Reconstructing Functions on Finite Abelian Groups with Autocorrelations

Bobby Orozco (Advisor: Dr. Casper)

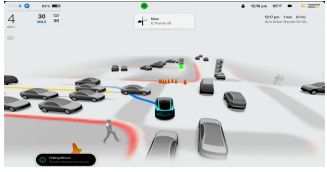
Department of Mathematics
California State University Fullerton

Janurary 6, 2026

Motivation

- Image recognition and computer vision
- Apple, Tesla, Microsoft, and Google are companies that want computers to understand images





Potential Problems



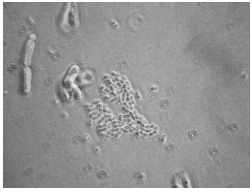
Counting number of holes in a picture

Potential Problems



Counting number of holes in a picture





Count the number of coins or the number of bacteria in images

Autocorrelations

Definition

Autocorrelations of a data set are correlations between a set of data and the shift of the data. In other words, the correlations between $f(x)$ and $f(x + y)$.

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Definition

In general, **higher-order autocorrelations** can be defined as

$$M_n(f; x_1, \ldots, x_{n-1}) = \sum_{y \in G} f(y) f(y + x_1) \cdots f(y + x_{n-1}),$$

for $n \geq 1$ and $x_1, \ldots, x_{n-1} \in G$, where G is a finite Abelian group.

Autocorrelations of an Image

Autocorrelations of an Image

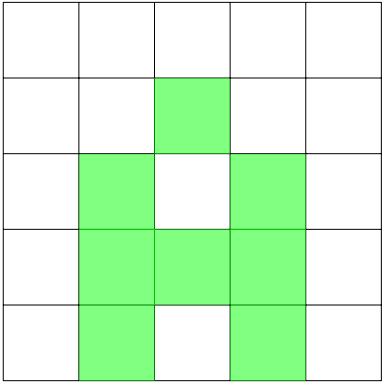


Image of the letter A

Autocorrelations of an Image

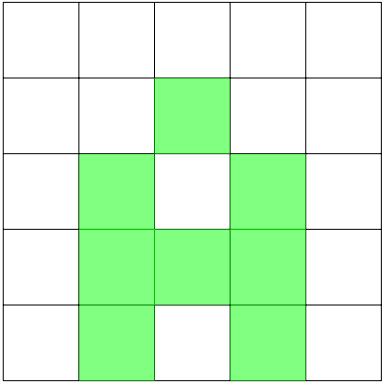


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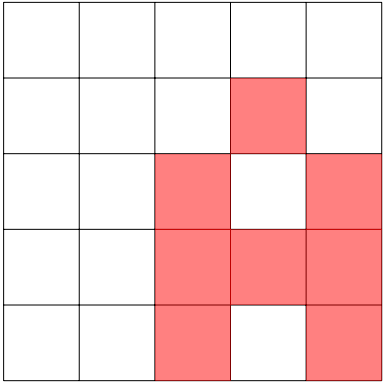


Image of the letter A shifted right by one

Autocorrelations of an Image

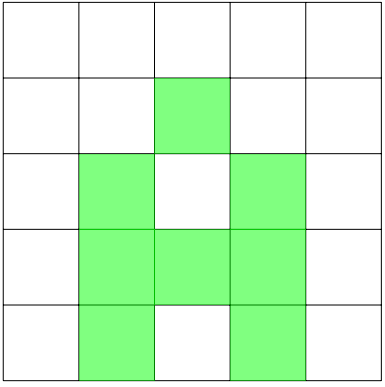


Image of the letter A

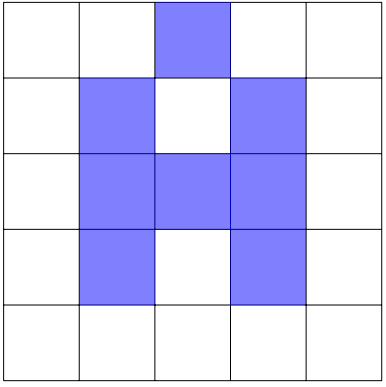


Image of the letter A shifted up by one

Autocorrelations of an Image

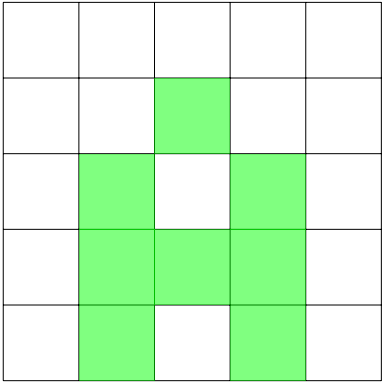


Image of the letter A

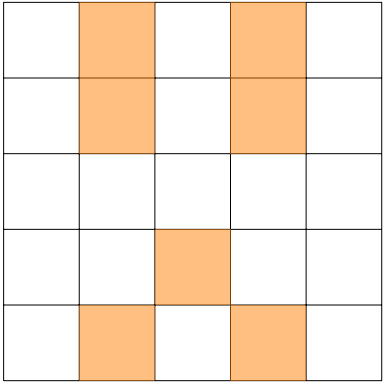


Image of the letter A shifted up by three

Computing the Matrix

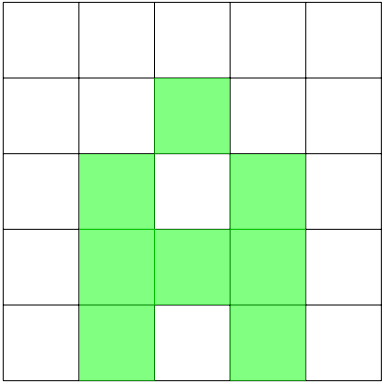


Image of the letter A

$$\begin{bmatrix} & \\ & \\ & \\ & \\ & \end{bmatrix}$$

Computing the Matrix

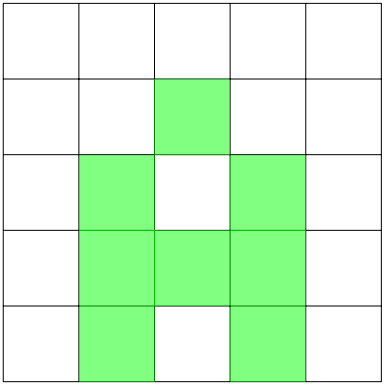
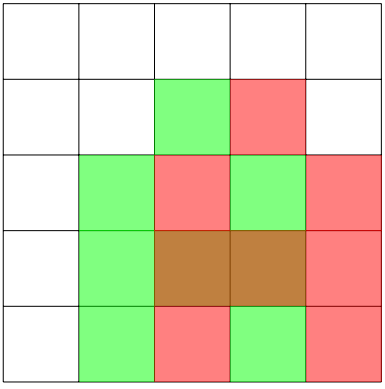


Image of the letter A

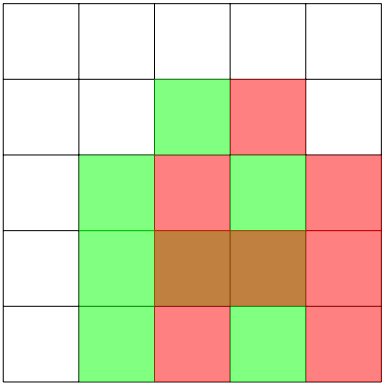
$$\begin{bmatrix} 8 & & \\ & & \\ & & \\ & & \end{bmatrix}$$

Computing the Matrix



Overlapping of the image of A and A shifted right by one

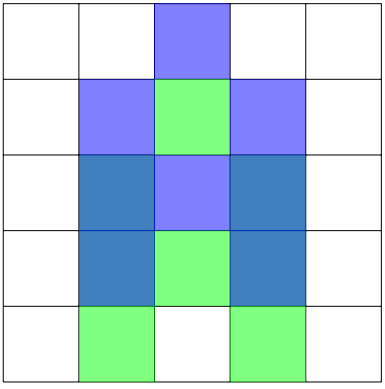
Computing the Matrix



Overlapping of the image of A and A shifted right by one

$$\begin{bmatrix} 8 & 2 & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

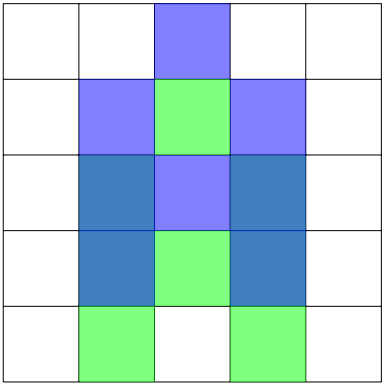
Computing the Matrix



Overlapping of the image of A and A shifted up by one

$$\begin{bmatrix} 8 & 2 & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Computing the Matrix



Overlapping of the image of A and A shifted up by one

$$\begin{bmatrix} 8 & 2 & & \\ & & & \\ & & & \\ 4 & & & \end{bmatrix}$$

Computing the Matrix

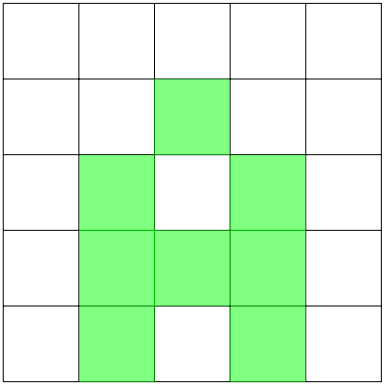


Image of the letter A

$$\begin{bmatrix} 8 & 2 \\ & \\ & \\ 4 & \end{bmatrix}$$

Computing the Matrix

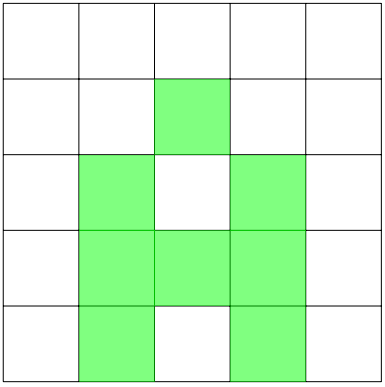


Image of the letter A

$$\begin{bmatrix} 8 & 2 & 3 & 3 & 2 \\ 4 & 3 & 2 & 2 & 3 \\ 3 & 2 & 1 & 1 & 2 \\ 3 & 2 & 1 & 1 & 2 \\ 4 & 3 & 2 & 2 & 3 \end{bmatrix}$$

Higher Autocorrelations of an Image

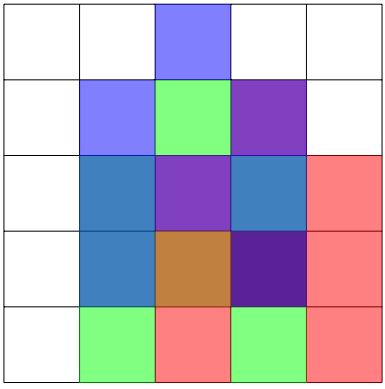


Image of the triple overlap

Our Problem

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Question Posed

Can we reconstruct a rational-valued function on a finite Abelian group G using its **autocorrelations**?

Why It Matters

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- Useful in computer vision applications, such as object tracking and video surveillance

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- There are also applications in spectroscopy (studying how matter interacts with light, X-rays, radio waves)
- Using autocorrelations when trying to figure out the structure of certain molecules using X-ray tomography

Main Theorem

Theorem

Let f be a rational-valued function on G where G is a finite Abelian group, r-dimensional grid with number of points being $|G|$. If $|G|$ is odd, then f can be reconstructed from its autocorrelations, up to order $2r + 2$. If $|G|$ is even, then f can be reconstructed with autocorrelations up to order $3r + 3$.

Discrete Fourier Transform

Definition

Let G be a finite Abelian group. The (non-normalized) **discrete Fourier transform** of a function $f : G \to \mathbb{C}$ is the function $\widehat{f} : G \to \mathbb{C}$ defined by

$$\widehat{f}(x) = \sum_{y \in G} f(y) \exp\left(-2\pi i \sum_{k=1}^{r} \frac{x[k] \cdot y[k]}{a_k}\right)$$

and where the a_k's are the invariant factors of G.

Field Automorphisms

Definition

Let F be a field. The **field automorphisms** are bijective functions $\sigma : F \to F$ which respects addition

$$\sigma(x + y) = \sigma(x) + \sigma(y), \quad \text{for all } x, y \in F,$$

and respects multiplication

$$\sigma(xy) = \sigma(x)\sigma(y), \quad \text{for all } x, y \in F.$$

Example

Example

$$M_2(f; ae_1 + be_2) = \begin{bmatrix} 24 & 18 & 16 & 16 & 16 & 16 & 18 \\ 17 & 17 & 14 & 14 & 14 & 14 & 17 \\ 10 & 10 & 11 & 10 & 10 & 11 & 10 \\ 5 & 6 & 7 & 8 & 8 & 7 & 6 \\ 5 & 6 & 7 & 8 & 8 & 7 & 6 \\ 10 & 10 & 11 & 10 & 10 & 11 & 10 \\ 17 & 17 & 14 & 14 & 14 & 14 & 17 \end{bmatrix}$$

Example

$$M_3(f; ae_1, be_1) = \begin{bmatrix} 24 & 17 & 10 & 5 & 5 & 10 & 17 \\ 17 & 17 & 10 & 5 & 0 & 5 & 10 \\ 10 & 10 & 10 & 5 & 0 & 0 & 5 \\ 5 & 5 & 5 & 5 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 5 & 5 & 5 \\ 10 & 5 & 0 & 0 & 5 & 10 & 10 \\ 17 & 10 & 5 & 0 & 5 & 10 & 17 \end{bmatrix}$$

Example

$$M_3(f; ae_2, be_2) = \begin{bmatrix} 24 & 18 & 16 & 16 & 16 & 16 & 18 \\ 18 & 18 & 15 & 13 & 13 & 13 & 15 \\ 16 & 15 & 16 & 13 & 12 & 12 & 13 \\ 16 & 13 & 13 & 16 & 13 & 12 & 13 \\ 16 & 13 & 12 & 13 & 16 & 13 & 13 \\ 16 & 13 & 12 & 12 & 13 & 16 & 15 \\ 18 & 15 & 13 & 13 & 13 & 15 & 18 \end{bmatrix}$$

Example

$$M_3(f; ae_1, be_2) = \begin{bmatrix} 24 & 18 & 16 & 16 & 16 & 16 & 18 \\ 17 & 14 & 12 & 12 & 12 & 12 & 14 \\ 10 & 7 & 7 & 8 & 8 & 7 & 7 \\ 5 & 2 & 3 & 4 & 4 & 3 & 2 \\ 5 & 2 & 2 & 2 & 2 & 2 & 2 \\ 10 & 7 & 5 & 5 & 5 & 5 & 7 \\ 17 & 14 & 11 & 11 & 11 & 11 & 14 \end{bmatrix}$$

Example

$$\widehat{M_2}(f; ae_1 + be_2) = \begin{bmatrix} 576 & 12.988 & 6.220 & 0.792 & 0.792 & 6.220 & 12.988 \\ 140.693 & 28.196 & 10.543 & 2 & 2 & 10.543 & 28.196 \\ 5.249 & 2 & 9.232 & 2.418 & 2.418 & 9.232 & 2 \\ 0.058 & 0.039 & 2 & 10.572 & 10.572 & 2 & 0.039 \\ 0.058 & 0.039 & 2 & 10.572 & 10.572 & 2 & 0.039 \\ 5.249 & 2 & 9.232 & 2.418 & 2.418 & 9.232 & 2 \\ 140.693 & 28.196 & 10.543 & 2 & 2 & 10.543 & 28.196 \end{bmatrix}$$

Example

$$\widehat{M_3}(f; ae_1, be_1) = \begin{bmatrix} 336 & 113.966 & 12.380 & 13.654 & 13.654 & 12.380 & 113.966 \\ 113.966 & -5.912 & 7 & -7.939 & 7 & -5.912 & 113.966 \\ 12.380 & 7 & -0.149 & -0.149 & 7 & 12.380 & -5.912 \\ 13.654 & -7.939 & -0.149 & -7.939 & 13.654 & 7 & 7 \\ 13.654 & 7 & 7 & 13.654 & -7.939 & -0.149 & -7.939 \\ 12.380 & -5.912 & 12.380 & 7 & -0.149 & -0.149 & 7 \\ 113.966 & 113.966 & -5.912 & 7 & -7.939 & 7 & -5.912 \end{bmatrix}$$

Example

$$\widehat{M_3}(f; ae_2, be_2) = \begin{bmatrix} 720 & 38.778 & 22.193 & 13.030 & 13.030 & 22.193 & 38.778 \\ 38.778 & -1.604 & -1 & 1.110 & -1 & -1.604 & 38.778 \\ 22.193 & -1 & 4.494 & 4.494 & -1 & 22.193 & -1.604 \\ 13.030 & 1.110 & 4.494 & 1.110 & 13.030 & -1 & -1 \\ 13.030 & -1 & -1 & 13.030 & 1.110 & 4.494 & 1.110 \\ 22.193 & -1.604 & 22.193 & -1 & 4.494 & 4.494 & -1 \\ 38.778 & 38.778 & -1.604 & -1 & 1.110 & -1 & -1.604 \end{bmatrix}$$

Example

$$\widehat{M_3}(f; ae_1, be_2) = \begin{bmatrix} 444 & 38.372 & 30.934 & 16.694 & 16.694 & 30.934 & 38.372 \\ 179.629 - 15.480i & 17.548 + 9.652i & 4.692 & -0.970 - 1.912i & -0.970 - 1.912i & 4.692 & 17.548 + 9.652i \\ 20.146 + 5.130i & 0.044 - 3.707i & 1.800 + 1.142i & 4.357 & 4.357 & 1.800 + 1.142i & 0.044 - 3.707i \\ 12.225 + 0.233i & -0.049 & 2.925 - 0.327i & 3.652 + 0.210i & 3.652 + 0.210i & 2.925 - 0.327i & -0.049 \\ 12.225 - 0.233i & -0.049 & 2.925 + 0.327i & 3.652 - 0.210i & 3.652 - 0.210i & 2.925 + 0.327i & -0.049 \\ 20.146 - 5.130i & 0.044 + 3.707i & 1.800 - 1.142i & 4.357 & 4.357 & 1.800 - 1.142i & 0.044 + 3.707i \\ 179.629 + 15.480i & 17.548 - 9.652i & 4.692 & -0.970 + 1.912i & -0.970 + 1.912i & 4.692 & 17.548 - 9.652i \end{bmatrix}$$

Example

$$\widehat{f}(e_j)^{2^{\phi(N)}-1} = \prod_{k=0}^{\phi(N)-1} \left(\frac{\widehat{M}_3(f; 2^k e_j, 2^k e_j)}{\widehat{M}_2(f; 2^{k+1} e_j)} \right)^{2^{\phi(N)-1-k}}$$

Example

$$\widehat{f}(e_j)^7 = \prod_{k=0}^{2} \left(\frac{\widehat{M}_3(f; 2^k e_j, 2^k e_j)}{\widehat{M}_2(f; 2^{k+1} e_j)} \right)^{2^{2-k}}$$

Example

$$\widehat{f}(e_1)^7 = \prod_{k=0}^{2} \left(\frac{\widehat{M}_3(f; 2^k e_1, 2^k e_1)}{\widehat{M}_2(f; 2^{k+1} e_1)} \right)^{2^{2-k}} = 30795559.2510875 - 11951007.0711647i$$

Example

$$\widehat{f}(e_1)^7 = \prod_{k=0}^{2} \left(\frac{\widehat{M}_3(f; 2^k e_1, 2^k e_1)}{\widehat{M}_2(f; 2^{k+1} e_1)} \right)^{2^{2-k}} = 30795559.2510875 - 11951007.0711647i$$

$$\widehat{f}(e_2)^7 = \prod_{k=0}^{2} \left(\frac{\widehat{M}_3(f; 2^k e_2, 2^k e_2)}{\widehat{M}_2(f; 2^{k+1} e_2)} \right)^{2^{2-k}} = 7895.6598724$$

Example

Find $\alpha_k = \sum_{j=0}^{6} c_{kj} \xi_7^j$ satisfying $\alpha_k^7 = \widehat{f}(e_j)^7$ for both e_1 and e_2.

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Find $\alpha_k = \sum_{j=0}^{6} c_{kj}\xi_7^j$ satisfying $\alpha_k^7 = \widehat{f}(e_j)^7$ for both e_1 and e_2.

We found $\alpha_1 = 11.8448 - 0.62698i$, & $\alpha_2 = 3.60388$.

Example

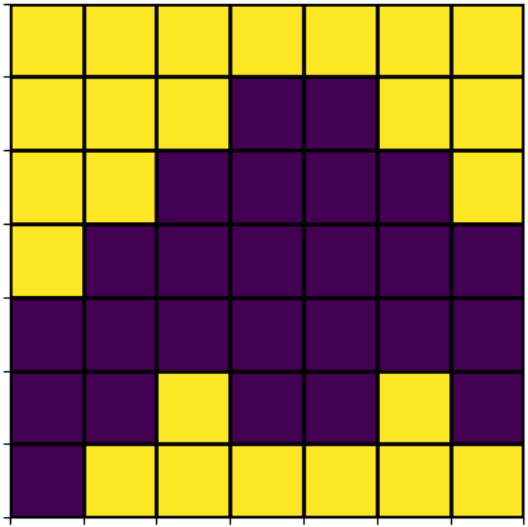
$$\overline{\widehat{g}((a,b))} = \frac{\widehat{M}_3(f; ae_1, be_2)}{\sigma_a(\alpha_1)\sigma_b(\alpha_2)}$$

$$= \begin{bmatrix}
24 & -3.25 - 1.56i & -1.55 - 1.95i & -0.20 - 0.87i & -0.20 + 0.87i & -1.55 + 1.95i & -3.25 + 1.56i \\
-3.25 - 11.41i & 5.30 + 0.39i & -3.25 & 1.02 - 0.97i & -1.35 + 0.43i & 0.72 + 3.17i & 3.60 - 3.90i \\
-1.55 - 1.68i & -1.18 - 0.78i & -1.14 - 2.82i & 1.40 - 0.67i & -1.55 & -2.49 + 1.74i & -1.35 + 0.43i \\
-0.20 + 0.14i & -0.12 - 0.15i & -1.18 + 0.78i & -2.16 - 2.43i & 0.89 + 3.13i & 1.02 + 0.97i & -0.20 \\
-0.20 - 0.14i & -0.20 & 1.02 - 0.97i & 0.89 - 3.13i & -2.16 + 2.43i & -1.18 - 0.78i & -0.12 + 0.15i \\
-1.55 + 1.68i & -1.35 - 0.43i & -2.49 - 1.74i & -1.55 & 1.40 + 0.67i & -1.14 + 2.82i & -1.18 + 0.78i \\
-3.25 + 11.41i & 3.60 + 3.90i & 0.72 - 3.17i & -1.35 - 0.43i & 1.02 + 0.97i & -3.25 & 5.30 - 0.39i
\end{bmatrix}$$

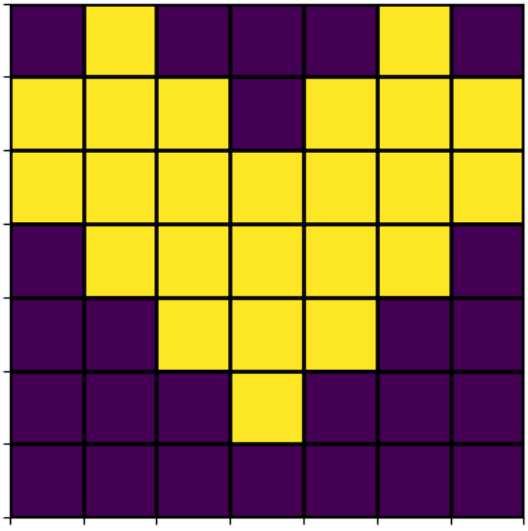
Example

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Example



Example

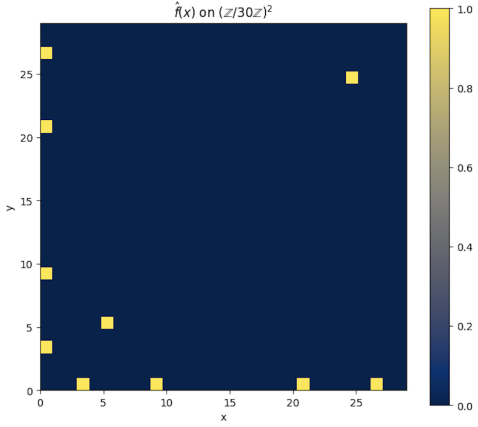


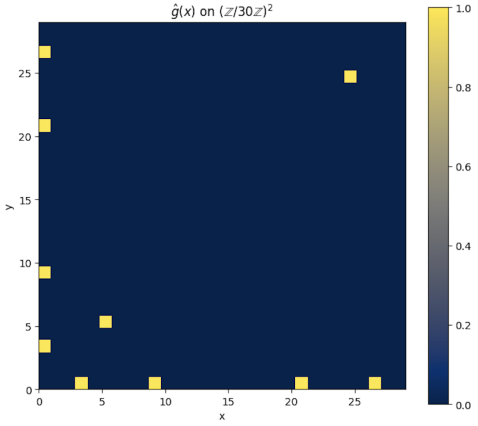
Worst Case When Things Go Wrong

Consider the group $G = (\mathbb{Z}/30\mathbb{Z})^2$. Let the two-dimensional discrete Fourier transforms of f and g be defined by: $\widehat{f}(k,0) = \widehat{f}(0,k) = \widehat{g}(k,0) = \widehat{g}(0,k) = 1$ for $k \in \{3, 9, 21, 27\}$; $\widehat{f}(5,5) = \widehat{f}(25,25) = 1$; $\widehat{g}(5,5) = \widehat{g}(25,25) = -1$; and $\widehat{f}(x) = \widehat{g}(x) = 0$ elsewhere.

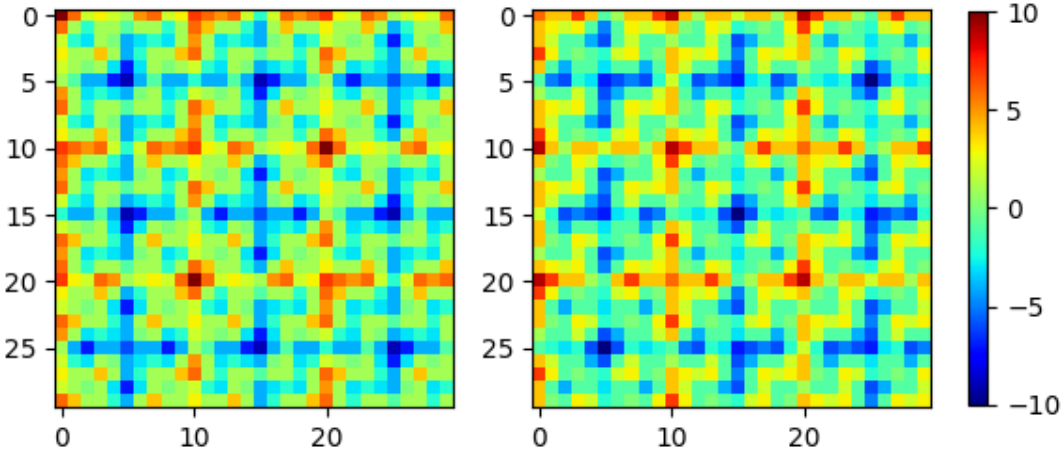
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Worst Case When Things Go Wrong



Potential Directions For Future Research

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- Can we consider the case where the functions are not rational-valued, but instead real or complex-valued? More generally, is it possible to carry this out over a different field?

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- Generalize the reconstruction framework from finite abelian groups to more general structures, such as topological groups, rings, or even modules over rings

Additional Resources


H. Bartelt, Adolf W. Lohmann, and B Wirnitzer.
Phase and amplitude recovery from bispectra.
Applied optics, 23(18):3121–3129, 1984.


W Riley Casper and Bobby Orozco.
Reconstructing gridded data from higher autocorrelations.
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F Alberto Grünbaum and Calvin C. Moore.
The use of higher-order invariants in the determination of generalized patterson cyclotomic sets.
Acta Crystallographica Section A: Foundations of Crystallography, 51(3):310–323, 1995.


Akinori Hidaka, Takio Kurita, and Nobuyuki Otsu.
Object detection by selective integration of hlac mask features.
In *2008 Bio-inspired, Learning and Intelligent Systems for Security*, pages 46–50, 2008.


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Paper

