

The Integral Test

Given a series $\sum_{n=k}^{\infty} a_n$, if $f(x)$ is a function such that $f(n) = a_n$ for each $n \geq k$ and:

- f is continuous
 - f is positive
 - f is decreasing
- } on $[k, \infty)$

then $\sum_{n=k}^{\infty} a_n$ converges if and only if

$\int_k^{\infty} f(x) dx$ converges.

Important:

This means if $\int_k^{\infty} f(x) dx$ converges then

can conclude $\sum_{n=k}^{\infty} a_n$ converges

Also, if $\int_k^{\infty} f(x) dx$ diverges, then

can conclude $\sum_{n=k}^{\infty} a_n$ diverges.

ex: $\sum_{n=1}^{\infty} n e^{-n^2}$ converges or diverges?

Use Int. Test: Let $f(x) = x e^{-x^2}$.

Check f is continuous, positive,
decreasing for $x \geq 1$.

← don't need to prove

not necessary

• $f(x) = x e^{-x^2}$ is continuous b/c it's a product of continuous functions ✓

• $x > 0$ for all $x \geq 1$
 $e^{-x^2} > 0$ for all x

So $f(x) > 0$ for all $x \geq 1$ ✓

• Show decreasing using the derivative.

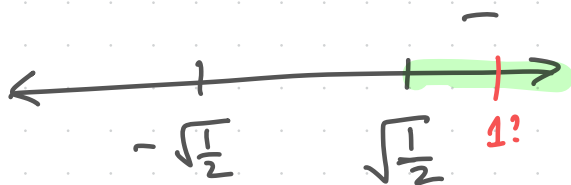
$$\begin{aligned} f'(x) &= e^{-x^2} + x(e^{-x^2}(-2x)) \\ &= e^{-x^2} - 2x^2 e^{-x^2} \\ &= e^{-x^2}(1 - 2x^2) \end{aligned}$$

$$f'(x) = e^{-x^2}(1 - 2x^2) = 0$$

$$e^{-x^2} = 0 \Rightarrow \text{no solutions} \quad \text{or} \quad 1 - 2x^2 = 0$$

$$2x^2 = 1 \quad x^2 = \frac{1}{2}$$

$$x = \pm \sqrt{\frac{1}{2}}$$



sign of f' on $(\sqrt{\frac{1}{2}}, \infty)$

test pt: $f'(1) = \underbrace{e^{-100}}_{\text{always positive}} (1 - 200) < 0$

f is decreasing on $(\sqrt{\frac{1}{2}}, \infty)$
so $[1, \infty)$.

We showed the 3 conditions are true.

So we can use the Integral Test.

$$\int_1^{\infty} f(x) dx = \int_1^{\infty} x e^{-x^2} dx$$

$$= \lim_{t \rightarrow \infty} \int_1^t x e^{-x^2} dx$$

$$\boxed{\begin{array}{l} u = -x^2 \\ du = -2x dx \end{array}}$$

$$x=1 \Rightarrow u=-1$$

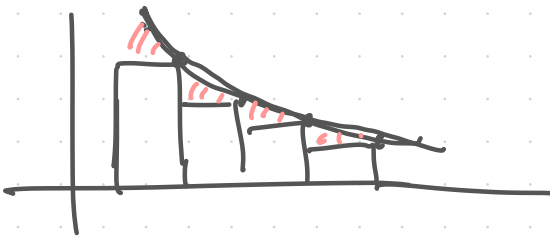
$$x=t \Rightarrow u=-t^2$$

$$\begin{aligned}
&= \lim_{t \rightarrow \infty} \int_{-1}^{-t^2} e^u \left(-\frac{1}{2}\right) du \\
&= -\frac{1}{2} \lim_{t \rightarrow \infty} \int_{-1}^{-t^2} e^u du \\
&= -\frac{1}{2} \lim_{t \rightarrow \infty} (\underbrace{e^{-t^2}}_{\downarrow 0} - e^{-1}) \\
&= -\frac{1}{2} (0 - e^{-1}) \\
&= -\frac{1}{2} \left(-\frac{1}{e}\right) \\
&= \frac{1}{2e} \quad \text{converges}
\end{aligned}$$

We got a finite number $\Rightarrow$ improper integral converges.

{ Therefore, the series converges by the Integral Test.

Note: the series is not equal to $\frac{1}{2e}$.



ex: $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ converges or diverges?

Int Test: Let $f(x) = \frac{\ln x}{x}$

continuous on $(2, \infty)$ ✓

positive on $(2, \infty)$ ✓

decreasing?

$$f'(x) = \frac{\left(\frac{1}{x}\right)x - \ln x \cdot 1}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

f decreasing if

$$1 - \ln x < 0$$

$$1 < \ln x$$

$$e^1 < e^{\ln x} = x$$

$$\Rightarrow x > e \approx 2.7$$

negative if numerator is negative

So f is decreasing for $x > e$, which is good enough. ✓

$$\int_2^{\infty} \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{\ln x}{x} dx$$

$$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array}}$$

$$= \lim_{t \rightarrow \infty} \int_{\ln 2}^{\ln t} u \, du$$

$$= \lim_{t \rightarrow \infty} \left[\frac{1}{2} u^2 \right]_{\ln 2}^{\ln t}$$

$$= \lim_{t \rightarrow \infty} \frac{1}{2} (\ln t)^2 - \frac{1}{2} (\ln 2)^2$$

$$= \infty - \frac{1}{2} (\ln 2)^2$$

$$= \infty$$

The improper integral diverges, so the series diverges by the Integral Test.

p-Series

$$\int_1^{\infty} \frac{1}{x} dx \text{ diverges}$$

$$\int_1^{\infty} \frac{1}{x^2} dx \text{ converges}$$

Recall: $\int_1^{\infty} \frac{1}{x^p} dx$ converges if $p > 1$
diverges if $p \leq 1$

A p-series is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

By the Integral Test, a p-series converges if $p > 1$ and diverges if $p \leq 1$.

p-series Test

ex: $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$$p = \frac{3}{2} > 1$$

so $\sum_n$ converges.
the series

ex: $\sum_{n=1}^{\infty} n^{-5} = \sum_{n=1}^{\infty} \frac{1}{n^5}$

$$p = 5 > 1$$

so converges.