

Surface Integrals over Planes

We can write a surface integral of a VF as

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS \quad \text{where } \vec{n} \text{ is the unit normal vector}$$

If the surface S is a plane $ax+by+cz=d$, we know

$$\vec{n} = \frac{\langle a, b, c \rangle}{\|\langle a, b, c \rangle\|} \quad (\text{constant throughout } S)$$

We can plug this in for $\vec{n}$, but what is dS ?

dS depends on the parameterization but say we use the graph parameterization:

$$\text{if } ax+by+cz=d, \text{ then } \begin{cases} x=x \\ y=y \\ z = \frac{d}{c} - \frac{a}{c}x - \frac{b}{c}y \end{cases} \text{ with domain } D$$

$$\vec{r}(x,y) = \langle x, y, \frac{d}{c} - \frac{a}{c}x - \frac{b}{c}y \rangle$$

$$\vec{r}_x = \langle 1, 0, -a/c \rangle$$

$$\vec{r}_y = \langle 0, 1, -b/c \rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle \frac{a}{c}, \frac{b}{c}, 1 \rangle = \frac{1}{c} \langle a, b, c \rangle$$

$$\Rightarrow dS = \|\vec{r}_x \times \vec{r}_y\| dA = \left| \frac{1}{c} \right| \cdot \|\langle a, b, c \rangle\| dA$$

$$\text{So } \iint_S \vec{F} \cdot \vec{n} dS = \iint_D \vec{F}(\vec{r}(x,y)) \cdot \frac{\langle a, b, c \rangle}{\|\langle a, b, c \rangle\|} \cdot \left| \frac{1}{c} \right| \cdot \|\langle a, b, c \rangle\| dA$$

$$= \iint_D \vec{F}(\vec{r}(x,y)) \cdot \langle a, b, c \rangle \cdot \left| \frac{1}{c} \right| dA$$

$$(dA = dydx \text{ or } dx dy)$$