial condition e . Hence by uniqueness of integral curves we get $\sigma(s) = \Sigma(s)$. Thus setting $s = 1$ we get $\exp(t\xi) = \gamma_\xi(t)$.

2. Let $\gamma : \mathbb{R} \rightarrow G$ be a one-parameter subgroup, then $\gamma(0) = e$ and $\gamma(s+t) = \gamma(s)\gamma(t)$ for all $s, t \in \mathbb{R}$. Thus γ is an integral curve of X^ξ for $\xi = \dot{\gamma}(0)$, since

$$\frac{d}{dt}\gamma(t) = \frac{d}{ds}\Big|_{s=0} \gamma(t+s) = \frac{d}{ds}\Big|_{s=0} \gamma(t)\gamma(s) = T_e L_{\gamma(t)}(\dot{\gamma}(0)) = X^\xi(\gamma(t))$$

and $\gamma(0) = e$. Hence, $\gamma(t) = \gamma_\xi(t)$ for all $t \in \mathbb{R}$ and thus $\gamma(t) = \exp(t\xi)$ by the first part.

3. We have $T_0 \exp(\xi) = X^\xi(e) = \xi$ for any $\xi \in \mathfrak{g}$. Hence, $T_0 \exp = id_{\mathfrak{g}}$. That implies that $\exp : \mathfrak{g} \rightarrow G$ is a local diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ onto a neighborhood of $e \in G$ by the inverse function theorem.

□

Lie Group Action

Definition 0.22. Let M be a manifold and G be a Lie Group. Then a (left) action of a Lie group $G \curvearrowright M$ is a smooth map $\phi : G \times M \rightarrow M$ such that

- 1) $\phi(e, x) = x$
- 2) $\phi(g, \phi(h, x)) = \phi(gh, x)$

for any $g, h \in G$ and $x \in M$.

Remark. For every $g \in G$ let $\phi_g : M \rightarrow M$, $x \mapsto \phi(g, x)$, then (1) and (2) above are equivalent to the following two properties

$$1') \quad \phi_e = id_M$$

$$2') \quad \phi_g \circ \phi_h = \phi_{gh}$$

Thus a (left) action of a Lie group $G \curvearrowright M$ is a Lie group homomorphism

$$\phi : G \rightarrow \text{Diff}(M), \quad g \mapsto \phi_g$$

where $\text{Diff}(M)$ denotes the group of diffeomorphisms of M .

Example 0.23. As an example, consider $SO(3)$ acting on $\mathbb{R}^3$ by matrix multiplication

$$\phi : SO(3) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad (R, x) \mapsto Rx$$

It is easy to see that it is a Lie group action, since for any $R_1, R_2 \in SO(3)$ and $x \in \mathbb{R}^3$ we have

$$1) \quad \phi(I, x) = Ix = x$$

$$2) \quad \phi(R_1, \phi(R_2, x)) = R_1(R_2x) = (R_1R_2)x = \phi(R_1R_2, x)$$

See that, it leaves the two-sphere S^2 invariant, since for any $R \in SO(3)$ and $x \in S^2$ we have

$$\|\phi(R, x)\|^2 = x^T R^T R x = x^T I x = x^T x = \|x\|^2 = 1$$

Thus the restricted action $\phi : SO(3) \times S^2 \rightarrow S^2$ is also a Lie group action.

Definition 0.24. Let $\phi : G \rightarrow \text{Diff}(M)$, $g \mapsto \phi_g$ be any Lie group action.

The **orbit** of G through $x \in M$ is defined by $O_x = \{\phi_g(x) : g \in G\} \subset M$

The **stabilizer** (or **isotropy**) of $x \in M$ is defined by $G_x := \{g \in G : \phi_g(x) = x\} \subset G$

Remark. Notice that $G_x = \phi_g^{-1}(x)$ is a closed subgroup of G , since ϕ_g is a continuous map and $\{x\}$ is closed in M . Thus G_x is a regular Lie subgroup.¹¹

Let us define a orbit equivalence relation by

$$x \sim y \iff \exists g \in G \text{ s.t. } \phi_g(x) = y, \quad \text{i.e. } x \text{ and } y \text{ are in the same orbit}$$

¹¹A regular Lie subgroup $H \subset G$ is a subgroup of a Lie group G that is also a submanifold of G such that the inclusion map $i : H \rightarrow G$ is an injective immersion. Moreover, closedness of H is equivalent to being a regular Lie subgroup, see [MR99, Theorem 9.1.14.].

Then we can define the **orbit space** as the space of orbits $M/\sim = M/G$ with the point-orbit projection

$$\pi : M \rightarrow M/G, \quad x \mapsto O_x$$

We can naturally equip the orbit space M/G with the structure of a topological space by endowing it with the quotient topology induced by the point-orbit projection map π , i.e. a subset $U \subset M/G$ is open iff $\pi^{-1}(U)$ is open in M . Nevertheless, in that topology M/G is not necessarily a smooth manifold. For instance, for a Lie group action $\phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $(g, x) \mapsto \phi_g(x) = e^g x$ one obtain three distinct orbits, namely

$$O_0 = \{0\}, \quad O_{x>0} = (0, \infty), \quad O_{x<0} = (-\infty, 0)$$

Thus $\mathbb{R}/\mathbb{R} = \{[0], [1], [-1]\}$. So the orbit $[0]$ has no open neighborhood other than the whole quotient space $\mathbb{R}/\mathbb{R}$ and the orbits $[1]$ and $[-1]$ can not be separated by disjoint open sets in the quotient. Thus $\mathbb{R}/\mathbb{R}$ is not Hausdorff. To overcome this issue and to get a smooth manifold structure on the orbit space, we will see that it is sufficient to assume that the action ϕ is proper and free. Now, let us define these properties.

Definition 0.25. An action ϕ is said to be

transitive, if $\forall x, y \in M$, $\exists g \in G$ s.t. $gx = y$, equivalently, there is just one orbit.

free, if $\phi_g(x) = x$ implies $g = e$, equivalently, all stabilizers are trivial $G_x = \{e\}$, $\forall x \in M$.

faithful if $\phi_g = id_M$ implies $g = e$, that is ϕ is injective.

proper, if the map $\tilde{\phi} : G \times M \rightarrow M \times M$ defined by

$$\tilde{\phi}(g, x) = (x, \phi(g, x))$$

is proper.¹²

Remark. Notice that every free action is faithful.

Proposition 0.26. *If $\phi : G \times M \rightarrow M$ is a proper and free Lie group action, then M/G is a smooth manifold and the point-orbit projection $\pi : M \rightarrow M/G$ is a smooth submersion.*

¹²A map is proper if the preimage of any compact set is compact.

Proof. See [[MR99], Proposition 9.3.2]. $\square$

Definition 0.27. For each $\xi \in \mathfrak{g}$ the **infinitesimal generator** is defined by left invariant vector field $\xi^M \in \mathfrak{X}(G)$ given by

$$\xi_x^M = \left. \frac{d}{dt} \right|_{t=0} \phi_{\exp(t\xi)}(x), \quad \forall x \in M$$

where $\phi : G \times M \rightarrow M$ is a Lie group action

Example 0.28. Let $\phi : G \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(g, x) \mapsto gx$ be a Lie group action for a matrix Lie group G , then the infinitesimal generator is given by

$$\xi_x^M = \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi)x = \xi x, \quad \forall x \in \mathbb{R}^n$$

For any $g \in G$, define the **inner automorphism** by

$$I_g : G \rightarrow G, \quad h \mapsto L_g R_{g^{-1}}(h) = ghg^{-1}$$

The **adjoint representation** of G on $\mathfrak{g}$ is the homomorphism $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$, $g \mapsto \text{Ad}_g$ defined by differentiating I_g at e

$$\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}, \quad \xi \mapsto T_e I_g \xi = T_e (L_g R_{g^{-1}}) \xi$$

We define **coadjoint representation** $\text{Ad}^* : G \rightarrow \text{GL}(\mathfrak{g}^*)$ by the map $g \mapsto \mathfrak{Ad}_g^*$ defined by

$$\mathfrak{Ad}_g^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*, \quad \langle \mathfrak{Ad}_g^* \alpha, \xi \rangle := \langle \text{Ad}_{g^{-1}}^* \alpha, \xi \rangle = \langle \alpha, \text{Ad}_{g^{-1}} \xi \rangle, \quad \forall \alpha \in \mathfrak{g}^*, \xi \in \mathfrak{g}$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. By differentiating the adjoint representation $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ at $e \in G$ along a direction $\xi \in \mathfrak{g}$ we get the **adjoint Lie algebra representation** $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$, $\xi \mapsto \text{ad}_\xi$ defined by

$$\text{ad}_\xi : \mathfrak{g} \rightarrow \mathfrak{g}, \quad \eta \mapsto \text{ad}_\xi \eta := [\xi, \eta]$$

Again by differentiating the coadjoint representation $\text{Ad}^* : G \rightarrow \text{GL}(\mathfrak{g}^*)$ at $e \in G$ we obtain the **coadjoint Lie algebra representation** $\text{ad}^* : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}^*)$ by the map $\xi \mapsto \mathfrak{ad}_\xi^*$ defined by

$$\mathfrak{ad}_\xi^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*, \quad \langle \mathfrak{ad}_\xi^* \alpha, \eta \rangle := -\langle \text{ad}_\xi^* \alpha, \eta \rangle = -\langle \alpha, \text{ad}_\xi \eta \rangle, \quad \forall \alpha \in \mathfrak{g}^*, \eta \in \mathfrak{g}$$

Example 0.29. Take the matrix Lie group $G = SO(n)$ and take a path $A : (-\varepsilon, \varepsilon) \rightarrow G$ with $A(0) = I$ and $A'(0) = \xi \in \mathfrak{so}(n)$. Then the adjoint representation of G is given by

$$\begin{aligned} \text{Ad}_B \xi &= T_e I_{A(t)} \xi = \left. \frac{d}{dt} \right|_{t=0} I_B(A(t)) = \left. \frac{d}{dt} \right|_{t=0} BA(t)B^{-1} \\ &= B\xi B^{-1} \end{aligned}$$

for any $B \in G$. The coadjoint representation is given by

$$\langle \mathfrak{Ad}_B^* \alpha, \xi \rangle = \langle \alpha, \text{Ad}_{B^{-1}} \xi \rangle = \langle \alpha, B^{-1} \xi B \rangle, \quad \forall \alpha \in \mathfrak{so}(n)^*, \xi \in \mathfrak{so}(n)$$

Now identify $\alpha \in \mathfrak{so}(n)^*$ with a matrix $\alpha \in \mathfrak{so}(n)$ via trace pairing we get

$$\langle \mathfrak{Ad}_B^* \alpha, \xi \rangle = \langle \alpha, B^{-1} \xi B \rangle = \text{tr}(\alpha B^T \xi^T B^{-T}) = \text{tr}(B^{-T} \alpha B^T \xi^T) = \langle B^{-T} \alpha B^T, \xi \rangle$$

Hence $\mathfrak{Ad}_B^* \alpha = B^{-T} \alpha B^T = B \alpha B^T$. As we already have given above, the adjoint Lie algebra representation is given by $\text{ad}_\xi \eta = [\xi, \eta]$ for all $\xi, \eta \in \mathfrak{so}(n)$. Then the coadjoint Lie algebra representation is given by

$$\langle \mathfrak{ad}_\xi^* \alpha, \eta \rangle = -\langle \alpha, \text{ad}_\xi \eta \rangle = -\langle \alpha, \xi \eta - \eta \xi \rangle = \langle \alpha, \eta \xi - \xi \eta \rangle, \quad \forall \alpha \in \mathfrak{so}(n)^*, \eta \in \mathfrak{so}(n)$$

Again identifying $\alpha \in \mathfrak{so}(n)^*$ with a matrix $\alpha \in \mathfrak{so}(n)$ via trace pairing we get

$$\begin{aligned} \langle \mathfrak{ad}_\xi^* \alpha, \eta \rangle &= \langle \alpha, \eta \xi - \xi \eta \rangle = \text{tr}(\alpha(\eta \xi - \xi \eta)^T) \\ &= \text{tr}(\alpha(\xi^T \eta^T - \eta^T \xi^T)) = \text{tr}((\xi^T \alpha - \alpha \xi^T) \eta^T) \\ &= \langle \xi^T \alpha - \alpha \xi^T, \eta \rangle \end{aligned}$$

Hence $\mathfrak{ad}_\xi^* \alpha = \xi^T \alpha - \alpha \xi^T = -[\alpha, \xi^T]$. Since $\xi = -\xi^T$ we get $\mathfrak{ad}_\xi^* \alpha = [\alpha, \xi]$.

Now we can relate the Lie algebra bracket on $\mathfrak{g}$ with the usual Lie bracket of vector fields on M by the following Theorem

Theorem 0.30. *Let the Lie group G act on the manifold M . Then the map $\mathfrak{g} \rightarrow \mathfrak{X}(M)$, $\xi \mapsto \xi^M$ is a Lie algebra antihomomorphism, i.e.*

$$(a\xi + b\eta)^M = a\xi^M + b\eta^M, \quad [\xi^M, \eta^M] = -[\xi, \eta]^M$$

for all $\xi, \eta \in \mathfrak{g}$ and all $a, b \in \mathbb{R}$.

Proof. Denote the action by ϕ and denote $\phi^x(g) = \phi(g, x)$ for all $g \in G$ and $x \in M$.

Then we have for all $x \in M$

$$\begin{aligned}
 \xi_x^M &= \left. \frac{d}{dt} \right|_{t=0} \phi_{\exp(t\xi)}(x) \\
 &= \left. \frac{d}{dt} \right|_{t=0} \phi^x(\exp(t\xi)) \\
 &= T_e \phi^x \left(\left. \frac{d}{dt} \right|_{t=0} \exp(t\xi) \right) \\
 &= T_e \phi^x(\xi)
 \end{aligned}$$

Hence

$$(a\xi + b\mu)_x^M = T_e \phi^x(a\xi + b\mu) = aT_e \phi^x(\xi) + bT_e \phi^x(\mu) = a\xi_x^M + b\mu_x^M$$

Furthermore

$$[\xi^M, \eta^M] = [-\eta^M, \xi^M] = \mathfrak{L}_{\xi^M}(-\eta^M) = \left. \frac{d}{dt} \right|_{t=0} \phi_{\exp(-t\eta)}^* \xi^M$$

Now claim $(\text{Ad}_g \xi)^M = \phi_{g^{-1}}^* \xi^M$. Indeed, we have for all $x \in M$

$$\begin{aligned}
 (\text{Ad}_g \xi)_x^M &= \left. \frac{d}{dt} \right|_{t=0} \phi_{\exp(t \text{Ad}_g \xi)}(x) \\
 &= \left. \frac{d}{dt} \right|_{t=0} \phi(\exp(tg\xi g^{-1}), x) \\
 &= \left. \frac{d}{dt} \right|_{t=0} \phi(g \exp(t\xi) g^{-1}, x) \\
 &= \left. \frac{d}{dt} \right|_{t=0} (\phi_g \circ \phi_{\exp(t\xi)} \circ \phi_{g^{-1}})(x) \\
 &= T_{(\phi_g \circ \phi_{g^{-1}})(x)} \phi_g \left(\left. \frac{d}{dt} \right|_{t=0} \phi_{\exp(t\xi)}(\phi_{g^{-1}}(x)) \right) \\
 &= T_{\phi_{g^{-1}}(x)} \phi_g(\xi_{\phi_{g^{-1}}(x)}^M) \\
 &= (\phi_{g^{-1}}^* \xi^M)_x
 \end{aligned}$$

Thus the claim. So

$$\begin{aligned}
 [\xi^M, \eta^M] &= \left. \frac{d}{dt} \right|_{t=0} \phi_{\exp(-t\eta)}^* \xi^M \\
 &= \left. \frac{d}{dt} \right|_{t=0} (\text{Ad}_{\exp(t\eta)} \xi)^M
 \end{aligned}$$

$$\begin{aligned}
&= [\eta, \xi]^M \\
&= -[\xi, \eta]^M
\end{aligned}$$

□

Using this theorem we can now define an action of the Lie algebra $\mathfrak{g}$

Definition 0.31. Let $\mathfrak{g}$ be a Lie algebra and M be a manifold. Define the left **Lie algebra action** as a Lie algebra anti-homomorphism

$$\mathfrak{g} \rightarrow \mathfrak{X}(M), \quad \xi \mapsto \xi^M$$

such that the map $\mathfrak{g} \times M \rightarrow TM, \quad (\xi, x) \mapsto \xi_x^M$ is smooth.

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