

Idea behind Remainder Estimates

$$s = \sum a_n = s_N + R_N$$

$s \approx s_N \leftarrow$ approx. s with s_N
and R_N is the error

AST Remainder Estimate:

$$|R_N| = |s_N - s| \leq b_{N+1}$$

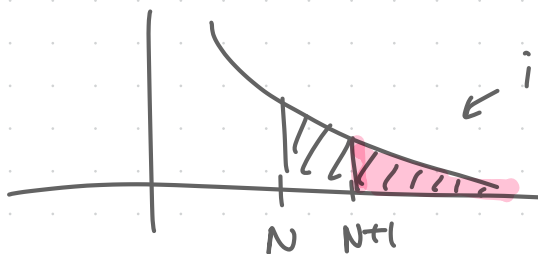
↑
where b_n is from
the AST, e.g.
for $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, $b_n = \frac{1}{n^2}$

Integral Test Remainder Estimate

Suppose that $\sum a_n$ is a ^{convergent} series for which the Integral Test applies (there exists a function f such that $f(n) = a_n$, and f is continuous, positive and decreasing). Then the remainder R_N

Satisfies:

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx$$



← integral starting from $N+1$ is smaller

$$S_N + \int_{N+1}^{\infty} f(x) dx \leq S_N + R_N \leq S_N + \int_N^{\infty} f(x) dx$$

$$S_N + \int_{N+1}^{\infty} f(x) dx \leq S \leq S_N + \int_N^{\infty} f(x) dx$$

N th partial sum

true value of series $\sum_{n=1}^{\infty} a_n$

$$\sum_{n=1}^N a_n = a_1 + a_2 + \dots + a_N$$

Also $R_N = \sum_{n=N+1}^{\infty} a_n$ remainder

ex: $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(We happen to know this series converges to $\pi^2/6 \approx 1.644934\dots$)

Approximating $\sum_{n=1}^{\infty} \frac{1}{n^2}$ using s_4 (4th partial sum):

$$\sum_{n=1}^4 \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \\ = 1.423611\dots$$

$$|\text{error}| = |s - \underbrace{s_4}_{R_4}| = 0.22\dots$$

But if you don't know the true value, you want an upper bd. for the error, so a remainder est.

$$|R_N| \leq \text{"upper bd"}$$

Integral Test Remainder Est. Example

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx$$

(a) Approximate $\sum_{n=1}^{\infty} \frac{1}{n^5}$ using the 1st 10 terms and estimate the error in this approximation.

using S_{10}

(b) How many terms are needed to approx. $\sum_{n=1}^{\infty} \frac{1}{n^5}$ to within 0.0001?

(a) First check Int. Test applies.

$$f(x) = \frac{1}{x^5}$$

continuous ✓ except at 0
so cts on $[1, \infty)$

positive ✓ on $[1, \infty)$

decreasing ✓ because x^5 is increasing on $[1, \infty)$ so $\frac{1}{x^5}$ is decreasing.

$$S_{10} = \sum_{n=1}^{10} \frac{1}{n^5}$$

Desmos link
for plotting S_n

$$= \frac{1}{1} + \frac{1}{2^5} + \frac{1}{3^5} + \dots + \frac{1}{10^5}$$

$$= 1.036907\dots$$

Now estimate error:

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^5} dx = \lim_{t \rightarrow \infty} \int_{10}^t x^{-5} dx$$

$$= \lim_{t \rightarrow \infty} \left(-\frac{1}{4} x^{-4} \right) \Big|_{10}^t$$

$$= \lim_{t \rightarrow \infty} \left(-\frac{1}{4} t^{-4} + \frac{1}{4} 10^{-4} \right)$$

$$= \lim_{t \rightarrow \infty} \left(-\frac{1}{4t^4} + \frac{1}{4} \cdot 10^{-4} \right)$$

$$= \frac{1}{4} \cdot \frac{1}{10000}$$

$$= \frac{1}{40000}$$

$$\uparrow$$

$$0.000025$$

$$N=10$$

(b) Solving for N . (approx. to within 0.0001)

set remainder estimate less than desired accuracy:

$$R_N \leq \underbrace{\int_N^{\infty} \frac{1}{x^5} dx}_{\text{evaluate}} < 0.0001$$

$$\begin{aligned} \int_N^{\infty} \frac{1}{x^5} dx &= \lim_{t \rightarrow \infty} \int_N^t x^{-5} dx \\ &= \lim_{t \rightarrow \infty} \left(-\frac{1}{4} x^{-4} \right) \Big|_N^t \\ &= \lim_{t \rightarrow \infty} \left(-\frac{1}{4} t^{-4} + \frac{1}{4} N^{-4} \right) \\ &= 0 + \frac{1}{4} \cdot \frac{1}{N^4} \\ &= \frac{1}{4N^4} \end{aligned}$$

$$\frac{1}{4N^4} \leq 0.0001$$

$$\frac{1}{4N^4} \leq \frac{1}{10000}$$

$$10000 \leq 4N^4$$

$$\frac{10000}{4} \leq N^4$$

$$7.071 \approx \sqrt[4]{\frac{10000}{4}} \leq N$$

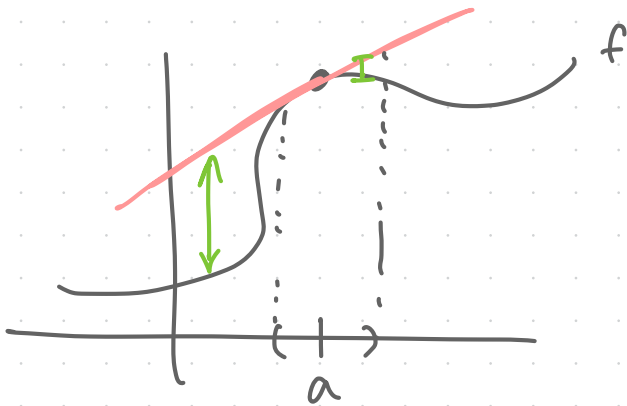
Choose $N=8$.

integer
 ← strictly larger than
 7.071

Intro to Activity

Idea behind linearization (of f at a)
point $x=a$

We want a linear function $L(x)$ that approximates f well near the point a



For a good approx of f :

- L has the same value as f at a

$$L(a) = f(a)$$

- L has the same slope as f at a :

$$L'(a) = f'(a)$$

For x near a , $f(x) \approx L(x)$