

The determinant of a 3×3 matrix.

Expanding along the first row:

$$\det \begin{pmatrix} \begin{bmatrix} \textcolor{red}{a} & \textcolor{red}{b} & \textcolor{red}{c} \\ d & e & f \\ g & h & k \end{bmatrix} \end{pmatrix} = \textcolor{red}{a} \begin{vmatrix} e & f \\ h & k \end{vmatrix} - \textcolor{red}{b} \begin{vmatrix} d & f \\ g & k \end{vmatrix} + \textcolor{red}{c} \begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ek - fh) - b(dk - fg) + c(dh - eg)$$

The first term is the first entry in Row 1 times the determinant of the matrix you get by deleting the row and column containing that entry.

The terms alternate sign, so the second term is $-b$ times the determinant of the matrix you get by deleting the 1st row and 2nd column.

The determinant of a 4×4 matrix, expanding along the first row.

$$\det \begin{pmatrix} \begin{bmatrix} \textcolor{red}{a} & \textcolor{red}{b} & \textcolor{red}{c} & \textcolor{red}{d} \\ e & f & g & h \\ j & k & l & m \\ q & r & s & t \end{bmatrix} \end{pmatrix} = \textcolor{red}{a} \begin{vmatrix} f & g & h \\ k & l & m \\ r & s & t \end{vmatrix} - \textcolor{red}{b} \begin{vmatrix} e & g & h \\ j & l & m \\ q & s & t \end{vmatrix} + \textcolor{red}{c} \begin{vmatrix} e & f & h \\ j & k & m \\ q & r & t \end{vmatrix} - \textcolor{red}{d} \begin{vmatrix} e & f & g \\ j & k & l \\ q & r & s \end{vmatrix}$$

Example: Compute $\det \begin{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 3 & 1 & -1 \\ 1 & 2 & 1 & -1 \\ 1 & -1 & 3 & 0 \end{bmatrix} \end{pmatrix}$.

Computing Determinants using Cofactor Expansion.

You can evaluate a determinant by expanding along any row or column (called the cofactor expansion along that row or column).

Definitions. Let A be an $n \times n$ matrix.

- (a) The **(i, j) -minor** of A , denoted A_{ij} , is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i th row and j th column.
- (b) The **(i, j) -cofactor** of A , denoted C_{ij} , is given by the formula

$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$

The sign of the cofactor C_{ij} depends on the position of the entry a_{ij} in the matrix. (Here a_{ij} is the entry in the i th row and j th column.)

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \\ + & - & + & - & + & \\ - & + & - & + & - & \\ \vdots & & & & & \ddots \end{bmatrix}$$