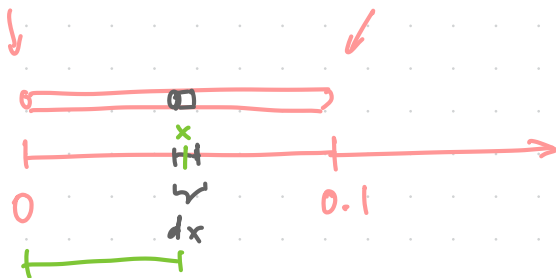


HWS #1 Hints

1. Consider two thin bars of constant cross-sectional area, each of length 10 cm , with respective mass linear-density functions $\rho(x) = \frac{1}{1+x^2} \frac{\text{kg}}{\text{m}}$ and $\delta(x) = e^{0.1x} \frac{\text{kg}}{\text{m}}$. $\Rightarrow 0.1\text{ m}$

- Interpret what it means for a bar to have mass linear-density $\rho(x) = \frac{1}{1+x^2} \frac{\text{kg}}{\text{m}}$, and a bar to have mass linear-density $\delta(x) = e^{0.1x} \frac{\text{kg}}{\text{m}}$.
- Find the mass of each bar.
- Explain why $\int_0^{0.1} x\rho(x) dx$ represents the total moment of mass for the first bar.
- Find the center of mass of each bar.



$$dm = \underbrace{\rho(x)}_{\text{density}} \cdot \underbrace{dx}_{\text{length}} \Rightarrow m = \int_0^{0.1} \rho(x) dx$$

$\uparrow$ total mass

moment = (distance relative to something) (mass)

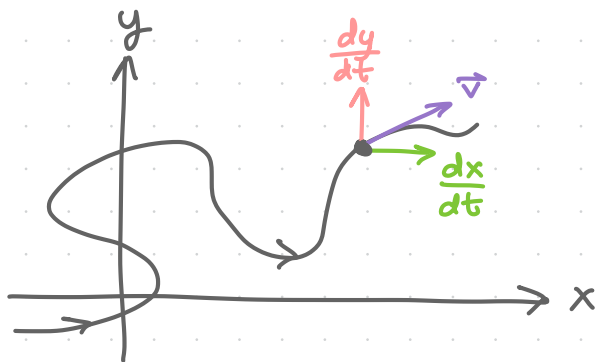
$\underbrace{\text{something}}_{\text{origin, in this case}}$

$$\bar{x} = \frac{\text{total (or combined) moment}}{\text{total mass}}$$

$$= \frac{\int_0^{0.1} x\rho(x) dx}{\int_0^{0.1} \rho(x) dx}$$

3.4 - Calculus with Parametric Curves

Think of the curve as the trajectory of a particle:



$(x(t), y(t))$ is the location of the particle at time t

$\frac{dy}{dt}$ is the instantaneous vertical velocity

$\frac{dx}{dt}$ is the instantaneous horizontal velocity

$\vec{V} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$ is the velocity vector
(combines horizontal & vertical velocity)

The tangent line points in the direction of the velocity vector, so

$$\text{slope of tangent line} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

← can pretend like the dt's cancel

y' is ambiguous, so use $\frac{dy}{dx}$ or $\frac{dy}{dt}$
 $y'(x)$ $y'(t)$

ex: Find the equation of the tangent line corresponding to $t=3$:

$$\begin{cases} x = 2t^2 + 1 \\ y = \frac{1}{3}t^3 - t \end{cases}$$

"in"
↓
 $t \in \mathbb{R}$
"t is a real number"

slope: $\left. \frac{dy}{dx} \right|_{t=3} = \left(\frac{t^2 - 1}{4t} \right) \Big|_{t=3} = \frac{8}{12} = \frac{2}{3}$

point: plug-in $t=3$

$$\begin{aligned} x &= 2 \cdot 9 + 1 = 19 \\ y &= 9 - 3 = 6 \end{aligned} \quad \} \quad (19, 6)$$

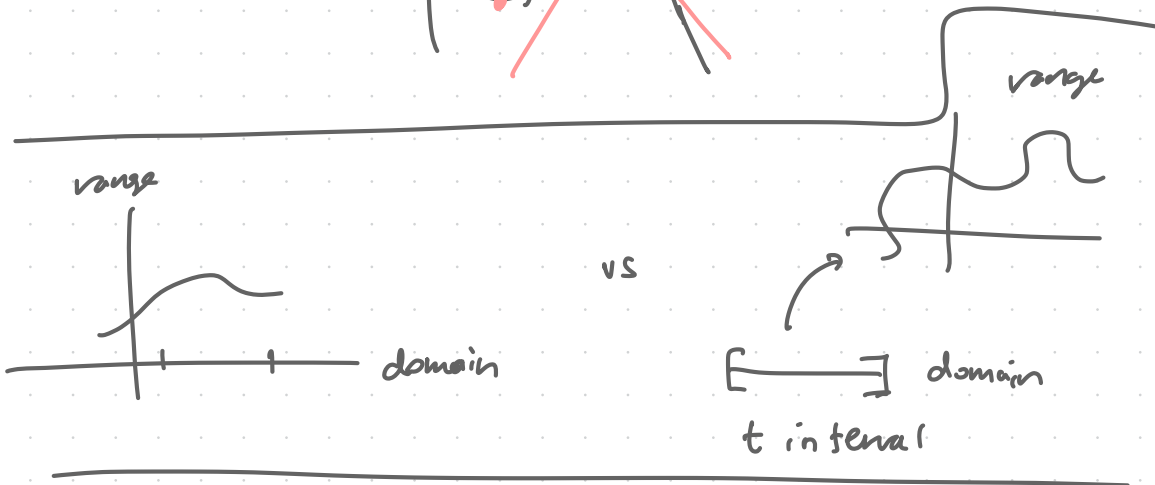
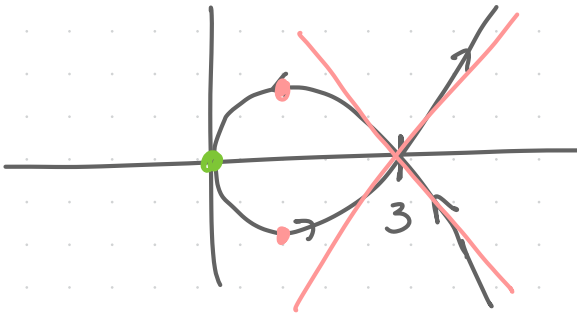
point-slope form

tangent line is $y - 6 = \frac{2}{3}(x - 19)$

example: $\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}, \quad t \in \mathbb{R}$

(a) Find all points where tangent is horizontal or vertical.

(b) Find the tangent lines at $(3,0)$.



(a) horizontal tangent $\Leftrightarrow \frac{dy}{dt} = 0$

$$\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}$$

(no vertical movement, so flat)

$$\frac{dy}{dt} = 3t^2 - 3 = 0$$

$$t^2 = 1$$

$$t = \pm 1$$

If $t=1$ the point is $(1, -2)$

$$x = (1)^2 = 1$$

$$y = 1^3 - 3(1) = -2$$

plug in to the formulae for x and y

If $t=-1$, the point is $(1, 2)$

vertical tangent $\Leftrightarrow \frac{dx}{dt} = 0$

(no horizontal movement)

$$\frac{dx}{dt} = 2t = 0$$

$$t = 0$$

If $t=0$, $x=0$ and $y=0$
So point is $(0,0)$

(b) Find the tangent lines thru $(3,0)$

① first find t corresponding to the point $(3,0)$

② then compute $\frac{dy}{dx}$ at the t you found in 1st step

$$x = t^2 = 3 \quad \text{AND}$$

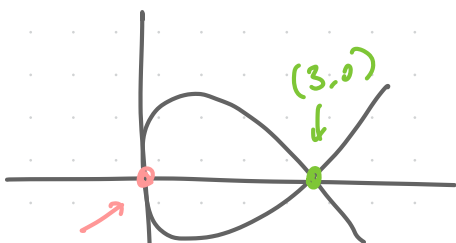
$$t = \pm\sqrt{3}$$

$$y = t^3 - 3t = 0$$

$$t(t^2 - 3) = 0$$

$$\cancel{t=0}, \quad t^2=3$$

$$t = \pm\sqrt{3}$$



$t=0$
corresponds
to this
point, not $(3,0)$
(y matches, but not x)

drop b/c this
doesn't work for
BOTH x and y

$$t = \sqrt{3} \quad \text{and} \quad t = -\sqrt{3}$$

$$\text{slope: } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}$$

$$\begin{aligned} \text{For } t = \sqrt{3}, \quad \frac{dy}{dx} \Big|_{t=\sqrt{3}} &= \frac{3 \cdot 3 - 3}{2\sqrt{3}} \\ &= \frac{6}{2\sqrt{3}} \quad \text{red arrow } \sqrt{3} \cdot \sqrt{3} \\ &= \frac{3}{\sqrt{3}} \\ &= \sqrt{3} \end{aligned}$$

$$\text{For } t = -\sqrt{3},$$

$$\frac{dy}{dx} \Big|_{t=-\sqrt{3}} = \frac{3 \cdot 3 - 3}{-2\sqrt{3}} = -\frac{6}{2\sqrt{3}} = -\sqrt{3}$$

tangent lines: point-slope form, point is (3,0)

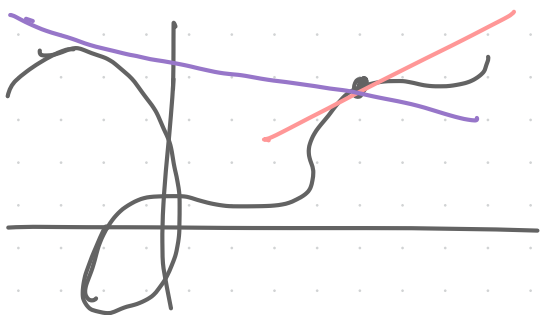
$$y - 0 = \sqrt{3}(x - 3)$$

$$y - 0 = -\sqrt{3}(x - 3)$$

Hard WA problem on Calc w/ Param. Eqs.
(last problem, #4)

Find tangent lines thru $(4, 3)$.

$$\begin{cases} x = 3t^2 + 1 \\ y = 2t^3 + 1 \end{cases}$$



$$\text{slope: } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t^2}{6t} = t$$

"General" equation of a tangent line
(for any t):

points have the form $(\underset{x_0}{3t^2+1}, \underset{y_0}{2t^3+1})$

slope is $t = m$

point-slope form: $y - y_0 = m(x - x_0)$

$$y - (2t^3 + 1) = t(x - (3t^2 + 1))$$

Means if $t=0$, tangent line is $y-1=0$
 $y=1$

if $t=2$, tangent line is $y-(17)=2(x-13)$
etc.

$$y-(2t^3+1) = t(x-(3t^2+1))$$

Plug in $x=4$, $y=3$ to see for what values of t we get the tangent line goes thru $(4,3)$.

$$3-(2t^3+1) = t(4-(3t^2+1))$$

$\therefore$ algebra

$$t^3 - 3t + 2 = 0$$

Notice: plug in $t=1$: $1-3+2=0$ ✓

$t=1$ is a root

Means $(t-1)(\underbrace{\quad? \quad}_{\text{quadratic}}) = 0$

$$\begin{array}{r}
 t-1 \overline{) \begin{array}{l} t^3 + 0t^2 - 3t + 2 \\ - \underline{t^3 - t^2} \\ t^2 - 3t + 2 \\ - \underline{t^2 - t} \\ -2t + 2 \\ - \underline{-2t + 2} \\ 0 \end{array}}
 \end{array}$$

$$\begin{aligned}
 t^3 - 3t + 2 &= (t-1)(t^2 + t - 2) \\
 &= (t-1)(t+2)(t-1) \\
 &= (t-1)^2(t+2)
 \end{aligned}$$

tangent
at (4,3)

$$t=1, t=-2$$

$$t=1 \Rightarrow \text{slope} = 1$$

@ $t=1$, get
the point (4,3)

$$t=-2 \Rightarrow \text{slope} = -2$$

tangent at
some other point
but goes thru (4,3)

Exam topics : areas between curves

volumes - known cross sections

Method of disks/washers

" " shells

work - spring, rope/cable/chain,
water pumping

center of mass