Hyperexponential and fixed-time stability of time-delay systems: Lyapunov-Razumikhin method

Artem N. Nekhoroshikh, Denis Efimov, Andrey Polyakov, Wilfrid Perruquetti and Igor B. Furtat

***Abstract*—Razumikhin-like theorems on hyperexponential and fixed-time stability of time-delay systems are proposed for both explicitly and implicitly defined Lyapunov functions. While the former method is useful for stability analysis, the latter approach is more suitable for control synthesis. Examples of systems that can be stabilized hyperexponentially and in fixed time are given. The control parameters tuning algorithm is presented in the form of linear matrix inequalities. The numerical simulations illustrate the theoretical results.**

***Index Terms*—Fixed-time and hyperexponential stability, implicit theorem, linear matrix inequalities, Lyapunov-Razumikhin theorem, time-delay systems**

I. Introduction

Stability analysis of time-delay systems is usually made by means of either Lyapunov-Krasovskii [11] or Lyapunov-Razumikhin [18] methods. The idea of the former approach consists in shifting the Lyapunov analysis to the functional space which the state vector belongs to. Thus, using this framework it is possible not only to prove (asymptotic) stability [9], [10] but also to find a rate of convergence: (hyper)exponential [6], [17] or finite-/(nearly) fixed-time [3], [13]. Besides, such a method provides both sufficient and necessary stability conditions of time-delay systems. Nevertheless, generally the choice of a Lyapunov–Krasovskii functional is difficult. Furthermore, it is even harder to find the functional satisfying restrictive conditions of the theorem on finite-time stability [4], [13].

Contrarily, the latter approach allows a conventional Lyapunov function to be used for the analysis [7]. For example, a Lyapunov function for the corresponding delay-free case can be chosen initially. However, such a paradigm does not provide necessary stability conditions or any quantitative estimates on the system convergence. The latter is the major drawback of Lyapunov-Razumikhin method since in many applications transient time is one of the main criteria. Yet recent extensions of this framework made it possible to analyze exponential and finite-/nearly fixed-time stability [3]. It is worth stressing that exponential stability of time-delays systems can also be proven without introduction of Lyapunov functionals by checking Halanay's inequality [8]. Based on this idea, in [19] global hyperexponential stability of a time-delay system with unbounded time-varying coefficients was investigated.

Notwithstanding, to the best of our knowledge, Razumikhin-like methods of hyperexponential and fixed-time stability analysis have not been proposed yet. Note that both types of convergence are superior to any exponential. It means that the transient time can be significantly reduced. For example, it can be also done by monotonous increasing feedback gains [5]. However, such a method makes the stability margin of time-delay systems drastically small. Another approach consists in introducing nonlinear feedback with comparatively small gains. Unfortunately, control design of nonlinear systems is a complicated task even for linear plants, e.g., a chain of integrators [1]. To overcome this difficulty the so-called Implicit Lyapunov method has been proposed for delay-free [12], [16] and time-delay systems [14], [15], [17]. This approach provides a constructive way for the control synthesis: it allows control parameters to be found as the solution of linear matrix inequalities (LMIs).

Therefore, the main objective of the paper is to formulate both explicit and implicit Razumikhin-like theorems on hyperexponential and fixed-time stability of time-delay systems. The contribution of the work is as follows:

- Development of an alternative way of stability analysis and control design of hyperexponentially/fixed-time stable time-delay systems: Lyapunov-Razumikhin method instead of Lyapunov-Krasovskii one [17]. It is shown how the problem of hyperexponential stabilization studied in [17] can be successfully solved using the proposed method. Moreover, differently from [17] the closed-loop system is *globally* stable.
- Extension of the results presented in [3], [14] to hyperexponential and fixed-time stability analysis and stabilization. Due to the complexity of the required analysis, two Lyapunov-Razumikhin functions are introduced.
- Introduction of a wider class of time-delay systems compared to the ones given in [14] for which the "necessary" condition on finite-time stabilization [4], [13] is fulfilled.

The paper is organized as follows. Preliminaries and stability definitions of time-delay systems are given in Sections II and III, respectively. Theorems on hyperexponential and fixed-time stability of time-delay systems are formulated in Sections IV (explicitly) and V (implicitly). The application of the proposed implicit methods for control synthesis is discussed in Section VI. The numerical simulation supporting theoretical results is presented in Section VII.

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A. N. Nekhoroshikh is with Faculty of Control Systems and Robotics, ITMO University, 197101, 49 Kronverkskiy av., Saint Petersburg, Russia (e-mail: annekhoroshikh@itmo.ru).

D. Efimov and A. Polyakov are with Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France (e-mail: denis.efimov@inria.fr, andrey.polyakov@inria.fr).

W. Perruquetti is with Univ. Lille, CNRS, Centrale Lille, UMR 9189 CRIStAL, F-59000 Lille, France (e-mail: wilfrid.perruquetti@centralelille.fr).

I. B. Furtat is with IPME RAS, V.O., Bolshoj pr., 61 St. Petersburg, 199178, Russia (e-mail: cainenash@mail.ru).

II. Preliminaries

A. Notations

- $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+^\star = \{x : x > 0\}$ and $\mathbb{R}_+ = \mathbb{R}_+^\star \cup \{0\}$;
- $\overline{1,n}$ is a series of natural numbers up to n, i.e. $1, \dots, n$;
- $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$;
- $\|\cdot\|_P$ is the weighted Euclidean norm in $\mathbb{R}^n$ for a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$;
- $\mathbb{C}_h$ is the space of continuous functions $\phi : [-h, 0] \to \mathbb{R}^n$ with the norm $\|\phi\|_h$ defined as $\|\phi\|_h := \max_{\tau\in[-h,0]} \|\phi(\tau)\|$;
- notations $P \succ 0$ ($P \prec 0$) and $P \succcurlyeq 0$ ($P \preccurlyeq 0$) mean that a symmetric matrix $P \in \mathbb{R}^{n\times n}$ is positive (negative) definite and semidefinite, respectively.
- $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the smallest and largest eigenvalues of a symmetric matrix $P \in \mathbb{R}^{n\times n}$, respectively;
- $\mathrm{diag}\{\lambda_i\}_{j=1}^n$ is a diagonal matrix with elements $\lambda_i \in \mathbb{R}$;
- $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix;
- $O_n \in \mathbb{R}^{n\times n}$ and $o_n \in \mathbb{R}^n$ are the zero matrix and the zero column, respectively;
- $\lceil s \rfloor^\omega := \mathrm{sign}(s)|s|^\omega$ is the signed power defined for any $s \in \mathbb{R}$ and $w \in \mathbb{R}_+^\star$;
- $\mathrm{sat}(s, \omega)$ is the saturation function defined for any $s \in \mathbb{R}$ and $w \in \mathbb{R}_+^\star$ such that $\mathrm{sat}(s, \omega) = s$ if $|s| \le \omega$ and $\mathrm{sat}(s, \omega) = \omega\,\mathrm{sign}(s)$ otherwise.

B. Comparison functions

A continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the *class* $\mathcal{K}$ if it is strictly increasing on $\mathbb{R}_+$ and $\psi(0) = 0$; if additionally it is unbounded then ψ belongs to $\mathcal{K}_\infty$.

A function $q : \mathbb{R}_+^\star \times \mathbb{R}_+^\star \to \mathbb{R}, (\sigma, s) \mapsto q(\sigma, s)$ is said to be of the *class* $\mathcal{IK}_\infty$ (*implicit* $\mathcal{K}_\infty$) if and only if [17]: 1) q is continuous on $\mathbb{R}_+^\star \times \mathbb{R}_+^\star$; 2) for any $s \in \mathbb{R}_+^\star$ there exists $\sigma \in \mathbb{R}_+^\star$ such that $q(\sigma, s) = 0$; 3) for any fixed $s \in \mathbb{R}_+^\star$ the function $q(\cdot, s)$ is strictly decreasing on $\mathbb{R}_+^\star$; 4) for any fixed $\sigma \in \mathbb{R}_+^\star$ the function $q(\sigma, \cdot)$ is strictly increasing on $\mathbb{R}_+^\star$; 5) for all $(\sigma, s) \in \Gamma = \{(\sigma, s) \in \mathbb{R}_+^\star \times \mathbb{R}_+^\star : q(\sigma, s) = 0\}$: $\lim\limits_{s\to 0^+} \sigma = 0$, $\lim\limits_{\sigma\to 0^+} s = 0$, $\lim\limits_{s\to+\infty} \sigma = +\infty$.

C. Auxiliary lemma

Lemma 1 [12]. *The function $g : \mathbb{R}_+ \times \mathbb{R}_+^* \to \mathbb{R}_+$, defined as $g(s, \epsilon) := |s^\epsilon - s|$, for any $\bar{s} \in \mathbb{R}_+$ and $(\bar{\epsilon}_1, \bar{\epsilon}_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ admits the following estimate:*

$$\max_{s\in[0,\bar{s}],\epsilon\in[\bar{\epsilon}_1,\bar{\epsilon}_2]} g(s, \epsilon) = \max\{\bar{g}(\bar{s}, \bar{\epsilon}_1), \bar{g}(\bar{s}, \bar{\epsilon}_2)\},$$

where

$$\bar{g}(\bar{s}, \bar{\epsilon}) := \begin{cases} 0, & \text{if } \bar{\epsilon} = 1, \\ g(\bar{s}, \bar{\epsilon}), & \text{if } \bar{\epsilon} \ne 1 \text{ and } \bar{s} \le \hat{s}, \\ \max\big(g(\hat{s}, \bar{\epsilon}), g(\bar{s}, \bar{\epsilon})\big), & \text{if } \bar{\epsilon} \ne 1 \text{ and } \bar{s} > \hat{s} \end{cases}$$

and $\hat{s} = \bar{\epsilon}^{1/(1-\bar{\epsilon})}$.

III. Definitions of stability

Consider a retarded time-delay system of the form [9]

$$\begin{cases} \dot{x}(t) = f(x_t), & t > 0, \\ x(\tau) = \Phi(\tau),\ \Phi \in \mathbb{C}_h, & \tau \in [-h, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the instantaneous state vector, $x_t \in \mathbb{C}_h$ is the functional state vector, defined by $x_t(\tau) := x(t + \tau)$ with $\tau \in [-h, 0]$ for $h > 0$ (time delay) and $f : \mathbb{C}_h \to \mathbb{R}^n$ is a continuous operator. Assume that the origin is an equilibrium point of (1), i.e., $f(0) = 0$ for all $t \ge 0$. A solution of (1) with initial conditions $\Phi \in \mathbb{C}_h$ is denoted by $x(t, \Phi)$.

In this subsection definitions of (hyper)exponential, finite-/(nearly) fixed-time stability for (1) are given. Note that in this work we will focus on hyperexponential stability only of degree one. For conventional definitions of Lyapunov stability the reader is referred to [6]. Let $\mathcal{D}$ be a neighborhood of the origin in $\mathbb{C}_h$. If $\mathcal{D} = \mathbb{C}_h$, then the corresponding properties are called *global*.

Definition 1 [9], [17]. *The origin of* (1) *is said to be* ***hyper-exponentially (exponentially) stable*** *if it is Lyapunov stable and there exist $\psi \in \mathcal{K}$ and $\vartheta > 0$ called* ***the decay rate*** *such that $\|x(t, \Phi)\| \le \psi(\|\Phi\|_h)e^{-e^{\vartheta t}}$ ($\|x(t, \Phi)\| \le \psi(\|\Phi\|_h)e^{-\vartheta t}$) for all $t \ge 0$ and any $\Phi \in \mathcal{D}$.*

Definition 2 [13], [17]. *The origin of* (1) *is said to be* ***finite-time stable*** *if it is Lyapunov stable and for any $\Phi \in \mathcal{D}$ there exists $0 \le T(\Phi) < \infty$ such that $x(t, \Phi) = 0$ for all $t \ge T(\Phi)$. The functional $T(\Phi) := \inf\{T_\Phi \ge 0 : x(t, \Phi) = 0\ \forall t \ge T_\Phi\}$ is called* ***the settling time*** *of the system* (1).

If, additionally, $\sup_{\Phi\in\mathcal{D}} T(\Phi) \le \mathcal{T} < +\infty$, then the origin if called ***fixed-time stable****.*

Definition 3 [3]. *The origin of* (1) *is said to be* ***nearly fixed-time stable*** *if it is Lyapunov stable and for any $\varrho > 0$ there exists $0 < T(\varrho) < \infty$ called* ***the transition time*** *such that $\|x(t, \Phi)\| \le \varrho$ for all $t \ge T(\varrho)$ and all $\Phi \in \mathcal{D}$.*

IV. Explicit Lyapunov-Razumikhin theorems

Now we are ready to formulate Razumikhin-like theorems on hyperexponential and fixed-time stability of system (1). Denote values of V_k, $k = 1, 2$, at time instants t and $t + \tau$ and time derivative of V_k along trajectories of (1) at time instant t as follows: $V_{t,k} := V_k(x_t(0))$, $V_{t+\tau,k} := V_k(x_t(\tau))$, $\dot{V}_{t,k} := \partial V_k(x)/\partial x\big|_{x=x_t(0)} f(x_t)$.

A. Hyperexponential stability

Theorem 2. *Let there exist two locally Lipschitz continuous Lyapunov-Razumikhin functions $V_k : \mathbb{R}^n \to \mathbb{R}_+$, $k = 1, 2$, such that*

C_e1) for some $\sigma_{1,k}, \sigma_{2,k} \in \mathcal{K}_\infty$ and all $x \in \mathbb{R}^n$

$$\sigma_{1,k}(\|x\|) \le V_k(x) \le \sigma_{2,k}(\|x\|);$$

C_e2) there are constants $c_1 > 0$ and $c_2 > 0$ such that $c_1\|x\| \le \sigma_{1,1}(\|x\|)$ for all $c_1\|x\| \le 1$ and $c_2\|x\| \le \sigma_{1,2}(\|x\|)$ for all $c_2\|x\| > 1$;

C_e3) $V_1(x) \le 1$ for all $x \in \mathbb{R}^n$ such that $V_2(x) \le 1$;

C_e4) for some $\chi > 1$, $\vartheta' > 0$ and for all $x_t \in \mathbb{C}_h$ we have:

(a) $$V_{t,2} > 1 \text{ and } \max_{\tau\in[-h,0]} V_{t+\tau,2} \le V_{t,2}^\chi e^{\chi-1} \Rightarrow \dot{V}_{t,2} \le -\vartheta' \ln(eV_{t,2})V_{t,2};$$

(b) $$V_{t,1} \le 1 \text{ and } \max_{\tau\in[-h,0]} V_{t+\tau,1}^\chi \le V_{t,1}e^{\chi-1} \Rightarrow \dot{V}_{t,1} \le \vartheta' \ln(V_{t,1}/e)V_{t,1}.$$

Then system (1) *is globally hyperexponentially stable at the origin with the decay rate $\vartheta = \min\left\{\frac{\ln\chi}{h}, \vartheta'\right\}$.*

Proof. Following the ideas of [3], the objective of the proof is to show that the Lyapunov-Razumikhin functions $V_{t,k}$, $k = 1, 2$, decrease hyperexponentially regardless of whether the relations between $\max_{\tau\in[-h,0]} V_{t+\tau,k}$ and $V_{t,k}$ given in condition C_e*4)* are satisfied or not. To this end, we will consider two cases: $V_{0,2} > 1$ and $V_{0,2} \le 1$.

Case 1: $V_{0,2} > 1$

I. Condition C_e*4a)* **check.** First, assume that

$$\max_{\tau\in[-h,0]} V_{t+\tau,2} > V_{t,2}^{\chi} e^{\chi-1}, \quad \forall t \in [0, t_1], \tag{2}$$

where $t_1 \in [0, +\infty)$ is the moment of time such that either condition C_e*4a)* starts to hold or $V_{t_1,2} = 1$. It is worth mentioning that (2) implies $\max_{\tau\in[-h,0]} V_{t+\tau,2} \neq V_{t,2}$ for all $t \in [0, t_1]$ since $\chi > 1$. Introduce $\theta_{t',2} := \min\{\theta \in [-h, 0) : V_{t'+\theta,2} = \max_{\tau\in[-h,0]} V_{t'+\tau,2}\}$ for all $t' \in [0, t]$. Thus,

$$\ln(eV_{t,2}) < \chi^{-1} \ln(eV_{t+\theta_{t,2},2}) = e^{-\ln\chi} \ln(eV_{t+\theta_{t,2},2}).$$

This estimate is successively applied until the sum $t + \theta_{t,2} + \theta_{t+\theta_{t,2},2} + \dots$ belongs to the interval $[-h, 0]$. Clearly, this process will stop after a *finite* number of iterations $m_2 \ge t/h$ since $\theta_{t,2}, \theta_{t+\theta_{t,2},2}, \dots < 0$. Eventually, it yields:

$$\ln(eV_{t,2}) < e^{-\vartheta t} \max_{\tau\in[-h,0]} \ln(eV_{\tau,2}), \quad \forall t \in [0, t_1],$$

where $\vartheta = \min\{\frac{\ln\chi}{h}, \vartheta'\}$. Evidently, it follows that $t_1 < T' := \vartheta^{-1} \ln(\max_{\tau\in[-h,0]} \ln(eV_{\tau,2})) + h < +\infty$.

Now suppose that condition C_e*4a)* holds for $t \in [t_1, t_2)$, where $t_2 \in [t_1, T')$ is the moment of time such that either $\max_{\tau\in[-h,0]} V_{t_2+\tau,2} > V_{t_2,2}^{\chi} e^{\chi-1}$ or $V_{t_2,2} = 1$. Solving the corresponding differential equation, it yields:

$$\ln(eV_{t,2}) \le e^{-\vartheta(t-t_1)} \ln(eV_{t_1,2}), \quad \forall t \in [t_1, t_2).$$

Summarizing both cases, we get

$$\ln(eV_{t,2}) < e^{-\vartheta t} \max_{\tau\in[-h,0]} \ln(eV_{\tau,2}), \quad \forall t \in [0, t_2). \tag{3}$$

Repeating the same steps, it can be shown that estimate (3) holds for all $t \in [0, T)$, where $T \in [t_2, T')$ such that $V_{T,2} = 1$, and $V_{t,2} \le 1$ for all $t \ge T$. Moreover, let us prove that for all $t \in [0, T)$ the following inequality holds:

$$e^{-\vartheta t} \max_{\tau\in[-h,0]} \ln(eV_{\tau,2}) \le 1 - e^{\vartheta t} + \max_{\tau\in[-h,0]} \ln(eV_{\tau,2}).$$

Indeed, since the function $y(t) := p(e^{-\vartheta t} - 1) + e^{\vartheta t} - 1$, where $p := \max_{\tau\in[-h,0]} \ln(eV_{\tau,2}) > 1$, is convex with respect to time t, then it is sufficient to check that $y(0)$ and $y(T)$ are nonpositive. Obviously, $y(0) = p(1 - 1) + 1 - 1 = 0$. On the other hand, $1 = \ln(eV_{T,2}) < pe^{-\vartheta T}$ due to (3) and, therefore, $y(T) = p(e^{-\vartheta T} - 1) + e^{\vartheta T} - 1 = -(pe^{-\vartheta T} - 1)(e^{\vartheta T} - 1) < 0$. Hence, taking into account C_e*1)* and C_e*2)*, one can see that

$$c_2\|x(t, \Phi)\| \le V_{t,2} < e^{1-e^{\vartheta t}} \max_{\tau\in[-h,0]} V_{\tau,2}, \ \forall t \in [0, T). \tag{4}$$

II. Condition C_e*4b)* **check.** Condition C_e*3)* implies $V_{t,1} \le 1$ for all $t \ge T$. Assume that

$$\max_{\tau\in[-h,0]} V_{t+\tau,1}^{\chi} > V_{t,1} e^{\chi-1}, \quad \forall t \in [T, t_3], \tag{5}$$

where $t_3 \ge T$ (possibly *infinite*) is the moment of time such that condition C_e*4b)* starts to hold or $V_{t_3,1} = 0$. It is worth pointing out that (5) implies $\max_{\tau\in[-h,0]} V_{t+\tau,1} \neq V_{t,1}$ for all $t \in [T, t_3]$ since $V_{t,1} \le 1$ for all $t \ge T$. Introduce $\theta_{t',1} := \min\{\theta \in [-h, 0) : V_{t'+\theta,1} = \max_{\tau\in[-h,0]} V_{t'+\tau,1}\}$ for all $t' \in [T, t]$. Thus,

$$\ln(V_{t,1}/e) < \chi \ln(V_{t+\theta_{t,1},1}/e) = e^{\ln\chi} \ln(V_{t+\theta_{t,1},1}/e).$$

This estimate is successively applied until the sum $t + \theta_{t,1} + \theta_{t+\theta_{t,1},1} + \dots$ belongs to the interval $[T, T + h]$. Clearly, this process will stop after a *finite* number of iterations $m_1 \ge (T - t + h)/h$ since $\theta_{t,1}, \theta_{t+\theta_{t,1},1}, \dots < 0$. Eventually, it yields:

$$\ln(V_{t,1}/e) < e^{\vartheta(t-T+h)} \max_{\tau\in[0,h]} \ln(V_{T+\tau,1}/e), \quad \forall t \in [T, t_3].$$

Now, suppose that condition C_e*4b)* holds for $t \in [t_3, t_4)$, where $t_4 \ge t_3$ (again possibly *infinite*) is the moment of time such that $\max_{\tau\in[-h,0]} V_{t_4+\tau,1}^{\chi} > V_{t_4,1} e^{\chi-1}$ or $V_{t_4,1} = 0$. Solving the corresponding differential equation, it yields:

$$\ln(V_{t,1}/e) \le e^{\vartheta(t-T-t_3)} \ln(V_{T+t_3,1}/e), \quad \forall t \in [t_3, t_4).$$

Summarizing both cases, for all $t \in [T, t_4)$ we get

$$\ln(V_{t,1}/e) < e^{\vartheta(t-T+h)} \max_{\tau\in[0,h]} \ln(V_{T+\tau,1}/e), \forall t \in [T, t_4). \tag{6}$$

Again, repeating the same steps, it can be shown that estimate (6) holds for all $t \ge T$. Moreover, $e^{\vartheta(t-T+h)} \max_{\tau\in[0,h]} \ln(V_{T+\tau,1}/e) \le 1 - e^{\vartheta(t-T)} + \max_{\tau\in[0,h]} \ln(V_{T+\tau,1}/e)$ since $\max_{\tau\in[0,h]} V_{T+\tau,1} \le 1$ for all $t \ge T$. Thus, taking into account C_e*1)* and C_e*2)*, one can see

$$c_1\|x(t, \Phi)\| \le V_{t,1} < e^{1-e^{\vartheta(t-T)}} \max_{\tau\in[0,h]} V_{T+\tau,1}, \ \forall t \ge T. \tag{7}$$

Finally, combining (4) and (7) and taking into account condition C_e*1)*, we conclude that system (1) is globally hyperexponentially stable at the origin with the decay rate ϑ and function $\psi(\|\Phi\|_h) = \sigma_{2,2}(\|\Phi\|_h)e/\min\{c_1, c_2\}$.

Case 2: $V_{0,2} \le 1$

If $V_{0,2} \le 1$ then $V_{0,1} \le 1$ due to condition C_e*3)*. Therefore, all results obtained in sections II and III of Case 1 remain valid with $T = 0$ and $\psi(\|\Phi\|_h) = \sigma_{2,1}(\|\Phi\|_h)e/c_1$. □

Example 1. Consider a scalar system of the form

$$\dot{x}(t) = -2ex^{1/3}(t) - 2ex^3(t) + x(t - h).$$

Choose $V_1 := x^{2/3}$ and $V_2 := x^2$. Obviously, conditions C_e*1)*–C_e*3)* hold with $\sigma_{1,1}(s) = \sigma_{2,1}(s) := s^{2/3}$, $\sigma_{1,2}(s) = \sigma_{2,2}(s) := s^2$ and $c_k = 1$. Selecting $\chi = 3$, it follows from C_e*4a)* and C_e*4b)*, respectively, that $|x(t-h)| \le ex^3(t)$ implies $\dot{V}_{t,2} \le -2eV_{t,2}^2 \le -2e\ln(eV_{t,2})V_{t,2}$ for all $V_{t,2} > 1$ and $|x(t-h)| \le ex^{1/3}(t)$ implies $\dot{V}_{t,1} \le -\frac{2}{3}e \le \frac{2}{3}e\ln(V_{t,1}/e)V_{t,1}$ for all $V_{t,1} \le 1$. Thus, the decay rate is $\vartheta = \min\left\{\frac{\ln 3}{h}, \frac{2}{3}e\right\}$. ■

Corollary 3. *Let there exists a locally Lipschitz continuous Lyapunov-Razumikhin function $V_0 : \mathbb{R}^n \to \mathbb{R}_+$ such that condition C_e1) holds with $k = 0$ and, furthermore:*

C_e2*) there is a constant $c_0 > 0$ such that $c_0\|x\| \le \sigma_{1,0}(\|x\|)$ for all $x \in \mathbb{R}_+$;

C_e3*) for some $\kappa > 1$, $\vartheta' > 0$ and all $x_t \in \mathbb{C}_h$ we have:

$$\max_{\tau\in[-h,0]} V_{t+\tau,0} \le \kappa V_{t,0} \Rightarrow \dot{V}_{t,0} \le -\vartheta' V_{t,0}.$$

Then system (1) *is globally exponentially stable at the origin with the decay rate $\vartheta = \min\{\frac{\ln\kappa}{h}, \vartheta'\}$.*

Example 2. It is easy to verify that the following scalar system

$$\dot{x}(t) = -2\kappa x(t) + x(t-h),$$

where $\kappa > 1$, is exponentially stable with $\vartheta = \min\{\frac{\ln \kappa}{h}, \kappa\}$. Indeed, choosing $V_0 := |x|$, conditions $C_e1)$ and $C_e2^\star)$ hold with $\sigma_{1,0}(s) = \sigma_{2,0}(s) := s$ and $c_0 = 1$. Moreover, it follows from $C_e3^\star)$ that $|x(t-h)| \le \kappa |x(t)|$ implies $\dot{V}_{t,0} \le -\kappa V_{t,0}$. ■

B. Fixed-time stability

As it has been mentioned in the introduction, fixed-time stabilization imposes more severe restrictions on system (1).

Theorem 4. *Let there exist two locally Lipschitz continuous Lyapunov-Razumikhin functions $V_k : \mathbb{R}^n \to \mathbb{R}_+$, $k = 1, 2$, such that conditions $C_e1)$ and $C_e3)$ hold and, furthermore, $C_e4^*)$ for some $\mu \in (-1, 0)$, $\nu > 0$, $\alpha > 0$, $\rho > 0$ and for all $x_t \in \mathbb{C}_h$ we have:*

(a) $V_{t,2} > 1$ *and* $(\max\limits_{\tau\in[-h,0]} V_{t+\tau,2})^{-\nu} + \rho \ge V_{t,2}^{-\nu}$
$\Rightarrow \dot{V}_{t,2} \le -\alpha V_{t,2}^{1+\nu}$;

(b) $V_{t,1} \le 1$ *and* $(\max\limits_{\tau\in[-h,0]} V_{t+\tau,1})^{-\mu} \le V_{t,1}^{-\mu} + \rho$
$\Rightarrow \dot{V}_{t,1} \le -\alpha V_{t,1}^{1+\mu}$.

Then system (1) *is globally fixed-time stable at the origin with the settling time* $\mathcal{T} = \frac{1}{\min\{\alpha\nu, \rho/h\}} + \frac{1}{\min\{-\alpha\mu, \rho/h\}}$.

Proof. The proof is based on the fact that system (1) is nearly fixed-time stable with respect to the set $\mathcal{B}_2 := \{x \in \mathbb{R}^n : V_2(x) \le 1\}$ (Theorem 4 of [3]) and finite-time stable (Theorem 3 of [3]) with the domain of attractivity $\mathcal{B}_1 := \{x \in \mathbb{R}^n : V_1(x) \le 1\}$. Since condition $C_e3)$ implies $\mathcal{B}_2 \subseteq \mathcal{B}_1$, then the system is fixed-time stable. Moreover, the following estimates hold

$$\begin{aligned}
V_{t,2} &< (\max_{\tau\in[-h,0]} V_{\tau,2}^{-\nu} + \min\{\alpha\nu, \rho/h\}t)^{-1/\nu}, && \forall t \in [0, T),\\
V_{t,1} &< \frac{1}{(\max\limits_{\tau\in[0,h]} V_{T+\tau,1}^{-\mu} - \min\{-\alpha\mu, \rho/h\}(t-T))^{1/\mu}}, && \forall t \in [T, \mathcal{T}),\\
V_{t,1} &= 0, && \forall t \ge \mathcal{T},
\end{aligned}$$

where $T \in [0, \frac{1}{\min\{\alpha\nu, \rho/h\}}]$ is the moment of time such that $V_{T+\tau,2} = 1$, and $\mathcal{T} \in [T, T + \frac{1}{\min\{-\alpha\mu, \rho/h\}}]$. □

Example 3. Let us consider a scalar *bilinear* system

$$\dot{x}(t) = -2\lceil x(t)\rfloor^{1/2} - 2\lceil x(t)\rfloor^2 + x(t)\,\mathrm{sat}(x(t-h), b),$$

where $b \in (4, +\infty)$.

It is worth stressing that due to boundedness of the delay term, it is always possible to analyze stability only of the corresponding delay-free system by letting $\mathrm{sat}(x(t-h), b) = b$. However, it yields a very conservative result since parameter b can be large enough. For example, the delay-free system is only locally finite-time stable at the origin if $x(0) \in \mathcal{D} := \{x \in \mathbb{R} : |x|^{-1/2} + |x| > b/2, |x| \le 1\}$ and nearly fixed-time stable with respect to the set $\mathcal{B} := \{x \in \mathbb{R} : |x|^{-1/2} + |x| \le b/2\}$ otherwise. Therefore, the delay-free analysis does not allow us to prove the fixed-time stability of the original system.

Having said that, now we will show how to apply Theorem 4. We choose the same Lyapunov-Razumikhin functions as in Example 1, i.e., $V_1 = x^{2/3}$ and $V_2 = x^2$. Obviously, for all $V_{t,2} = |x(t)|^2 \ge b^2$ we have $\dot{V}_{t,2} \le -2|x(t)|^3 = -2V_{t,2}^{1+\nu}$ for $\nu = 1/2$. On the other hand, it follows from $C_e4^*a)$ that $|\mathrm{sat}(x(t-h), b)| \le |x(t-h)| = V_{t-h,2}^{\nu} \le (V_{t,2}^{-\nu} - \rho)^{-1} = (|x(t)|^{-1} - \rho)^{-1} \le 2|x(t)|^{-1/2} + |x(t)|$ for all $V_{t,2} \in (1, b^2)$ if $\rho \le 2/(2b + b^{5/2})$. Clearly, if the delay term is unbounded, i.e., $b = +\infty$, then $\rho = 0$ and condition $C_e4^*a)$ cannot be satisfied. Hence, $\dot{V}_{t,2} \le -2V_{t,2}^{1+\nu}$ for all $V_{t,2} > 1$. Analogously, taking $\mu = -3/4$, it follows from $C_e4^*b)$ that $|\mathrm{sat}(x(t-h), b)| \le |x(t-h)| = V_{t-h,1}^{-2\mu} \le (V_{t,1}^{-\mu} + \rho)^2 = (|x(t)|^{1/2} + \rho)^2 \le |x(t)|^{-1/2} + 2|x(t)|$ implies $\dot{V}_{t,1} \le -\frac{2}{3}|x(t)|^{1/6} = -\frac{2}{3}V_{t,1}^{1+\mu}$ for all $V_{t,1} \le 1$ if $\rho \le \sqrt{2} - 1$. Hence, the system is globally fixed-time stable with the settling time $\mathcal{T} = \frac{1}{\min\{1, \rho/h\}} + \frac{1}{\min\{1/2, \rho/h\}}$, where $\rho \in (0, 2/(2b + b^{5/2})]$. ■

V. Implicit Lyapunov-Razumikin theorems

In this section implicit counterparts of Theorems 2 and 4 are introduced. Denote values of V_k and partial derivatives of functions $Q_k(V_k, x)$, $k = 1, 2$, at time instant t as follows: $V_{t,k} := V_k(t)$, $Q_{x,k}(t) := \partial Q_k(V_k, x)/\partial x\big|_{\substack{V_k = V_{t,k}\\ x = x_t(0)}}$ and $Q_{V,k}(t) := \partial Q_k(V_k, x)/\partial V\big|_{\substack{V_k = V_{t,k}\\ x = x_t(0)}}$.

Theorem 5. *If there exist two continuous functions $Q_k : \mathbb{R}_+^\star \times \mathbb{R}^n \to \mathbb{R}$ such that:*
$C_i1)$ Q_k are continuously differentiable outside the origin;
$C_i2)$ for any $x \in \mathbb{R}^n \setminus \{0\}$ there exists $V_k \in \mathbb{R}_+^\star$ such that $Q_k(V_k, x) = 0$;
$C_i3)$ $\frac{\partial Q_k(V_k, x)}{\partial V_k} < 0$ for all $V_k \in \mathbb{R}_+^\star$ and $x \in \mathbb{R}^n \setminus \{0\}$;
$C_i4)$ there exist $q_{1,k}, q_{2,k} \in \mathcal{IK}_\infty$ such that for all $V_k \in \mathbb{R}_+^\star$ and $x \in \mathbb{R}^n \setminus \{0\}$:

$$q_{1,k}(V_k, \|x\|) \le Q_k(V_k, x) \le q_{2,k}(V_k, \|x\|);$$

$C_i5)$ there are constants $c_1 > 0$ and $c_2 > 0$ such that $q_{1,1}(c_1\|x\|, \|x\|) \ge 0$ for all $c_1\|x\| \le 1$ and $q_{1,2}(c_2\|x\|, \|x\|) \ge 0$ for all $c_2\|x\| > 1$;
$C_i6)$ $Q_1(1, x) = Q_2(1, x)$ for all $x \in \mathbb{R}^n$;
$C_i7)$ for some $\chi > 1$, $\vartheta' > 0$ we have:

(a) $(V_{t,2}, x_t) \in \Omega_2 := \big\{(s, \varphi) \in \mathbb{R}_+^\star \times \mathbb{C}_h : Q_2(s, \varphi(0)) = 0,\ Q_2\big(1, \varphi(0)\big) > 0,\ \max\limits_{\tau\in[-h,0]} Q_2\big(s^\chi e^{\chi-1}, \varphi(\tau)\big) \le 0\big\}$
$\Rightarrow Q_{x,2}(t) f(x_t) \le \vartheta' \ln(eV_{t,2}) V_{t,2} Q_{V,2}(t)$;

(b) $(V_{t,1}, x_t) \in \Omega_1 := \big\{(s, \varphi) \in \mathbb{R}_+^\star \times \mathbb{C}_h : Q_1(s, \varphi(0)) = 0,\ Q_1\big(1, \varphi(0)\big) \le 0,\ \max\limits_{\tau\in[-h,0]} Q_1\big(s^{1/\chi} e^{1-1/\chi}, \varphi(\tau)\big) \le 0\big\}$
$\Rightarrow Q_{x,1}(t) f(x_t) \le -\vartheta' \ln(V_{t,1}/e) V_{t,1} Q_{V,1}(t)$.

Then system (1) *is globally hyperexponentially stable at the origin with the decay rate $\vartheta = \min\big\{\frac{\ln \chi}{h}, \vartheta'\big\}$.*

Proof. In order to prove Theorem 5, let us show that there exist two unique functions $V_k : \mathbb{R}^n \to \mathbb{R}_+$ that satisfy conditions of Theorem 2. Firstly, it was proven in [16] that conditions $C_i1)$–$C_i3)$ imply existence of unique functions $V_k(x)$ defined by equality $Q_k(V_k(x), x) = 0$ with $V_k(0) = 0$.

Secondly, if follows from properties of $\mathcal{IK}_\infty$ functions that there exist $\sigma_{1,k}, \sigma_{2,k} \in \mathcal{K}_\infty$ such that $q_{1,k}(\sigma_{1,k}(\|x\|), \|x\|) = q_{2,k}(\sigma_{2,k}(\|x\|), \|x\|) = 0 = Q_k(V_k(x), x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Therefore, $C_i4) \Leftrightarrow C_e1)$ and $C_i5) \Leftrightarrow C_e2)$.

Thirdly, for any given $x \in \mathbb{R}^n$ such that $V_2(x) \le 1$ we have $Q_1(1, x) = Q_2(1, x) \le Q_2(V_2(x), x) = 0 = Q_1(V_1(x), x)$ due to $C_i3)$. Then $V_1(x) \le 1$ and, hence, $C_i6) \Rightarrow C_e3)$.

Finally, conditions $C_e4)$ and $C_i7)$ are equivalent. Indeed, for any $\tau \in [-h,0]$ take $(V_{t+\tau,k}, x_t(\tau)) \in \Omega_k$. Since functions $Q_k(s,x)$ are monotonically decreasing for any fixed $x \in \mathbb{R}^n \setminus \{0\}$ due to condition $C_i3)$, for any $s \in \mathbb{R}^\star_+$ we have:

$$\max_{\tau\in[-h,0]} Q_k(s, x_t(\tau)) \le 0 \Leftrightarrow \max_{\tau\in[-h,0]} V_{t+\tau,k} \le s.$$

Taking this into account and applying the implicit function theorem [2] for Euclidean spaces to condition $C_i7)$ ($\dot V_{t,k} = -Q_{x,k}(t)f(x_t)/Q_{V,k}(t)$), we finish the proof. □

Corollary 6. *If there exists a continuous function $Q_0 : \mathbb{R}^\star_+ \times \mathbb{R}^n \to \mathbb{R}$ such that conditions $C_i1)$-$C_i4)$ hold with $k = 0$ and, furthermore:*

$C_i5^\star)$ there is a constant $c_0 > 0$ such that $q_{1,0}(c_0\|x\|, \|x\|) \ge 0$ for all $x \in \mathbb{R}_+$;

$C_i6^\star)$ for some $\kappa > 1$, $\vartheta' > 0$ we have:

$$(V_{t,0}, x_t) \in \Omega_0 := \big\{(s,\varphi) \in \mathbb{R}^\star_+ \times \mathbb{C}_h : Q_0(s,\varphi(0)) = 0, \max_{\tau\in[-h,0]} Q_0\big(\kappa s, \varphi(\tau)\big) \le 0\big\} \Rightarrow Q_{x,0}(t)f(x_t) \le \vartheta' V_{t,0} Q_{V,0}(t).$$

Then system (1) *is globally exponentially stable at the origin with the decay rate $\vartheta = \min\{\frac{\ln\kappa}{h}, \vartheta'\}$.*

Theorem 7. *If there exist two continuous functions $Q_k : \mathbb{R}^\star_+ \times \mathbb{R}^n \to \mathbb{R}$ such that conditions $C_i1)$-$C_i4)$ and $C_i6)$ hold and, furthermore:*

$C_i7^*)$ for some $\mu \in (-1,0)$, $\nu > 0$, $\alpha > 0$, $\rho > 0$ we have:

(a) $(V_{t,2}, x_t) \in \Omega^*_2 := \big\{(s,\varphi) \in \mathbb{R}^\star_+ \times \mathbb{C}_h : Q_2(s,\varphi(0)) = 0, Q_2\big(1,\varphi(0)\big) > 0, \max_{\tau\in[-h,0]} Q_2\big((\max\{0, s^{-\nu} - \rho\})^{-1/\nu}, \varphi(\tau)\big) \le 0\big\}$
$\Rightarrow Q_{x,2}(t)f(x_t) \le \alpha V_{t,2}^{1+\nu} Q_{V,2}(t);$

(b) $(V_{t,1}, x_t) \in \Omega^*_1 := \big\{(s,\varphi) \in \mathbb{R}^\star_+ \times \mathbb{C}_h : Q_1(s,\varphi(0)) = 0, Q_1\big(1,\varphi(0)\big) \le 0, \max_{\tau\in[-h,0]} Q_1\big((s^{-\mu} + \rho)^{-1/\mu}, \varphi(\tau)\big) \le 0\big\}$
$\Rightarrow Q_{x,1}(t)f(x_t) \le \alpha V_{t,1}^{1+\mu} Q_{V,1}(t).$

Then system (1) *is globally fixed-time stable at the origin with the settling time $\mathcal{T} = \frac{1}{\min\{\alpha\nu, \rho/h\}} + \frac{1}{\min\{-\alpha\mu, \rho/h\}}$.*

Theorem 7 can be proven similarly to Theorem 5.

VI. Examples

A. Problem formulation

Let us consider a subclass of (1) with $n \ge 2$:

$$\begin{cases} \dot x(t) = A_0 x(t) + A_1 \xi(x_t) + Bu(t), & t > 0, \\ x(\tau) = \Phi(\tau), \quad \Phi \in \mathbb{C}_h, & \tau \in [-h, 0], \end{cases} \tag{8}$$

where $u(t) \in \mathbb{R}$ is a control input, $A_1 = \operatorname{diag}\{a_i\}_{i=1}^n$ is a known matrix, system matrices $A_0 \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$ are of the form:

$$A_0 := \begin{bmatrix} O_{n-1} & I_{n-1} \\ 0 & o_{n-1}^\top \end{bmatrix}, \quad B := \begin{bmatrix} o_{n-1} \\ 1 \end{bmatrix},$$

$\xi(x_t) = [\xi_1(x_{t,1}), \dots, \xi_n(x_{t,n})]^\top$ is a function which components $\xi_i(x_{t,i})$ satisfy the following conditions:

– in case of hyperexponential stabilization [17]:

$$|\xi_i(x_{t,i})| \le |x_i(t-h)|; \tag{9a}$$

– in case of fixed-time stabilization:

$$|\xi_i(x_{t,i})| \le |\operatorname{sat}(x_i(t), 1)||\operatorname{sat}(x_i(t-h), b)|, \tag{9b}$$

where $b \in (\max\{1, \max_i\{1/a_i\}\}, +\infty)$.

Note that many dynamical processes, such as vehicular traffic flow [6], regenerative chatter in metal cutting [7] or population dynamics [6], can be modeled by system (8) with conditions (9) fulfilled.

Remark 1. *It is worth mentioning that system* (8) *could be finite-time stable if $\dot x(t)|_{x(t)=0} = 0$ (see [4], [13]). Since in this paper we are looking for a feedback $u(x(t))$ such that $u(0) = 0$, then the delay term $A_1\xi(x_t)$ has to be identically zero whenever $x(t) = 0$. Obviously, condition* (9a) *does not guarantee this and function $\xi(x_t)$ has to satisfy more severe restrictions as, for example, given in* (9b)*. However, constraint* (9b) *is not very restrictive since it holds for many bounded functions (e.g., sine and different sigmoids) with a unit slope. Moreover, the amplitude of the time-delay term can be significantly large.*

B. Control design

For both hyperexponential and fixed-time stabilization we will define a continuous control law as follows [12]:

$$u(x) := \sum\nolimits_{i=1}^n K_i \lceil x_i \rfloor^{w_i(\zeta(\|x\|_P))}, \tag{10a}$$

$$w_i(\zeta) := \tfrac{1+\zeta}{1-(n-i)\zeta}, \quad i = \overline{1,n}, \tag{10b}$$

$$\zeta(\|x\|_P) := \begin{cases} \beta_2, & \text{if } \|x\|_P > M, \\ \beta_1, & \text{if } \|x\|_P < 1, \\ \frac{\beta_2-\beta_1}{M-1}\|x\|_P + \frac{M\beta_1-\beta_2}{M-1}, & \text{otherwise}, \end{cases} \tag{10c}$$

$$M := \sqrt{\lambda_{\min}(P^{-1/2} D_2(\delta) P D_2(\delta) P^{-1/2})} > 1, \tag{10d}$$

where $K_i < 0, i = \overline{1,n}$, are feedback gains, $\|x\|_P = \sqrt{x^\top P x}$, $0 \prec P = P^\top \in \mathbb{R}^{n\times n}$, $\delta > 1$ and

$$\beta_1 := -\tfrac{\beta}{1-(n-2)\beta}, \quad \beta_2 := \beta, \tag{11}$$

where $\beta \in (0, \frac{1}{n-1})$.

Remark 2. *To compare nonlinear and linear approaches, we will also introduce a linear counterpart of* (10) *with the same feedback gains $K^\top = [K_1, \dots, K_n]$, i.e. $\beta = 0$ and*

$$u(x) := K^\top x. \tag{12}$$

C. Control parameters tuning algorithm

In this subsection we will present restrictions on parameters β and δ such that sufficient conditions of global hyperexponential (fixed-time) stability given in Theorem 5 (7) are satisfied. Moreover, matrices P and K will be obtained as the solution of the corresponding LMIs. To this end, we will introduce Implicit Lyapunov-Razumikhin function (ILRFs) candidates $Q_k(V_k, x)$, $k = 1, 2$, [16]

$$Q_k(V_k, x) := x^\top D_k(V_k^{-1}) P D_k(V_k^{-1}) x - 1, \tag{13}$$

where $D_k(\lambda) := \operatorname{diag}\{\lambda^{r_{i,k}}\}_{i=1}^n$ is the dilation matrix with

$$r_{i,k} := 1 - (n-i)\beta_k, \quad i = \overline{1,n}. \tag{14}$$

Remark 3. *For the case of exponential stability we will introduce ILRF in the form $Q_0(V_0, x) := V_0^{-2} x^\top P x - 1$, which implicitly defines a conventional Lyapunov-Razumikhin function $V_0 = \|x\|_P$.*

Proposition 8. *Let there exist* $\beta \in (0, \frac{1}{n-1})$, $\delta > 1$, $\eta > 0$ *and* $\alpha \in (0, \eta/2]$ *such that LMIs:*

$$-\gamma X \preccurlyeq X H_k + H_k X \prec 0, \tag{15a}$$

$$\begin{bmatrix} R & \|\varepsilon\| B Y^\top \\ \|\varepsilon\| Y B^\top & -\alpha^2 \gamma X \end{bmatrix} \preccurlyeq 0, \tag{15b}$$

$$2\alpha I_n \preccurlyeq X \preccurlyeq \eta I_n, \quad 2\eta I_n \preccurlyeq Z, \tag{15c}$$

where

$$\gamma := 2(1 - n\beta_1),\ H_k := \operatorname{diag}\{-r_{i,k}\}_{i=1}^n,\ \varepsilon = [\varepsilon_1, \dots, \varepsilon_n]^\top,$$
$$\varepsilon_i := \max\big\{\bar{g}\big(\sqrt{\eta}, \tfrac{1+\beta_1}{r_{i,1}}\big), \bar{g}\big(\sqrt{\eta}, \tfrac{1+\beta_2}{r_{i,2}}\big)\big\} + \sqrt{\eta}(1 - \delta^{p_i}),$$
$$p_i := \tfrac{r_{i,2}}{r_{i,1}}(1 + \beta_1) - (1 + \beta_2),$$
$$R := A_0 X + B Y^\top + X A_0^\top + Y B^\top + 2\alpha\gamma X + \tfrac{\omega^2}{\alpha\gamma} A_1 Z A_1^\top,$$

$\omega = e^{-\beta_1}$ *and functions* $\bar{g}(\bar{s}, \bar{\epsilon})$ *defined in Lemma 1, are feasible for some* $0 \prec X = X^\top, 0 \prec Z = Z^\top \in \mathbb{R}^{n\times n}$ *and* $Y \in \mathbb{R}^n$.

Then the closed-loop system (8), (9a), (10) *with matrices* $P = X^{-1}$, $K^\top = Y^\top P$ *is* ***globally hyperexponentially stable*** *with the decay rate* $\vartheta = \min\big\{\frac{\ln(1+\beta)}{h}, \alpha\beta\big\}$.

Proposition 9. *Let there exist* $\beta \in (0, \frac{1}{n-1})$, $\delta > 1$, $\eta > 0$, $\alpha \in (0, \eta/2]$ *and* $\rho \in (0, \bar{\rho}]$ *such that LMIs* (15)*, where* $\omega = (1-\rho)^{-(1+\beta)/\beta}$ *and*

$$\bar{\rho} := \begin{cases} \frac{1 - b^{-\beta^2/r_{2,2}}}{b^{\beta/r_{2,2}} - 1}, & \text{if } \eta \le 1, \\ \min\big\{\frac{\sqrt{\eta}^{\beta/(1+\beta)} - 1}{\sqrt{\eta}},\ \frac{1 - b^{-\beta^2/r_{2,2}}}{b^{\beta/r_{2,2}} - 1}\big\}, & \text{if } \eta > 1, \end{cases}$$

are feasible for some $0 \prec X = X^\top, 0 \prec Z = Z^\top \in \mathbb{R}^{n\times n}$ *and* $Y \in \mathbb{R}^n$.

Then the closed-loop system (8), (9b), (10) *with matrices* $P = X^{-1}$, $K^\top = Y^\top P$ *is* ***globally fixed-time stable*** *with the settling time* $\mathcal{T} = \frac{1}{\min\{\alpha\beta_2, \rho/h\}} + \frac{1}{\min\{-\alpha\beta_1, \rho/h\}}$.

Corollary 10. *If Proposition 8 (or 9) holds, then the closed-loop system* (8), (9a) *(or* (9b)*),* (12) *with the same feedback gains* K *is* ***globally exponentially stable*** *with the decay rate* $\vartheta = \min\{\frac{\ln\kappa}{h}, \frac{\alpha\gamma}{2}\}$*, where* $\kappa = \omega\sqrt{2}$.

Proofs of Propositions 8, 9 and Corollary 10 are given in Appendices A, B and C, respectively.

VII. Numerical simulation

Let us illustrate the theoretical results obtained in the previous section by numerically simulating system (8), (9) governed either by feedback (10) or its linear counterpart (12). For both cases consider $n = 3$, $a_i = 0.02$, $i = \overline{1,3}$, $\delta = 1.1$ and constant initial conditions Φ, i.e., $\Phi(\tau_1) = \Phi(\tau_2)$ for any $\tau_1, \tau_2 \in [-h, 0]$. The numerical simulation has been done in MATLAB Simulink using Dormand-Prince method (*ode45*) with a variable step size $\Delta t \le 10^{-4}$ and absolute tolerance $e = 10^{-30}$.

A. Hyperexponential vs. exponential convergence

Let us choose parameters $\beta = 0.054$, $\eta = 1.79$ and $\alpha = 0.153$ such that LMIs (15) are feasible and $\vartheta = 0.0083$ [1/s] is maximal for $h = 1$ [s]. Thus, the solution of LMIs (15) is:

$$K = \begin{bmatrix} -1.92 \\ -4.12 \\ -2.92 \end{bmatrix},\ P = \begin{bmatrix} 1.27 & 1.04 & 0.50 \\ 1.04 & 2.32 & 0.69 \\ 0.50 & 0.69 & 0.92 \end{bmatrix}.$$

Besides, it follows from Corollary 10 that the system (8), (9a), (12) is globally exponentially stable.

The norm of the trajectories $\|x(t)\|$ of (8), (9a) for different initial conditions and time delays are depicted in Fig. 1a and Fig. 1b, respectively, in the logarithmic scale. Solid and dashed lines correspond to control (10) and its linear counterpart (12). Clearly, in both cases the hyperexponentially stable system converges faster to the origin than the exponentially stable one. Moreover, the numerical simulation confirms dependence of the convergence rate ϑ on the value of time delay h: ϑ is inversely proportional to h.

B. Fixed-time vs. exponential convergence

Let $b = 10^{20}$ in (9b). We choose parameters $\beta = 0.05$, $\eta = 1.91$, $\alpha = 0.161$ and $\rho = 0.011 = \bar{\rho}$ such that LMIs (15) are feasible and $\mathcal{T} = 243$ [s] is minimal for $h = 1$ [s]. Therefore, the solution of LMIs (15) is:

$$K = \begin{bmatrix} -1.98 \\ -4.20 \\ -2.99 \end{bmatrix},\ P = \begin{bmatrix} 1.20 & 1.00 & 0.48 \\ 1.00 & 2.19 & 0.66 \\ 0.48 & 0.66 & 0.87 \end{bmatrix}.$$

Besides, it follows from Corollary 10 that the system (8), (9b), (12) is globally exponentially stable.

The norm of the trajectories $\|x(t)\|$ of (8), (9b) for different initial conditions are depicted in Fig. 2 in the logarithmic scale. One can see that the fixed-time stable system converges faster to the origin than the exponentially stable one. It is worth mentioning that numerical simulation of system (8), (9b), (10) has shown the same settling time $\mathcal{T}$ for different values of time delay h (0.1, 1 and 2 [s]).

VIII. Conclusion

Lyapunov-Razumikhin methods for analysis and synthesis of hyperexponentially and fixed-time stable time-delay systems are proposed. Modified Razumikhin-like theorems have been formulated both explicitly and implicitly. Using the latter approach control parameters have been found as the solution of linear matrix inequalities. The numerical simulations have verified the obtained theoretical results. Consideration of parametric uncertainties and external disturbances as well as output stabilization are possible directions of future research.

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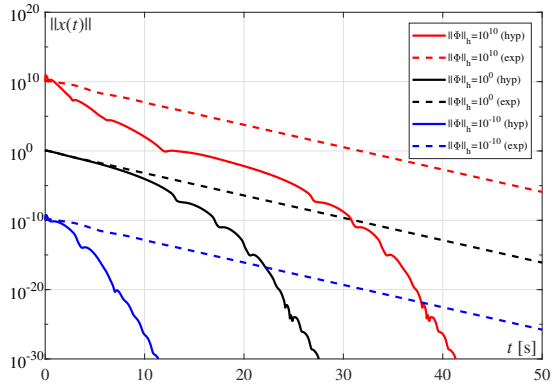
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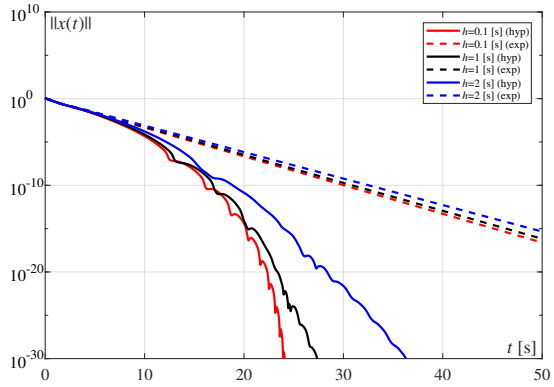
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(a) Different initial conditions ($h = 1$ [s])



(b) Different time delays ($\|\Phi\|_h = 10^{10}$)

Fig. 1: Hyperexponential (hyp) vs. exponential (exp) stabilization of system (8), (9a)

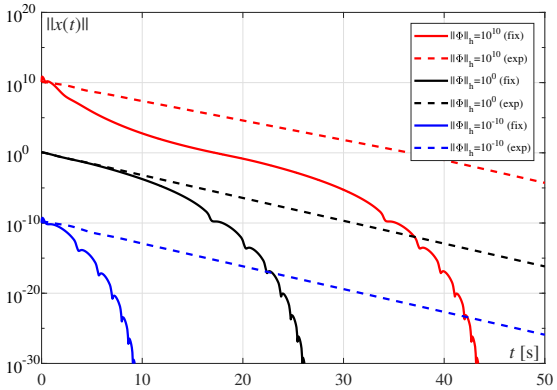


Fig. 2: Fixed-time (fix) vs. exponential (exp) stabilization of system (8), (9b)

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APPENDIX A
PROOF OF PROPOSITION 8

Let us show that ILRFs (13) satisfy all the conditions of Theorem 5 with respect to the system (8), (9a), (10).

A.I. Clearly, $Q_1(1,x) = Q_2(1,x)$ for all $x \in \mathbb{R}^n$. It can be shown that the following inequalities

$$q_{1,k}(V_k,\|x\|) = \frac{\lambda_{\min}(P)\|x\|^2}{\max\{V_k^{2-2(n-1)\beta_k}, V_k^2\}} - 1 \le Q_k(V_k,x) \le \frac{\lambda_{\max}(P)\|x\|^2}{\min\{V_k^{2-2(n-1)\beta_k}, V_k^2\}} - 1 = q_{2,k}(V_k,\|x\|)$$

hold for all $V_k \in \mathbb{R}^\star_+$ and $x \in \mathbb{R}^n$. Therefore, the function (13) satisfies conditions *C_i1)*, *C_i2)* and *C_i4)*. Moreover, there exist $c_1 = c_2 = \sqrt{\lambda_{\min}(P)}$ such that $q_{1,1}(c_1\|x\|,\|x\|) \ge 0$ for all $c_1\|x\| \le 1$ and $q_{1,2}(c_2\|x\|,\|x\|) \ge 0$ for all $c_2\|x\| > 1$.

A.II. One can see that (15a) implies that condition *C_i3)* holds for all $V_k \in \mathbb{R}^\star_+$ and $x \in \mathbb{R}^n \setminus \{0\}$ since by definition

$$Q_{V,k} := V_k^{-1} x^\top D_k(V_k^{-1})\big[H_k P + P H_k\big] D_k(V_k^{-1}) x < 0.$$

Furthermore, $V_k Q_{V,k} \ge -\gamma$ for all $(V_k, x) \in \Omega_k$.

A.III. Firstly, let us introduce the following change of variables $y_{t,k} := D_k(V_{t,k}^{-1})x_t(0)$, $z_{t,k} := D_k(V_{t,k}^{-1})\xi(x_t)$, $k = 1,2$. Secondly, note that $D_k(V_k^{-1})A_0 = V_k^{\beta_k} A_0 D_k(V_k^{-1})$, $D_k(V_k^{-1})A_1 = A_1 D_k(V_k^{-1})$ and $D_k(V_k^{-1})B = V_k^{-1}B$. Then adding and subtracting $2V_{t,k}^{\beta_k} y_{t,k}^\top PBK^\top y_{t,k}$, we obtain:

$$Q_{x,k}(t)f(x_t) = 2V_{t,k}^{\beta_k} y_{t,k}^\top P(A_0 + BK^\top) y_{t,k} + 2y_{t,k}^\top P A_1 z_{t,k} + 2V_{t,k}^{\beta_k} y_{t,k}^\top PBK^\top d_{t,k},$$

where $d_{t,k} := V_{t,k}^{-1-\beta_k} \lceil x_t(0) \rfloor^{w(\zeta(\|x_t(0)\|_P))} - y_{t,k}$ with $\lceil x_t(0) \rfloor^{w(\zeta)} = [\lceil x_{t,1}(0)\rfloor^{w_1(\zeta)}, \dots, \lceil x_{t,n}(0)\rfloor^{w_n(\zeta)}]^\top$.

Denote $\varsigma_{t,k} := [y_{t,k},\ V_{t,k}^{-\beta_k} z_{t,k}/\omega,\ d_{t,k}/\|\varepsilon\|]^\top$. Then adding and subtracting $2\alpha\gamma V_{t,k}^{\beta_k}$, $\frac{\alpha\gamma}{\omega^2} V_{t,k}^{-\beta_k}\|z_{t,k}\|_S^2$ and $\frac{\alpha^2\gamma}{\|\varepsilon\|^2} V_{t,k}^{\beta_k}\|d_{t,k}\|_P^2$, we get:

$$Q_{x,k}(t)f(x_t) = V_{t,k}^{\beta_k} \varsigma_{t,k}^\top \Psi \varsigma_{t,k} - 2\alpha\gamma V_{t,k}^{\beta_k} + \frac{\alpha\gamma}{\omega^2} V_{t,k}^{-\beta_k}\|z_{t,k}\|_S^2 + \frac{\alpha^2\gamma}{\|\varepsilon\|^2} V_{t,k}^{\beta_k}\|d_{t,k}\|_P^2,$$

where the matrix

$$\Psi := \begin{bmatrix} P(R - \frac{\omega^2}{\alpha\gamma} A_1 Z A_1^\top)P & bPA_1 & \varepsilon PBK^\top \\ bA_1^\top P & -\alpha\gamma S & O_n \\ \varepsilon KB^\top P & O_n & -\alpha^2\gamma P \end{bmatrix}$$

with $Z = S^{-1}$ is negative semidefinite due to (15b) and the Schur complement. Moreover, condition (15c) implies that $2\eta\|\cdot\|_S^2 \le \|\cdot\|^2$ and $2\alpha\|\cdot\|_P^2 \le \|\cdot\|^2$. Hence,

$$Q_{x,k}(t)f(x_t) \le -\alpha\gamma V_{t,k}^{\beta_k} - \tfrac{\alpha\gamma}{2\eta\omega^2}V_{t,k}^{-\beta_k}\big(\eta\omega^2 V_{t,k}^{2\beta_k} - \|z_{t,k}\|^2\big) - \tfrac{\alpha\gamma}{2\|\varepsilon\|^2}V_{t,k}^{\beta_k}\big(\|\varepsilon\|^2 - \|d_{t,k}\|^2\big).$$

If $\|z_{t,k}\| \le \sqrt{\eta}\omega V_{t,k}^{\beta_k} = \sqrt{\eta}e^{-\beta_1}V_{t,k}^{\beta_k}$ and $\|d_{t,k}\| \le \|\varepsilon\|$ for all $(V_{t,k}, x_t) \in \Omega_k$, then one can see

$$Q_{x,k}(t)f(x_t) \le -\alpha\gamma V_{t,k}^{\beta_k} \le \alpha V_{t,k}^{\beta_k} V_{t,k} Q_{V_t,k}(t) \le \alpha\beta_k \ln(e^{1/\beta_k}V_{t,k})V_{t,k}Q_{V_t,k}(t), \quad \forall (V_{t,k}, x_t) \in \Omega_k.$$

Since $\beta_2 \le |\beta_1| < 1$, then $e^{1/\beta_2} > e$ and $e^{1/\beta_1} < e^{-1}$, hence, condition $C_i7)$ holds with $\vartheta' = \alpha\beta_2$.

A.IV. Proof of $\|z_{t,k}\| \le \sqrt{\eta}\omega V_{t,k}^{\beta_k}$ in Ω_k

First of all, it has to be mentioned that sets Ω_k implicitly define that $V_{t,1} \le 1$ and $V_{t,2} > 1$, respectively. Furthermore, it yields for all $(V_{t,k}, x_t) \in \Omega_k$

$$\sqrt{\eta} \ge \|D_1(V_{t,1}^{-1/\chi}e^{1/\chi-1})x_t(-h)\| \ge (V_{t,1}/e)^{\frac{r_{1,1}(\chi-1)}{\chi}}\|z_{t,1}\|,$$
$$\sqrt{\eta} \ge \|D_2(V_{t,2}^{-\chi}e^{1-\chi})x_t(-h)\| \ge (V_{t,2}e)^{1-\chi}\|z_{t,2}\|.$$

Taking into account that $r_{1,1}\beta_2 = -\beta_1(1+\beta_2)$ and $|\beta_1| \ge \beta_2$, we deduce that the estimate indeed holds with $\chi = 1 + \beta$.

A.V. Proof of $\|d_{t,k}\| \le \|\varepsilon\|$ in Ω_k

Since $d_{t,k}$ depends only on time t we will omit subscripts t for the sake of shortness. Then adding and subtracting the term $V_k^{r_{i,k}w_i(\zeta)-1-\beta_k}y_{i,k}$ to $d_{i,k}$, we get

$$|d_{i,k}| \le V_k^{r_{i,k}w_i(\zeta)-1-\beta_k}\big||y_{i,k}|^{w_i(\zeta)} - |y_{i,k}|\big| + |y_{i,k}|\big|V_k^{r_{i,k}w_i(\zeta)-1-\beta_k} - 1\big|.$$

For the following analysis let us show that $\|x\|_P < 1$ and $\|x\|_P > M$ imply $V_2 < 1$ and $V_2 > \delta$, respectively. Indeed, $1 > x^\top Px = x^\top D_2(1)PD_2(1)x$ implies that $Q_2(1,x) < 0 = Q_2(V_2,x)$, therefore, $V_2 < 1$. Analogously, if $\lambda_{\min}(P^{-1/2}D_2(\delta)PD_2(\delta)P^{-1/2}) = M^2 < x^\top Px = x^\top D_2(\delta^{-1})P^{1/2}[P^{-1/2}D_2(\delta)PD_2(\delta)P^{-1/2}]P^{1/2}D_2(\delta^{-1})x$, then $Q_2(\delta,x) > 0 = Q_2(V_2,x)$ and $V_2 > \delta$. Besides, $\zeta(\|x\|_P) = \beta_1$ and $r_{i,1}w_i(\zeta) = 1 + \beta_1$ for all $V_1 \le 1$.

Then using Lemma 1 with $s = |y_{i,k}|$, $\bar{s} = \sqrt{\eta}$ (since $|y_{i,k}| \le \sqrt{\eta}$) and $\epsilon = w_i(\zeta)$, the term $\big||y_{i,k}|^{w_i(\zeta)} - |y_{i,k}|\big|$ can be bounded as follows:

$$\big||y_{i,k}|^{w_i(\zeta)} - |y_{i,k}|\big| \le \max\big\{\bar{g}(\sqrt{\eta}, \tfrac{1+\beta_1}{r_{i,1}}), \bar{g}(\sqrt{\eta}, \tfrac{1+\beta_2}{r_{i,2}})\big\}.$$

Secondly, it is clear that $V_k^{r_{i,k}w_i(\zeta)-1-\beta_k} = 1$ for all $V_1 \le 1$ and $V_2 > \delta$. On the other hand, it is easy to show that $r_{i,2}w_i(\zeta) - 1 - \beta_2 \le 0$ for all i. Moreover, since $\partial(r_{i,2}w_i(\zeta) - 1 - \beta_2)/\partial\zeta > 0$, then $\min_{\zeta\in[\beta_1,\beta_2]}(r_{i,2}w_i(\zeta) - 1 - \beta_2) = r_{i,2}w_i(\beta_1) - 1 - \beta_2 = p_i$. Therefore, we have $\delta^{p_i} \le V_2^{r_{i,2}w_i(\zeta)-1-\beta_2} \le 1$ and, finally, $|d_{i,k}| \le \varepsilon_i$. □

Appendix B
Proof of Proposition 9

Repeating the steps given in the proof of Proposition 8, one can see that conditions $C_i1)$-$C_i4)$ and $C_i6)$ are satisfied and, furthermore, condition $C_i7^*)$ holds with $\mu = \beta_1$ and $\nu = \beta_2$ if $\|z_{t,k}\| \le \sqrt{\eta}(1-\rho)^{-(1+\beta)/\beta}V_{t,k}^{\beta_k}$ and $\|d_{t,k}\| \le \|\varepsilon\|$ for all $(V_{t,k}, x_t) \in \Omega_k^*$. Since the latter is fulfilled (see the proof of Proposition 8), then we have to prove that the former holds. Recall that $|z_{t,i,k}| \le V_{t,k}^{-r_{i,k}}|\operatorname{sat}(x_{t,i}(0),1)||\operatorname{sat}(x_{t,i}(-h),b)|$.

For $k = 1$ we have for all $(V_{t,1}, x_t) \in \Omega_1^*$:

$$\|D_1((V_{t,1}^{-\mu} + \rho)^{1/\mu})x_t(-h)\| \le \sqrt{\eta}.$$

Since $|\operatorname{sat}(x_{t,i}(-h),b)| \le |x_{t,i}(-h)|$ then it is necessary to show that $V_{t,1}^{-r_{i,1}}|\operatorname{sat}(x_{t,i}(0),1)| \le (1-\rho)^{-(1+\nu)/\nu}(V_{t,1}^{-\mu} + \rho)^{r_{i,1}/\mu}V_{t,1}^{\mu}$ for all $V_{t,1} \in (0,1]$. We consider two cases: $V_{t,1} \in (0, \bar{V}_{t,1}]$ and $V_{t,1} \in (\bar{V}_{t,1}, 1]$, where $\bar{V}_{t,1} := \min\{1, \eta^{1/(2\mu)}\}$. Taking into account that $V_{t,1}^{-r_{i,1}}|\operatorname{sat}(x_{t,i}(0),1)| \le V_{t,1}^{-r_{i,1}}|x_{t,i}(0)| \le \sqrt{\eta}$, one can see that the inequality holds for all $V_{t,1} \in (0, \bar{V}_{t,1}]$ since it is satisfied for $V_{t,1} = \bar{V}_{t,1}$. On the other hand, since function $\phi_1(V_{t,1}^{\mu}) := (1+\rho V_{t,1}^{\mu})^{-r_{i,1}/\mu}$ is convex for all $V_{t,1} \in (\bar{V}_{t,1}, 1]$ and $|\operatorname{sat}(x_{t,i}(0),1)| \le 1$, it is enough to check the inequality only on boundaries, i.e., $(1+\rho)^{-r_{i,1}/\mu} \le (1-\rho)^{-(1+\nu)/\nu}$ and $(1+\rho\bar{V}_{t,1}^{\mu})^{-r_{i,1}/\mu} \le (1-\rho)^{-(1+\nu)/\nu}\bar{V}_{t,1}^{\mu}$. It is not difficult to verify that both conditions hold for all $\rho \in (0, \bar{\rho}]$, taking into account that $r_{1,1}\nu = -\mu(1+\nu)$.

For $k = 2$ we also consider two cases: $V_{t,2} \in (1, \bar{V}_{t,2})$ and $V_{t,2} \ge \bar{V}_{t,2}$, where $\bar{V}_{t,2} := (1-\rho)^{1/\nu}b^{1/r_{2,2}}$. Clearly, $V_{t,2}^{-r_{i,2}}|\operatorname{sat}(x_{t,i}(-h),b)| \le V_{t,2}^{-r_{1,2}}b \le (1-\rho)^{-(1+\nu)/\nu}V_{t,2}^{\nu}$ holds for all $V_{t,2} \ge \bar{V}_{t,2}$ since it is satisfied for $V_{t,2} = \bar{V}_{t,2}$. On the other hand, we have for all $(V_{t,2}, x_t) \in \Omega_2^*$:

$$\|D_2((\max\{0, V_{t,2}^{-\nu} - \rho\})^{1/\nu})x_t(-h)\| \le \sqrt{\eta}.$$

Hence, one has to prove that $V_{t,2}^{-r_{i,2}}|\operatorname{sat}(x_{t,i}(-h),b)| \le (1-\rho V_{t,2}^{\nu})^{-r_{i,2}/\nu} \le (1-\rho)^{-(1+\nu)/\nu}V_{t,2}^{\nu}$ for all $V_{t,2} \in (1, \bar{V}_{t,2})$. Since function $\phi_2(V_{t,2}^{\nu}) := (1-\rho V_{t,2}^{\nu})^{-r_{i,2}/\nu}$ is convex for all $V_{t,2}^{\nu} \in (1, \bar{V}_{t,2}^{\nu})$, it is enough to check the inequality only on boundaries, i.e., $(1-\rho)^{-r_{i,2}/\nu} \le (1-\rho)^{-(1+\nu)/\nu}$ and $(1-\rho\bar{V}_{t,2}^{\nu})^{-r_{i,2}/\nu} \le (1-\rho)^{-(1+\nu)/\nu}\bar{V}_{t,2}^{\nu}$. It is not difficult to verify that the last condition holds for all $\rho \in (0, \bar{\rho}]$. □

Appendix C
Proof of Corollary 10

Indeed, for ILRF $Q_0(V_0,x) = V_0^{-2}x^\top Px - 1$ conditions $C_i1)$-$C_i3)$ and $C_i5^\star)$ hold with $q_{1,0}(V_0,\|x\|) = V_0^{-2}\lambda_{\min}(P)\|x\|^2 - 1$, $q_{2,0}(V_0,\|x\|) = V_0^{-2}\lambda_{\max}(P)\|x\|^2 - 1$ and $c_0 = \sqrt{\lambda_{\min}(P)}$. Moreover, for all $V_0 \in \mathbb{R}_+^\star$ and $x \in \mathbb{R}^n \setminus \{0\}$ we have $Q_{V,0} := -2V_0^{-3}x^\top Px < 0$. Additionally, $V_0Q_{V,0} = -2$ for all $(V_0,x) \in \Omega_0$.

Taking into account that for (12) the term $d_{t,k}$ equals to zero, it follows from the proof of Proposition 8

$$Q_{x,0}(t)f(x_t) \le -2\alpha\gamma + \tfrac{\alpha\gamma}{\omega^2}V_{t,0}^{-2}\|\xi(x_t)\|_S^2 \le \vartheta' V_{t,0}Q_{V_t,0},$$

where $\vartheta' = \alpha\gamma/2$, if $\|\xi(x_t)\|_S \le \omega V_{t,0}$ for all $(V_{t,0}, x_t) \in \Omega_0$. Indeed, by definition of the set Ω_0 we have $Q_0(\kappa V_{t,0}, x_t(-h)) \le 0 \Leftrightarrow \|x_t(-h)\|_P \le \kappa V_{t,0}$. And since (15c) implies $\sqrt{2\eta}\|\xi(x_t)\|_S \le \|\xi(x_t)\| \le \|x_t(-h)\| \le \sqrt{\eta}\|x_t(-h)\|_P$, we conclude the proof with $\kappa = \omega\sqrt{2} > 1$. □