Valuative Criterion

For a klt Fano variety X,

X is k-stable $\Leftrightarrow$ for all the divisor E over X, $\beta_X(E) > 0$

" k-semistable " $\beta_X(E) \geq 0$

" k-polystable " $\beta_X(E) \geq 0$ and $\beta_X(E) = 0$ only when ord_E is a product valuation.

Throughout this talk, X klt Fano variety.

Def (β-invariant):

r: positive integer s.t. $-rK_X$ is Cartier.

v: $\mathbb{Z}$-valued valuation of X

$\beta_X(v) := A_X(v) - S_X(v)$

$A_X(v)$ log discrepancy.

$$S_X(v) := \lim_{m\to\infty} \sup \frac{\sum_{\lambda\in\mathbb{Z}} \lambda \cdot \dim \mathrm{gr}^{\lambda}_{F_v} H^0(X, -mrK_X)}{m \cdot \dim H^0(X, -mrK_X)}$$

(A decreasing filtration of V is

$\cdots \supseteq F^{\lambda}V \supseteq F^{\lambda+1}V \supseteq \cdots$.

$\mathrm{gr}^{\lambda}_{F_v} = F^{\lambda}V / F^{\lambda+1}V$)

$S_X(v)$ is actually a limit.

If $v = \mathrm{ord}_E$, $\beta_X(E) := A_X(E) - S_X(E)$

Prop 1 :
$\mu : Y \to X$ is a proper birational morphism
Y : regular
E : prime divisor over Y

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^\infty \mathrm{Vol}(-\mu^* K_X - tE)\, dt$$

(∞ can be replaced by the pseudo-effective threshold of $-K_X$ w.r.t. E
$\tau_L(E) := \sup\{ t \mid \mathrm{Vol}(L - tE) > 0 \}$)

Proof :
$$\sum_{\lambda \in \mathbb{Z}} \lambda \cdot \dim \mathrm{gr}_F^\lambda H^0(X, -m_r K_X)$$
$$= \sum_{\lambda \in \mathbb{Z}} \lambda \left(\dim F^\lambda H^0(X, -m_r K_X) - \dim F^{\lambda+1} H^0(X, -m_r K_X) \right)$$
$$= \sum_{\lambda=1}^{\infty} \dim F^\lambda H^0(X, -m_r K_X)$$
$$\| $$
$$H^0(Y, -\mu^* m_r K_X - \lambda E)$$
$$= \sum_{\lambda=1}^{\infty} \dim H^0(Y, -\mu^* m_r K_X - \lambda E)$$
$$= \int_0^\infty \dim H^0(Y, -\mu^* m_r K_X - \lceil t \rceil E)\, dt$$
$$= m_r \int_0^\infty \dim H^0(Y, -\mu^* m_r K_X - \lceil \tfrac{t}{m_r} \rceil E)\, dt$$

$$S_X(E) := \lim_{m\to\infty} \sup_\gamma \int_0^\infty \frac{\dim H^0(Y, -\mu^* m_\gamma K_X - \lfloor t m_\gamma \rfloor E)}{\dim H^0(X, -m_\gamma K)}\, dt$$

≤ 1

By definition of volume,

$$\lim_{m\to\infty} \sup \frac{\dim H^0(Y, -\mu^* m_\gamma K_X - \lfloor t m_\gamma \rfloor E)}{\dim H^0(X, -m_\gamma K)}$$

$$= \frac{\mathrm{Vol}(Y, -\mu^* m_\gamma K_X - tE)}{\mathrm{Vol}(-K_X)}$$

By dominant convergence Thm.

$$S_X(E) = \int_0^\infty \frac{\mathrm{Vol}(Y, -\mu^* m_\gamma K_X - tE)}{\mathrm{Vol}(-K_X)}\, dt$$

$= (-K_X)^n$, for $k(t$ Fano X of $\dim n$.

$$B_X(E) = A_X(E) - \int_0^\infty \frac{\mathrm{Vol}(Y, -\mu^* m_\gamma K_X - tE)}{\mathrm{Vol}(-K_X)}\, dt$$

Prop 2:
$(X, -Kx/\mathbb{A}^2)$: test configuration of klt Fano X with integral x_0.

$$DF(X, -Kx/\mathbb{A}^2) = \beta_x(Vx_0) = C\cdot(A_X(E) - S_X(E))$$
when $V = C\cdot ord_E$.

Proof: $DF(X, -Kx/\mathbb{A}^2) := -2F_1 = \dfrac{2(b_0a_1 - b_1a_0)}{a_0^2}$

$\dfrac{wt\, H^0(x_0, -mrKx/\mathbb{A}^2 | x_0)}{m\cdot \dim H^0(X, -mKx)} =: W_m$ with coeff. b_i $=: N_m$ .. a_i

$= F_0 + F_1 m^{-1} + \cdots$

$W_m = \chi(\bar{X}, m - \overline{Kx/\mathbb{A}^2}) - \chi(X, -mKx)$

$b_0 = \dfrac{(-\overline{Kx/\mathbb{A}^2})^{n+1}}{(n+1)!}$ $\Downarrow$ RR

Kf ample
$-K_f$
$V = -K_f$

$N_m = \dfrac{-Kx^n}{n!}m + \dfrac{-Kx\cdot(-Kx)^{n-1}}{2(n-1)!}m^{n-1} + O(m^{n-1})$

For Fano X,
$DF(X, -Kx/\mathbb{A}^2) = -F_0$

$$\mathrm{wt}\, H^0(\mathcal{X}_0, -mrK_{\mathcal{X}/\mathbb{A}^1}|_{\mathcal{X}_0})$$
$$:= \sum_{\lambda\in\mathbb{Z}} \lambda \cdot \dim \mathrm{gr}_F^{\lambda} H^0(X, -mrK_X)$$

For klt Fano X with integral $\mathcal{X}_0$

$$F^{\lambda}_{\mathcal{X},\mathcal{L}} H^0(X, mL) = F_{v_0}^{\lambda + m A_X(v_0)} H^0(X, mL)$$

$$= \sum_{\lambda\in\mathbb{Z}} \lambda \cdot \dim \mathrm{gr}_{F_v}^{\lambda + m A_X(v_0)} H^0(X, -mrK_X)$$

$$= \sum_{\lambda\in\mathbb{Z}} (\lambda - m A_X(v_0)) \dim \mathrm{gr}_{F_v}^{\lambda} H^0(X, -mrK_X)$$

$$= \sum_{\lambda\in\mathbb{Z}} \lambda \dim \mathrm{gr}_{F_v}^{\lambda} H^0(X, -mrK_X)$$
$$- m A_X(v_0) \sum_{\lambda\in\mathbb{Z}} \dim \mathrm{gr}_{F_v}^{\lambda} H^0(X, -mrK_X)$$

$$\dim H^0(X, -mrK_X)$$

$$\frac{\mathrm{wt}\, H^0(\mathcal{X}_0, -mrK_{\mathcal{X}/\mathbb{A}^1}|_{\mathcal{X}_0})}{m \cdot \dim H^0(X, -mK_X)}$$

$$= S_X(E) - A_X(v_0)$$

So $-F_0 = A_X(v_0) - S_X(E) = \beta_X(E)$

Ex: $X := Bl_p \mathbb{P}^2 \longrightarrow \mathbb{P}^2$, p pt.
$E :=$ exceptional divisor over X
$L :=$ pull back of a line on $\mathbb{P}^2$.

In general, $K_X = \pi^* K_{\mathbb{P}^2} + (n-1)E$, $n = \dim X$.

$$-K_X = 3L - E$$

Zariski decomposition ⇑ $= (3L-E-tE)^2$

$$S_X(E) = \frac{1}{(3L-E)^2} \int_0^2 Vol(3L-E-tE)\,dt$$

$$= \frac{1}{(9-1)} \int_0^2 (9-1-2t-t^2)\,dt$$

$$= \frac{28}{8}$$

$$\beta_X(E) = A_X(E) - S_X(E) = 1 - \frac{28}{3} < 0$$

X not K-stable