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Course: Grupos e Representações (MAT 889)
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Fitting subgroup

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November 14, 2019

1 Introduction

In this work, we will present the Fitting subgroup of a given group, named after Hans Fitting (1906-1938). We are mainly interested in the theory developed on finite groups, so we will deal only with such groups. The principal motivation to study the Fitting subgroup is the role played by this subgroup on the structure of solvable groups. How we will see, if G is nilpotent, then G is solvable, but the opposite is not true in general. However, the series builds on Fitting subgroups measures "how far" a solvable group is it to be nilpotent. As we will see this gives a criterion to determine if a group is solvable or not. When G isn't solvable a similar role is played by a generalization of Fitting subgroup, which has applications to classification of finite simple groups, but this runs away from our goals [12, page 223].

Before starting with the definition and properties of the Fitting subgroup, we dedicated to introducing, briefly, some basic concepts related to nilpotent groups. The notation that will be used throughout this work will be introduced in each definition. The following preliminaries not will be proved, however, such proofs could be found in respective citations.

A group G is called **nilpotent** if it admits a finite series

$$1 = G_0 < G_1 < G_2 < \cdots < G_n = G \tag{1}$$

in which $G_i \trianglelefteq G$ and $G_{i+1}/G_i \leqslant Z(G/G_i)$ for all i. Therefore a nilpotent group has nontrivial center. We call G **solvable** if it admits a series (1) in which G_{i+1}/G_i is abelian for all i. Note that every nilpotent group is solvable. Next follows some results about nilpotent that are often used in this work.

Theorem 1.1. *([7], Theorem 3.3 and 3.5)*

1. *Subgroups and homomorphic images of nilpotent groups are nilpotent.*
2. *Every p-group is nilpotent.*
3. *If $H \trianglelefteq G$ and G is a nilpotent group then G/H is nilpotent.*

4. *A group is nilpotent if and only if it is the direct product of it Sylows subgroups. In this case we have that each Sylow is a normal subgroup.*

Theorem 1.2 ([9], Theorem 5.1.2)**.** *Let G be a group, $Z \leq Z(G)$. Then G/Z is nilpotent if, and only if, G is nilpotent.*

Corollary 1.3. *Let G be a group, $H \leq Z(G)$ and $K \leq G$ such that $H \leq K$. Note that $H \leq Z(K)$. Then K/H is nilpotent if, and only if, K is nilpotent.*

Proof. Note that if $H \leq Z(G)$ and $H \leq K$, then $H \leq Z(K)$ and we can apply Theorem 1.2 to the quotient K/H. □

2 Fitting subgroup

In this section we shall derive some further properties of the *p-core* subgroups of G which are related to an important characteristic subgroup of G, the Fitting subgroup of G.

Definition 2.1. *Let G be a group and p be a prime. We define the* ***p*-core *subgroup*** *of G as $O_p(G) = \langle N \trianglelefteq G \mid N \text{ is } p\text{-subgroup of } G \rangle$.*

It follows from above definition that if G is a p-group then $O_p(G) = G$, and if $p \mid |G|$ and G is simple then $O_p(G) = 1$. On the other hand, if $p \nmid |G|$ we have $O_p(G) = 1$.

For a group G and a prime p we shall denote the set of Sylow p-subgroups of G by $Syl_p(G)$. We also set $\pi(G) = \{p \text{ prime} \mid p \text{ divides } |G|\}$. The next result shows some properties of $O_p(G)$ and could be found in [10, Theorem 2K] and [6, page 28].

Proposition 2.2. *For a group G and a prime p we have*

1. $O_p(G) = \bigcap_{P \in Syl_p(G)} P$;
2. $O_p(G)$ *char* G;
3. *$O_p(G)$ is the largest normal p-subgroup of G, that is, $O_p(G)$ is a normal p-subgroup of G and it contains all normal p-subgroups of G;*

Proof.

1. Let $X = \bigcap_{P \in Syl_p(G)} P$. Assume that $N \trianglelefteq G$ is a p-group. Let $P \in Syl_p(G)$. Then the subgroup NP also is a p-group because $|NP|$ divides $|N|\,|P|$. We have that $N \cap P \in Syl_p(N) = \{N\}$ (see Theorem 6.4(a) in [1]), so $N \leq P$. Thus $N \leq X$. By Definition 2.1 we see that $O_p(G) \leq X$. If $\alpha \in Aut(G)$, since automorphisms preserves orders, then $\alpha(P) \in Syl_p(G)$ for every $P \in Syl_p(G)$, following that

$$\alpha(X) = \bigcap_{P \in Syl_p(G)} \alpha(P) = \bigcap_{P \in Syl_p(G)} P = X.$$

Hence X *char* G and thus $X \trianglelefteq G$. Since X is an intersection of p-subgroups, X is a p-subgroup of G and $X \leq O_p(G)$.

2. Let $\alpha \in Aut(G)$. Since automorphisms preserves orders, we have $\alpha(P) \in Syl_p(G)$ for every $P \in Syl_p(G)$. From item *(1)* follows that

$$\alpha(O_p(G)) = \bigcap_{P \in Syl_p(G)} \alpha(P) = \bigcap_{P \in Syl_p(G)} P = O_p(G).$$

Hence X *char* G.

3. By the definition of $O_p(G)$ it is sufficient to show that $O_p(G)$ is a normal p-subgroup of G. By items (1) and (2) of the present proposition we have that $O_p(G) = \bigcap_{P \in Syl_p(G)} P$ is a normal p-subgroup of G and follows the result.

□

Definition 2.3. *The* ***Fitting subgroup*** *of a group G is defined as*

$$F(G) = \langle N \trianglelefteq G \mid N \text{ is nilpotent} \rangle .$$

We note that G is a nilpotent group if and only if $F(G) = G$, in particular if the group is abelian the same happens. For a trivial example consider the group Q_8 of quaternions. Note that Q_8 is a 2-group, so is nilpotent, hence $F(Q_8) = Q_8$.

One more interesting remark that we will use in the proof of Theorems 3.1 and 4.6 is:

Remark 2.4. *Suppose that $1 \neq G$ is solvable group. Let $G^{(m)}$ be the last nontrivial term of its derived series. We have that $G^{(m)}$ is a abelian and normal nontrivial subgroup of G, then $1 < G^{(m)} \leq F(G)$, that is, the Fitting subgroup is nontrivial.*

There are in the literature another definitions for the Fitting subgroup; see, for example, [15, page 81-82], [1, page 80], [6, page 29], [12, page 222], [5, page 5] and [8]. In the following proposition we exhibit and show the equivalence of some that definitions.

Proposition 2.5. *Let G be a group. The following statements are equivalent:*

1. $F(G) = \langle N \trianglelefteq G \mid N \text{ is nilpotent} \rangle$;
2. $F(G) = \langle O_p(G) \mid p \text{ is a prime} \rangle$;
3. $F(G) = \bigtimes_{p \in \pi(G)} O_p(G)$, *where* $\bigtimes$ *denotes the internal direct product;*
4. $F(G)$ *is the largest nilpotent normal subgroup of G, that is, $F(G)$ is a nilpotent normal subgroup of G and it contains all nilpotent normal subgroups of G.*

Proof. (1) ⇒ (2): Let $L = \langle O_p(G) \mid p \text{ is a prime} \rangle$. From Proposition 2.2 (2) we know that $O_p(G)$ is a normal p-subgroup of G for any prime p. Since every p-group is nilpotent, then $O_p(G)$ is nilpotent and $O_p(G) \leq F(G)$. Hence $L \leq F(G)$.

Now let N be a nilpotent normal subgroup of G. Since every nilpotent group is the direct product of its Sylow subgroups, we have $N = P_1 \times \cdots \times P_k$ where each P_i is the unique Sylow p_i-subgroup of N with $p_i \neq p_j$ for $1 \leq i \neq j \leq k$. It follows from item 1 of Proposition 2.2 that $O_{p_i}(N) = P_i$, then $N = O_{p_1}(N) \times \cdots \times O_{p_k}(N)$. By item 2 of Proposition 2.2 we have $O_{p_i}(N)$ *char* N and $O_{p_i}(N) \trianglelefteq G$ because $N \trianglelefteq G$, hence $O_{p_i}(N) \leq O_{p_i}(G)$. By the definition

of direct product $N = O_{p_1}(N) \cdots O_{p_k}(N)$, so $N \leq L$. It means that L contains every nilpotent normal subgroup of G, hence $F(G) \leq L$. Therefore $F(G) = L$.

(2) ⇒ (3): We assume that $F(G) = \langle O_p(G) \mid p \text{ is a prime} \rangle$. We note that we need consider only primes that divides $|G|$, so $F(G) = \langle O_p(G) \mid p \in \pi(G) \rangle$. Because the $O_p(G)$ are normal subgroups of G, their product $\prod_{p\in\pi(G)} O_p(G)$ is a finite product, following that $\prod_{p\in\pi(G)} O_p(G) \trianglelefteq G$ and in particular the order in which the factors are multiplied is irrelevant. Thus $F(G) = \langle O_p(G) \mid p \in \pi(G) \rangle = \prod_{p\in\pi(G)} O_p(G)$. Since the $O_p(G)$ are mutually of coprime order, so $O_p(G) \cap \prod_{q\neq p} O_q(G) = 1$, it follows that $F(G) = \prod_{p\in\pi(G)} O_p(G) = \bigtimes_{p\in\pi(G)} O_p(G)$.

(3) ⇒ (4): We assume *(3)*. If $F(G) = 1$ then $F(G) \trianglelefteq G$ and $F(G)$ is nilpotent. Now, we assume that $F(G) \neq 1$, then from *(3)* we have $F(G) = \prod_{i=1}^{m} O_{p_i}(G)$ where we consider $O_{p_i}(G) \neq 1$ and distinct primes $p_i \in \pi(G)$ for $1 \leq i \leq m$. By Proposition 2.2 (2) and the fact that $O_p(G) \cap O_q(G) = 1$ for primes $p \neq q$, we have $F(G) \trianglelefteq G$ with $|F(G)| = \prod_{i=1}^{m} |O_{p_i}(G)|$. On the other hand, since $F(G)$ is a finite group, we have $|F(G)| = q_1^{e_1} \cdots q_n^{e_n}$ with primes $q_k \neq q_l$ for $k \neq l$. It follows that $n = m$ and up to a permutation we have $p_i = q_i$ for $1 \leq i \leq n$, and thus $|O_{p_i}(G)| = p_i^{e_i}$. We get that, for each $p \in \pi(G)$, the subgroup $O_p(G)$ is the unique Sylow p-subgroup of $F(G)$. Thus every Sylow subgroup of $F(G)$ is normal, and hence $F(G)$ is nilpotent.

Now, we suppose that $N \trianglelefteq G$ is nilpotent, and we will show that $N \leq F(G)$. For $p \in \pi(N)$ let $Q_p \in Syl_p(N)$. Then Q_p *char* N and, since $N \trianglelefteq G$, we have $Q_p \trianglelefteq G$. Hence $Q_p \leq O_p(G)$. Since N is nilpotent, we have $N = \prod_{p\in\pi(N)} Q_p$ and it follows from item *(3)* that $N = \prod_{p\in\pi(N)} Q_p \leq \prod_{p\in\pi(G)} O_p(G) = F(G)$. Therefore we obtain *(4)*.

(4) ⇒ (1): We assume *(4)*. Let $Y = \langle N \trianglelefteq G \mid N \text{ is nilpotent} \rangle$. Since $F(G) \trianglelefteq G$ is nilpotent, then $F(G) \leq Y$. On the other hand, if $N \trianglelefteq G$ is nilpotent we have $N \leq F(G)$ by *(4)*, thus $Y \leq F(G)$. Therefore $F(G) = Y$. □

The following corollary is not completely obvious from first principles, but after showing these equivalences it has simple proof.

Corollary 2.6 (Fitting's Theorem)**.** *[8, Corollary 1] If H and K are nilpotent normal subgroups of a group G, then also $H \cdot K$ is a normal nilpotent subgroup of G.*

Remark 2.7. *The Corollary 2.6 also holds for a infinite group G, see [11, 5.2.8] and [12, Theorem 10.22].*

Proof of Corollary 2.6. Since H and K are normal subgroups of G, we have $HK \trianglelefteq G$. Each of H and K is contained in $F(G)$ and thus HK is a subgroup of the nilpotent group $F(G)$. It follows that HK is nilpotent. □

Corollary 2.8. *[5, page 5] The subgroup $F(G)$ may be described as the unique maximal nilpotent normal subgroup of G.*

Remark 2.9. *In Corollary 2.8 the concept of maximal is related to be normal and nilpotent, i.e., $F(G)$ is maximal among all normal nilpotent subgroups of G. Note that this concept came directly from the fact that $F(G)$ be the largest normal nilpotent. Sometimes, by abuse of language, we refer to these terms by "maximality".*

Proof. It follows from Proposition 2.5 (4) that $F(G)$ is a maximal nilpotent normal subgroup of G. Now, let M be a maximal nilpotent normal subgroup of G. Then $MF(G) \trianglelefteq G$ and, by

Corollary 2.6, $MF(G)$ is nilpotent. By the maximality of M and $F(G)$ we have $M = MF(G) = F(G)$. □

The properties presented before allows us to calculate the Fitting subgroup of some groups. In the next two examples, we calculate Fitting subgroup of S_n and D_n using the property of maximality, in the sense of Remark 2.9.

Example 2.10 (Fitting subgroup of S_n)**.** If $n = 2$, then $S_2 \simeq \mathbb{Z}_2$ is nilpotent, because is abelian. Thus, by maximality of Fitting subgroup, $F(S_2) = S_2$. We note that if $n \geq 3$, $Z(S_n) = 1$ and if $n \geq 4$, then $Z(A_n) = 1$, in particular, they are non-nilpotent [14, Lemma 1]. If $n = 3$, the unique nontrivial proper normal subgroup of S_3 is $A_3 = \{1, (123), (132)\} \simeq \mathbb{Z}_3$, which is nilpotent. Then, by maximality, $F(S_3) = A_3$.

If $n = 4$, all normal subgroups of S_4 are the trivial subgroup, $X = \langle (12)(34), (13)(24) \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, A_4 and S_4 [12, Section 8.1]. We have that X is abelian and A_4 and S_4 are non-nilpotent. Thus, by maximality of Fitting subgroup, $F(S_4) = X$.

If $n \geq 5$ the only normal subgroups of S_n are the trivial subgroup, A_n and S_n [14, Proposition 5], Proposition 5). Since A_n and S_n are non-nilpotent, $F(S_n) = 1$.

Example 2.11 (Fitting subgroup of D_n)**.** It is a well known result that D_n is nilpotent if, and only if, $n = 2^j$ for some $j \in \mathbb{N}$ [12, Problem 10.2]. In this case, $F(D_n) = D_n$.

Given that $D_n = \langle a, b \mid a^n = b^2 = 1\,, bab = a^{-1} \rangle$, 1 and the cyclic subgroups $\langle a^d \rangle$, where $d \,|\, n$, are normal. In addition, if n is even, $\langle a^2, ab \rangle$ and $\langle a^2, b \rangle$ are also normal [3, Theorem 3.8]. These are all proper and normal subgroups of D_n.

Suppose $n = 2^j m$ with $m \geq 3$ odd and $j \geq 0$. Note that $\langle a \rangle \leq F(D_n)$ because is normal and nilpotent. Since D_n isn't nilpotent in this cases, we have the following possibilities to $F(D_n)$: the trivial subgroup, $\langle a \rangle$ and if n is even, $G = \langle a^2, ab \rangle$ and $H = \langle a^2, b \rangle$. If n is odd, we can conclude that $F(D_n) = D_n$. If n is even we claim that $\langle a \rangle$ isn't contained in G or in H. Indeed, if $\langle a \rangle \leq G$, then $a \in G$ and $b \in G$. This implies $G = D_n$, which is a contradiction, because G is proper. If $\langle a \rangle \leq H$, then $a \in H$ and $a^{n-1}ab = b \in H$, this also implies $H = D_n$, which is a contradiction. Then $F(D_n) = \langle a \rangle \leq G$ if n is not a power of two.

3 Properties of Fitting subgroup

In this section, we will show some of the properties of the Fitting subgroup of a given group, including the relation between this subgroup and the Frattini subgroup. The next Theorem is a compile of results that could be found in [2, Corollary 2], [2, Lemma 4], [2, Theorem 3] and [9, Theorem 5.2.2].

Theorem 3.1. *Let G be a group. Then the following properties hold:*

1. $F(G)$ *char* G.
2. *If* $N \trianglelefteq G$, *then* $F(N) = N \cap F(G)$.
3. *If* $Z \leq Z(G)$, *then* $F\big(G/Z\big) = F(G)/Z$.
4. *If* $C = C_G(F(G))$, *then* $O_p\big(C/C \cap F\big) = 1$, *for all* $p \in \pi(G)$.

5. *If G is solvable, then $C_G(F(G)) \leq F(G)$.*

Proof.

1. Suppose $\alpha \in Aut(G)$. By item (2) of Theorem 2.5 and item (2) of Proposition 2.2:

$$\alpha(F(G)) = \prod_{p\in\pi(G)} \alpha(O_p(G)) = \prod_{p\in\pi(G)} O_p(G) = F(G)$$

2. Note that $F(N)$ is a nilpotent subgroup and since $F(N)\, char\, N \trianglelefteq G$, we have that $F(N) \trianglelefteq G$. This implies that $F(N) \leq N \cap F(G)$. On the other hand, $N \cap F(G) \trianglelefteq N$ and $N \cap F(G)$ is nilpotent because is subgroup of a nilpotent group, then $N \cap F(G) \leq F(N)$. Therefore, $N \cap F(G) = F(N)$.

3. Write $F(G/Z) = H/Z$. By the Correspondence Theorem, we have that $Z \leq H$. Since the Fitting subgroup is normal $H/Z \trianglelefteq G/Z$, hence $H \trianglelefteq G$. Since $Z \leq Z(G)$ and H/Z is nilpotent, then H is nilpotent (Corollary 1.3). Then $H \leq F(G)$. On the other hand, Z is a normal abelian subgroup of G, hence $Z \leq F(G)$. So, by Corollary 1.3, $F(G)/Z$ is nilpotent. Note that $F(G)/Z \trianglelefteq G/Z$, since $F(G) \trianglelefteq G$. This implies $F(G)/Z \leq F(G/Z) = H/Z$, thus $F(G) \leq H$. Then $F(G/Z) = F(G)/Z$.

4. Since $F(G) \trianglelefteq G$, we have that $C \trianglelefteq G$. By item (2) of the present Theorem, $F \cap C = F(C)$. Note that $F \cap C \leq Z(C)$, so by item (3) of the present Theorem, $F(C/C \cap F) = F(C)/C \cap F = 1_{C\cap F}$. Using item (2) of Proposition 2.5 we have that $O_p\big(C/C \cap F\big)$ must be the trivial group for all $p \in \pi(G)$.

5. By item (4) of the present Theorem, the Fitting subgroup of $C/C \cap F(G)$ is trivial. Since G is solvable, C and $C \cap F(G)$ are also solvable, thus $C/C \cap F(G)$ is solvable. By Remark 2.4, $C/C \cap F(G)$ must be trivial, so $C = C \cap F(G)$. Then $C = C_G(F(G)) \leq F(G)$.

□

Example 3.2 (Fitting subgroups of $PSL_n(\mathbb{F})$ and $SL_n(\mathbb{F})$)**.**

Recall that $PSL_n(\mathbb{F}) = SL_n(\mathbb{F})/Z(SL_n(\mathbb{F}))$ and consider $(n, |\mathbb{F}|) \neq (2,2), (2,3)$ with $\mathbb{F}$ a finite field. We have that $SL_n(\mathbb{F})$ is a perfect group (see [4], Theorem 4.2), in particular, $SL_n(\mathbb{F})$ is non-nilpotent. Note that $F(PSL_n(\mathbb{F}))$ is a normal subgroup, but $PSL_n(\mathbb{F})$ is simple, then $F(PSL_n(\mathbb{F})) = 1$ or $F(PSL_n(\mathbb{F})) = PSL_n(\mathbb{F})$. The second case would imply $SL_n(\mathbb{F})$ nilpotent, because

$$\frac{SL_n(\mathbb{F})}{Z(SL_n(\mathbb{F}))} = PSL_n(\mathbb{F}) = F(PSL_n(\mathbb{F})) = F\left(\frac{SL_n(\mathbb{F})}{Z(SL_n(\mathbb{F}))}\right) = \frac{F(SL_n(\mathbb{F}))}{Z(SL_n(\mathbb{F}))}$$

and $F(SL_n(\mathbb{F}))/Z(SL_n(\mathbb{F}))$ is nilpotent. Then $F(PSL_n(\mathbb{F})) = 1$. To calculate $F(SL_n(\mathbb{F}))$ we use item (3) of Proposition 3.1:

$$F\left(\frac{SL_n(\mathbb{F})}{Z(SL_n(\mathbb{F}))}\right) = F(PSL_n(\mathbb{F})) = 1, \text{ then } F(SL_n(\mathbb{F})) = Z(SL_n(\mathbb{F}))$$

The following Theorem will be useful to show some of Properties of Fitting subgroup, in special, the properties involving the Frattini subgroup.

Theorem 3.3. *[7, Theorem 1.4] Let M be a maximal subgroup of G and set $L = \bigcap_{x\in G} M^x$, so that $L \trianglelefteq G$. Consider the canonical projection $\pi : G \to G/L$, and set $\overline{G} = G/L$, $\overline{F} = F(\overline{G})$, and let $\overline{M}$ be the image of M in $\overline{G}$. Then either $\overline{F} = \overline{1}$ or the following conditions hold:*

(i) *$|G : M| = p^n$ for some n.*

(ii) *$\overline{F}$ is an elementary abelian p-group for some prime p.*

(iii) *$C_{\overline{G}}(\overline{F}) = \overline{F}$.*

(iv) *$\overline{F} \cap \overline{M} = \overline{1}$.*

(v) *$\overline{F}$ is a minimal normal subgroup of $\overline{G}$.*

Proof. First we will show that $|G : M| = p^n$ for some n. Assume that $\overline{F} \neq \overline{1}$, so $Z(\overline{F}) \neq \overline{1}$ since $\overline{F}$ is nilpotent. Let $\overline{P} = \{x \in Z(\overline{F}) |\ x^p = \overline{1}\}$ be a nontrivial p-subgroup of $Z(\overline{F})$, for some prime p dividing $|Z(\overline{F})|$. Because every element has prime order we have $\overline{P}$ *char* $Z(\overline{F})$ and $\overline{P}$ is abelian since $\overline{P} \subseteq Z(\overline{F})$. Hence $\overline{P}$ is an elementary abelian p-group. We get that

$$\overline{1} \neq \overline{P} \text{ char } Z(\overline{F}) \text{ char } \overline{F} = F(\overline{G}) \text{ char } \overline{G}$$

so $\overline{P} \trianglelefteq \overline{G}$. Note that $\overline{P} \cdot \overline{M} \leqslant \overline{G}$, where $\overline{M}$ is maximal in $\overline{G}$. Therefore if we show that $\overline{P} \nsubseteq \overline{M}$ then we can conclude that $\overline{G} = \overline{P} \cdot \overline{M}$. For this, we will show that $\overline{M}$ has no nontrivial normal subgroups of $\overline{G}$. Suppose $\overline{N} \neq \overline{1}$ is a subgroup of $\overline{M}$ which is normal in $\overline{G}$, then $\overline{N} \subseteq \overline{M}^x$ for all $x \in \overline{G}$, hence $\overline{N} \subseteq \bigcap_{x\in\overline{G}} \overline{M}^x = \bigcap_{g\in G} \overline{M}^{\pi(g)}$, so

$$N \cdot L = \pi^{-1}(\overline{N}) \subseteq \pi^{-1}\left(\bigcap_{g\in G} \overline{M}^{\pi(g)}\right) \subseteq \bigcap_{g\in G} \pi^{-1}(\overline{M}^{\pi(g)}) \subseteq \bigcap_{g\in G} M^g \cdot L = L$$

which implies $N \subseteq L$. Since $\overline{G} = G/L$ this gives $\overline{N} = \overline{1}$, a contradiction. Then $\overline{M}$ has no nontrivial normal subgroups of $\overline{G}$, in particular $\overline{P} \nsubseteq \overline{M}$, therefore $\overline{G} = \overline{P} \cdot \overline{M}$. Consequently $|\overline{G} : \overline{M}|$ divide $|\overline{P}|$, so it is a power of p. Because $\overline{G}/\overline{M} \simeq G/M$ we have $|G : M| = p^n$ for some n.

Now set $\overline{C} := C_{\overline{G}}(\overline{P}) = \{x \in \overline{G} \mid x^{-1}px = p,\ \forall p \in \overline{P}\}$. Since $\overline{P} \subseteq Z(\overline{F})$ we get $\overline{F} \subseteq \overline{C}$. Note that $\overline{P} \trianglelefteq \overline{G}$, so we have that $\overline{C} \trianglelefteq \overline{G}$, which implies $\overline{C} \cap \overline{M} \trianglelefteq \overline{M}$. We have that a group K centralizes another group H if and only if $[H, K] = 1$. Note that $\overline{P}$ centralizes $\overline{C}$ and also $\overline{C} \cap \overline{M}$. Hence $\overline{P} \leqslant C_{\overline{G}}(\overline{C} \cap \overline{M}) \subseteq N_{\overline{G}}(\overline{C} \cap \overline{M})$. Then $\overline{C} \cap \overline{M} \trianglelefteq \overline{P} \cdot \overline{M} = \overline{G}$, which implies that $\overline{C} \cap \overline{M} = \overline{1}$, by preceding paragraph. In particular $\overline{P} \cap \overline{M} = \overline{1}$. Therefore $\overline{G} = \overline{C} \cdot \overline{M} = \overline{F} \cdot \overline{M} = \overline{P} \cdot \overline{M}$. Since $\overline{P} \subseteq \overline{F} \subseteq \overline{C}$ and $|\overline{G}| = |\overline{C}||\overline{M}| = |\overline{P}||\overline{M}|$ we can conclude that $\overline{P} = \overline{F} = \overline{C}$.

This gives the result (ii) because $\overline{P} = \overline{F}$ and $\overline{P}$ is an elementary abelian p-group. Also the result (iii) and (iv), since $C_{\overline{G}}(\overline{F}) = C_{\overline{G}}(\overline{P}) = \overline{C} = \overline{F}$ and $\overline{F} \cap \overline{M} = \overline{C} \cap \overline{M} = \overline{1}$, respectively. To conclude suppose that $\overline{Q}$ is a nontrivial subgroup of $\overline{F} = \overline{P}$ which is normal at $\overline{G}$. Because $\overline{P} \nsubseteq \overline{M}$ we have $\overline{Q} \nsubseteq \overline{M}$, so $\overline{Q} \cdot \overline{M} = \overline{G} = \overline{P} \cdot \overline{M}$. Then $\overline{Q} = \overline{P}$. Consequently $\overline{F} = \overline{P}$ is a minimal normal subgroup of $\overline{G}$ and we have the result (v).

□

If G is a group, then its **Frattini subgroup** $\Phi(G)$ is defined as the intersection of all the maximal subgroups of G. An element $x \in G$ is called a **nongenerator** if it can be omitted from

any generating set: if $G = \langle x, Y \rangle$, then $G = \langle Y \rangle$. We have that the Frattini subgroup is the set of all nongenerators (see [13], Theorem 5.47). Moreover $\Phi(G)$ is a characteristic nilpotent subgroup of G (see [13], Theorem 5.48).

Theorem 3.4. *Let G be a group and $\Phi(G)$ the Frattini subgroup of G. Then the following properties hold:*

1. $[F(G), F(G)] \subseteq \Phi(G) \subseteq F(G)$.
2. $F(G/\Phi(G)) = F(G)/\Phi(G)$.

Proof.

1. The Frattini subgroup is normal and nilpotent, so $\Phi(G) \subseteq F(G)$. Consider the conditions of Theorem 3.3. The image of $F(G)$ is a nilpotent normal subgroup of $\overline{G}$, so $\pi(F(G)) \subseteq F(\overline{G})$. By the cited theorem $F(\overline{G})$ is abelian hence so is $\pi(F)$. Then $1 = \pi(F(G))' = \pi(F(G)')$, which implies $F(G)' \subseteq \ker(\pi) = L$, so $[F(G), F(G)] \subseteq L \subseteq M$. Since this holds for each maximal subgroup M of G we conclude that $[F(G), F(G)] \subseteq \Phi(G)$.
2. Consider the natural projection $f : G \to G/\Phi(G)$ and let K be the inverse image of $F(G/\Phi(G))$ in G. If $P \in Syl_p(K)$, then $f(P)$ is in $Syl_p(F(G/\Phi(G)))$ since f is a homomorphism. Because $F(G/\Phi(G))$ is a normal nilpotent subgroup of $G/\Phi(G)$ we have that $f(P) \trianglelefteq G/\Phi(G)$. Note that $\Phi(G) \cdot P = f^{-1}(f(P)) \trianglelefteq G$, because $f(P) \trianglelefteq G/\Phi(G)$. Since the Frattini subgroup is normal and nilpotent, we have $f(\Phi(G)) \leq F(G/\Phi(G))$, so $\Phi(G) \subseteq K$, which gives $P \in Syl_p(\Phi(G) \cdot P)$. By Frattini's argument $G = \Phi(G) \cdot P \cdot N_G(P) = \Phi(G)N_G(P)$ (see [13], Theorem 4.18). Because the Frattini subgroup is generated by the nongenerators of G, we have $G = N_G(P)$. Then P is normal in G. This is true for each Sylow subgroup of K, hence every Sylow subgroup of K is normal in $G/\Phi(G)$. Then K is nilpotent, hence $K \subseteq F(G)$. On the other hand $\Phi(G) \trianglelefteq G$ and in particular normal in $F(G)$, so $F(G)/\Phi(G)$ is nilpotent, by nilpotence of $F(G)$. Since the f preserves normality and $F(G)$ is normal, we have that $F(G)/\Phi(G)$ is a normal nilpotent subgroup of $G/\Phi(G)$, thus $F(G)/\Phi(G) \subseteq F(G/\Phi(G))$. Then $F(G)/\Phi(G) = F(G/\Phi(G))$.

□

4 Fitting Series

Until here we have seen the properties of Fitting subgroups and the relation that it has with the Frattini subgroup. Now we will give an application, which can be seen as a motivation to study the Fitting subgroup. In this section we will define the Fitting series, also called nilpotent series. The role played by this series for solvable groups is analogous to the central series for nilpotent groups. We will introduce the definition of the Fitting series and the concept of Fitting length, which intuitively measures "how far" a solvable group is it to be nilpotent. This last idea is given by the main theorem of this section.

Definition 4.1. *The **Fitting series**, or nilpotent series, of a group G is a normal series*

$$1 = G_0 \leq \cdots \leq G_n = G$$

such that G_{i+1}/G_i is nilpotent for each i.

Similar to central series we have **upper Fitting series** and **lower Fitting series**. For our purpose we will deal only with upper Fitting series, which, by abuse of language we just call of Fittings series. For a group G, let $F_0 = 1$, $F_1 = F(G)$ and F_{i+1} is such that $F(G/F_i) = F_{i+1}/F_i$. Hence F_{i+1}/F_i is nilpotent. Moreover, $F(G/F_i) \trianglelefteq G/F_i$ and $F_i \leq F_{i+1}$. Therefore $F_{i+1} \trianglelefteq G$ for all i.

$$\{e\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset G$$

Example 4.2. *(Fitting Series of S_n)* Using the computations in Example 2.10 we can construct the Fitting series for S_n. If $n = 2$, $F_0 = 1$, $F_1 = S_2$ and the series stabilizes in this term. For $n = 3$, $F_0 = 1$, $F_1 = A_3$ and $F_2/A_3 = F(S_3/A_3) = S_3/A_3$, because $S_3/A_3 \simeq \mathbb{Z}_2$. Then $F_2 = S_3$ and the series ends. For S_n with $n \geq 5$, $F_0 = 1$ and $F(S_n) = 1$.

For S_4, $F_0 = 1$, $F_1 = X$ and $F_2/X = F(S_4/X)$. Note that S_4/X is a group of order 6. This implies S_4/X is cyclic or S_3, but the first case would imply $A_4 = S_4' \leq X$, which is a contradiction. Then $S_4/X \simeq S_3$ and $F_2/X \simeq F(S_3) = A_3$. We have that F_2 has order 12, and a subgroup of order 12 must be normal in S_4, therefore $F_2 = A_4$. Finally, $F_3/F_2 = F_3/A_4 = F(S_4/A_4) = S_4/A_4$, so $F_3 = S_4$ and the series ends.

Example 4.3. *(Fitting Series of D_n)* Using the results from Example 2.11, if n is a power of two, $F_0 = 1$ and $F_1 = D_n$. If n isn't a power of two, $F_0 = 1$, $F_1 = \langle a \rangle$ and $F_2/F_1 = F_2/\langle a \rangle = F(D_n/\langle a \rangle) = D_n/\langle a \rangle$, because $D_n/\langle a \rangle \simeq \mathbb{Z}_2$. Then $F_2 = D_n$

Example 4.4. *(Fitting Series of $SL_n(\mathbb{F})$ and $PSL_n(\mathbb{F})$)* In the conditions of Example 3.2, the Fitting Series for $PSL_n(\mathbb{F})$ is trivial, because $F_1 = F(PSL_n(\mathbb{F})) = 1$. If we take $SL_n(\mathbb{F})$, $F_0 = 1$, $F_1 = Z(SL_n(\mathbb{F}))$ and $F_2/F_1 = F_2/Z(SL_n(\mathbb{F})) = F(SL_n(\mathbb{F})/Z(SL_n(\mathbb{F})) = F(PSL_n(\mathbb{F})) = 1$, then $F_2 = Z(SL_n(\mathbb{F}))$ and the series stabilizes in this subgroup.

The next theorem shows that Fitting series bounds any other nilpotent series in the same spirit of upper central series. A similar result is valid for lower nilpotent series [5, page 6].

Theorem 4.5. *Let G be a group. If $1 = G_0 \leq \cdots \leq G_n = G$ is a nilpotent series, then $G_i \leq F_i$ for all i.*

Proof. We proceed by induction on n. If $n = 0$, $G_0 = 1 = F_0$. Suppose true for $n = k$, that is, $G_k \leq F_k$, and define the map $\pi : G/G_k \to G/F_k$ such that $\pi(gG_k) = gF_k$. Since $G_k \leq G_{k+1} \trianglelefteq G$, $G_{k+1}/G_k \trianglelefteq G/G_k$ and, by hipothesis, G_{k+1}/G_k is nilpotent. Because G_{k+1}/G_k is nilpotent and normal in G/G_k, we can conclude that $\pi(G_{k+1}/G_k) \leq F(G/F_k)$. But $\pi(G_{k+1}/G_k) = G_{k+1}F_k/F_k$ and $F(G/F_k) = F_{k+1}/F_k$, thus $G_{k+1} \leq G_{k+1}F_k \leq F_{k+1}$ □

The smallest k for which $F_k = G$ is called **Fitting length**. The trivial group has Fitting length 0, and a group has Fitting length 1 if and only if it is nontrivial and nilpotent. If $F_k \neq G$ for every k we say that the group has no Fitting length or the Fitting length is infinite.

Theorem 4.6. *A group G is solvable if, and only if, has Fitting length.*

Proof. Suppose G is solvable. Since G is finite, its Fitting series stabilizes in some term, say $F_k = F_{k+1}$. This implies that $F(G/F_k) = F_{k+1}/F_k = 1$. Since G is solvable, G/F_k is also solvable. By Remark 2.4, G/F_k must be the trivial subgroup, then $G = F_k$ and the Fitting length is finite. Suppose Fitting length is finite. We claim that F_i is solvable for each i. By induction, if $i = 0$, $F_0 = 1$. For $i = k$, suppose F_k solvable. Since that F_{k+1}/F_k is nilpotent, is also solvable. Because F_{k+1}/F_k and F_k are solvable, the subgroup F_{k+1} is solvable. Thus F_i is solvable for all i. In particular there exists j such that $F_j = G$, therefore G is solvable. □

Applying Theorem 4.6 in examples 4.2, 4.3 and 4.4, we recover that D_n is solvable for all n while $SL_n(\mathbb{F})$ and $PSL_n(\mathbb{F})$ aren't solvable under those conditions. Moreover S_n is solvable if, and only if, $n = 2, 3$ or 4.

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