

Approximating Sums and Remainder Estimates

Suppose $\sum_{n=1}^{\infty} a_n$ converges, so has some value S .

$$S = \sum_{n=1}^{\infty} a_n = \underbrace{(a_1 + a_2 + \dots + a_N)}_{\substack{\text{Nth partial sum} \\ S_N = \sum_{n=1}^N a_n}} + \underbrace{a_{N+1} + a_{N+2} + \dots}_{\substack{\text{remainder} \\ R_N = \sum_{n=N+1}^{\infty} a_n}}$$

$$S = S_N + R_N$$

Diagram illustrating the components of the equation $S = S_N + R_N$:

- S is labeled "true value" with a red arrow pointing to it.
- S_N is labeled "approximation" with a red arrow pointing to it.
- R_N is labeled "error" with a red arrow pointing to it.

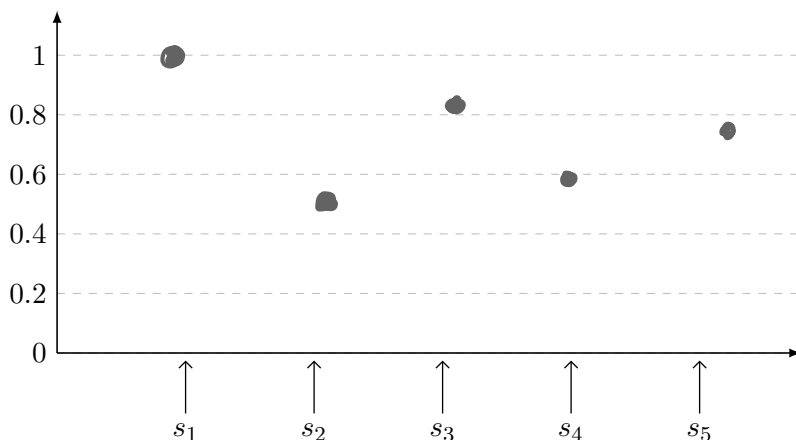
$$S - S_N = R_N \quad \leftarrow \text{size of the remainder is the size of the error}$$

We'll use Remainder Estimates to find out what the max error is when using S_N to approximate S .

Goal: To estimate the value of a convergent alternating series when we can't find its precise value.

1. We will try to estimate the value of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ using partial sums.

(a) Compute and plot the first five partial sums on the following graph.



$$s_1 = 1$$

$$s_2 = 1 - \frac{1}{2} = 0.5$$

$$s_3 = 1 - \frac{1}{2} + \frac{1}{3} \approx 0.83$$

$$s_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \approx 0.5833$$

$$s_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \approx 0.7833$$

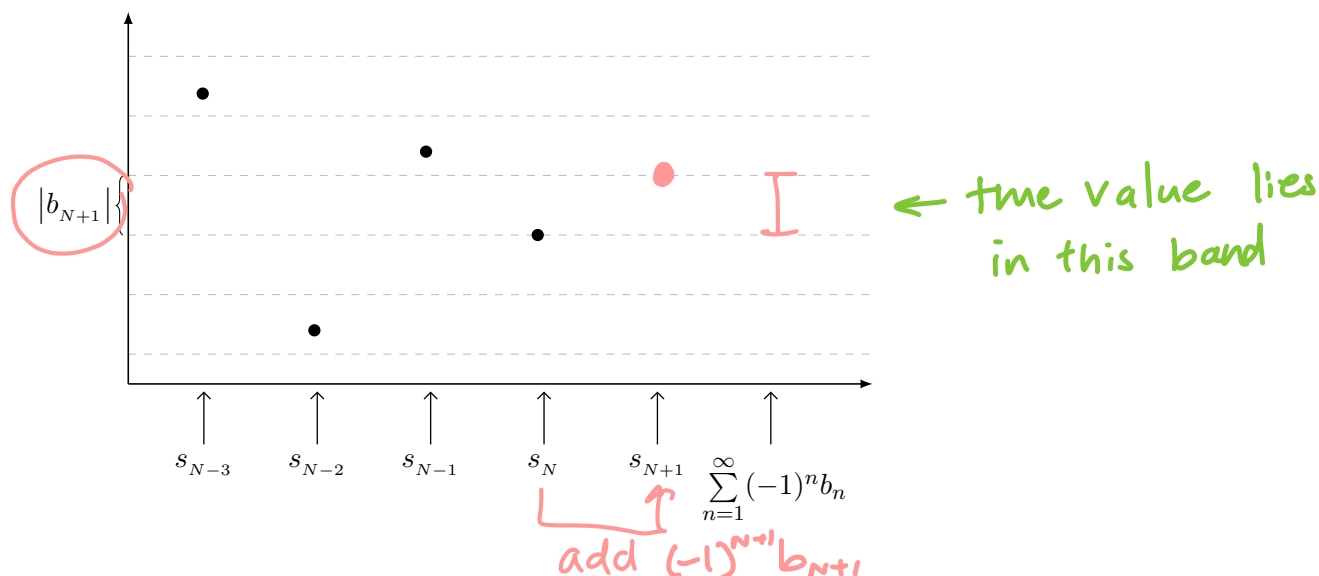
- (b) As you plot these, a pattern should emerge. Make a guess about where the sixth partial sum would fall in relation to the first five, e.g. “greater than all of them,” “less than all of them,” “between first and second,” etc.

- (c) Where do you think the value of the series would fall in relation to the first five partial sums?

2. Now, suppose that $\sum_{n=1}^{\infty} (-1)^n b_n$ is some unknown alternating series with the properties that

- $\lim_{n \rightarrow \infty} b_n = 0$
- b_n is decreasing (i.e. $b_{n+1} \leq b_n$).

Some partial sums of this unknown series have been graphed below.



- (a) Plot s_{N+1} on the graph. Notice that the distance between the dotted lines on the graph is exactly the size of the next term, $|b_{N+1}|$.
- (b) What are the possible values of the series $\sum_{n=1}^{\infty} (-1)^n b_n$? Draw a line segment that represents these possible values on the graph above.
- (c) Use your answer in (b) to complete the following statement about the error that occurs when using the partial sums of an alternating series to estimate its value.

Alternating Series Remainder Estimate:

If $\sum_{n=1}^{\infty} (-1)^n b_n$ is an alternating series such that

- $\lim_{n \rightarrow \infty} b_n = 0$ and
- b_n is decreasing

} converges by AST

then the remainder (i.e. the difference between the N th partial sum and the value of the series) satisfies

$$|R_N| = |s_N - s| = \left| \sum_{n=1}^N (-1)^n b_n - \sum_{n=1}^{\infty} (-1)^n b_n \right| \leq \underline{b_{N+1}}$$

$$|R_N| \leq b_{N+1}$$

Now put this result to use in estimating the values of alternating series.

3. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

(a) Show that the series is convergent.

Apply AST: $b_n = \frac{1}{n^2}$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0 \checkmark$$

b_n is decreasing because $\frac{1}{(n+1)^2} \leq \frac{1}{n^2}$ for all n

(b) Use the Alternating Series Remainder Estimate to determine how many terms we need to add in order to estimate the value of the series to within 0.01 (i.e. so that $|\text{error}| < 0.01$).

$$|\text{error}| = |R_N| \leq b_{N+1} = \frac{1}{(N+1)^2} < \underbrace{0.01}_{\text{desired error}}$$

bound that we can guarantee

(★ always set the bound less than your desired error)

$$\frac{1}{(N+1)^2} < \frac{1}{100}$$

$$100 < (N+1)^2$$

$$10 < N+1$$

$$9 < N$$

$$N = 10$$

4. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n}$.

(a) Show that the series is convergent.

AST: $b_n = \frac{1}{n5^n}$

$\lim_{n \rightarrow \infty} b_n = 0$ ✓ *larger denominator*

and $\frac{1}{(n+1)5^{n+1}} \leq \frac{1}{n5^n}$ so b_n is decreasing.

(b) Use the Alternating Series Remainder Estimate to determine how many terms we need to add in order to estimate the value of the series to within 0.0001 (i.e. so that $|\text{error}| < 0.0001 = 10^{-4}$).

$$|R_N| \leq b_{N+1} = \frac{1}{(N+1)5^{N+1}} < 0.0001$$

$$\frac{1}{(N+1)5^{N+1}} < \frac{1}{10000}$$

$$10000 < (N+1)5^{N+1}$$

↑ hard to solve, so find N numerically

$$b_n = \frac{1}{n5^n}$$

$$b_1 = 0.2$$

$$b_2 = 0.02$$

$$b_3 = 0.002\bar{6}$$

$$b_4 = 0.0004$$

→ $b_5 = 0.000064 < 0.0001$
small enough!

so $N+1=5$
 $N=4$

5. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$.

(a) Show that the series is convergent.

(b) Use the Alternating Series Remainder Estimate to determine how many terms we need to add in order to estimate the value of the series to within 0.005.