

6.6 Applications to Physics and Engineering - Work

← if a force acts on an object and causes it to move, we say work is done

In Physics, a **force** is a push or pull on an object.

Newton's 2nd Law of Motion: If an object moves along a straight line and its position is given by a function $s(t)$, then the force F on the object is given by

$$F = ma$$

"Force equals mass times acceleration."

When a force *moves* an object, we say the force does **work** on the object. Work can be thought of as the amount of energy it takes to move an object.

The work W done by exerting a constant force F over a distance d is given by

$$W = Fd.$$

"Work equals force times distance."

Relevant Units

In the SI metric system:

- Force is measured in Newtons ($N = \text{kg} \cdot \text{m/s}^2$)
- Mass is measured in kilograms (kg)
- Distance is measured in meters (m)
- Work is measured in Newton-meters (N-m) a.k.a. joules (J)*

In the US Customary system:

- Force is measured in pounds (lbs)
- Distance is measured in feet (ft)
- Work is measured in foot-pounds (ft-lb)

Note:

- The **weight** of an object is the force on the object due to gravity, NOT its mass.
- In the US system, the fundamental unit is the unit of force. (There is a unit of mass, called a slug, but it is not commonly used.)
- Acceleration due to gravity, denoted g , is approximately 9.8 m/s^2 in SI units and approximately 32 ft/s^2 in US units.
- The density of water, denoted ρ , is approximately 1000 kg/m^3 in SI units.

*One joule is the amount of work done by a force of 1 N over a distance of 1 meter.

← 2 systems

→ $\text{weight} = \text{mass} \cdot g$ ← acceleration due to gravity

ex: a 100 kg object weighs $100 \cdot (9.8) = 980 \text{ N}$ ← units of force
 vs. a 100 lb object weighs 100 lbs

$$g = 9.8$$

$$\rho = 1000$$

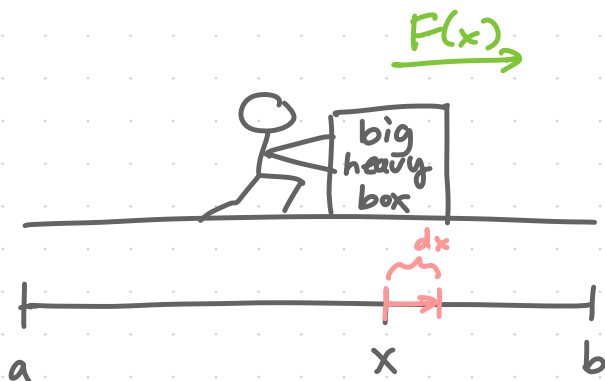
Simple example: How much work is done to lift a 200kg rock to a height of 3m?

$$\text{weight} = \text{mass} \cdot g = 200 \cdot (9.8) \\ = 1960 \text{ N}$$

$$W = F \cdot d = (1960) \cdot (3) = 5880 \text{ J}$$

But what if the force is not constant?

Say you push a box from a to b, but you get tired halfway through and can't push as hard.



the force you exert is a function of x !

Consider the work done over a small interval of length dx . Call this dW (a tiny bit of work).

so small that the force is essentially constant

$$dW = \underbrace{F(x)}_{\text{force}} \cdot \underbrace{dx}_{\text{dist}}$$

Integrate to get the total work: $W = \int_a^b dW = \int_a^b F(x) dx$

$$W = \int_a^b F(x) dx$$

← work is the integral
of force

(if you can find $F(x)$)

More useful (we'll see in examples):

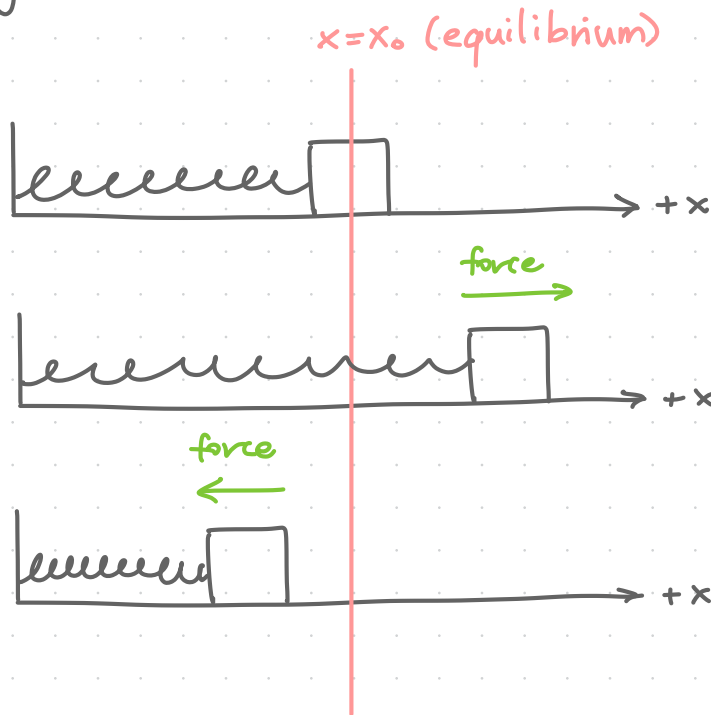
$$W = \int_a^b dW$$

"adding" up

tiny bits of work

Work needed to stretch/compress a spring * good example of a non-constant force

Consider a block sitting on a frictionless surface, attached to a spring:



pull on the spring to stretch it

push on it to compress

Hooke's Law: The force needed to stretch/compress the spring from x_0 to x is

$$F(x) = k(x - x_0)$$

↑
"spring constant"

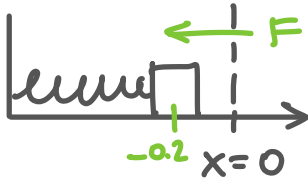
Usually, we can set up our coordinate system so that

$$x_0 = 0 \Rightarrow F(x) = kx.$$

Hooke's Law if equilibrium is at 0

6.6 Work Examples

Example 1: It takes a force of 10 N to compress a spring 0.2 m from equilibrium. How much work is done to stretch the spring 0.5 m from equilibrium?



equilibrium at $x=0$

$\Rightarrow$ Hooke's Law is $F(x) = kx$

Note: Force is in the negative direction so $F = -10\text{ N}$

Also, displacement is negative

Since $F = kx$, $-10 = k(-0.2)$

$$k = \frac{10}{0.2} = 50$$

So for this spring $F(x) = 50x$.

$$W = \int_0^{0.5} 50x \, dx = 50 \left(\frac{1}{2} x^2 \right) \Big|_0^{0.5} = 25(0.25^2 - 0) = 6.25 \text{ J}$$

Example 2: If 10 J of work is needed to stretch a spring 0.5 m beyond its natural length, how much work is needed to stretch it from 1 to 2 m beyond its natural length?



Again, set equilibrium at $x=0$.

$$W = \int_a^b kx \, dx$$

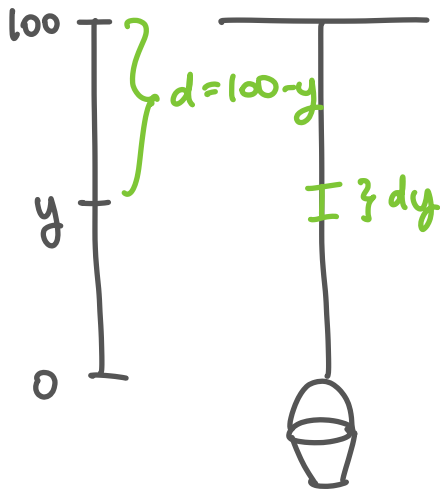
$$10 = \int_0^{0.5} kx \, dx \quad \leftarrow \text{given}$$

$$10 = k \left(\frac{1}{2} x^2 \right) \Big|_0^{0.5}$$

$$10 = k \left(\frac{1}{2} (0.5)^2 \right) \Rightarrow k = 80$$

$$W = \int_1^2 80x \, dx = 80 \left(\frac{1}{2} x^2 \right) \Big|_1^2 = 40(4 - 1) = 120 \text{ J}$$

Example 3: A cable that weighs 2 lbs/ft is attached to a bucket filled with water that weighs 40 lbs. The bucket is initially at the bottom of a 100 ft well. Determine the amount of work necessary to lift the bucket to the top of the well.



To lift the bucket (and cable), need to fight gravity, so

$$\text{Work} = (\underbrace{\text{weight}}_{\text{AKA force due to gravity}})(\text{dist. traveled})$$

Also $\text{total work} = W_{\text{bucket}} + W_{\text{cable}}$

For the bucket, weight = 40 lbs and it travels 100 ft so

$$W_{\text{bucket}} = 40 \cdot 100 = 4000 \text{ ft-lbs}$$

For the cable, consider a tiny chunk of cable (length dy) at height y between 0 and 100

Let dW = work need to raise the chunk to the top

$$= (\text{weight of chunk}) \cdot (\text{distance to the top})$$

$$= (2 dy) \cdot (100 - y) = 2(100 - y) dy$$

↑
↑ from diagram

$$\text{weight of chunk} = (2 \text{ lbs/ft}) \cdot (\underbrace{dy \text{ ft}}_{\text{length} = dy}) = 2 dy \text{ lbs}$$

$$\text{So } W_{\text{cable}} = \int_0^{100} dW = \int_0^{100} 2(100 - y) dy = 10,000 \text{ ft-lbs}$$

Answer: 14000 ft-lbs

About linear densities:

2 lbs/ft is a "linear weight density".

Tells you how the weight is distributed along a line.

If instead we were told that the cable was 100 ft long and 200 lbs total, we'd have to compute it ourselves:

$$\frac{\text{total weight}}{\text{total length}} = \frac{200 \text{ lbs}}{100 \text{ ft}} = 2 \text{ lbs/ft}$$

If given total mass, compute the linear mass density

ex: a 75 kg cable that is 50 m long

$$\text{linear density} = \frac{75 \text{ kg}}{50 \text{ m}} = \frac{3}{2} \text{ kg/m}$$

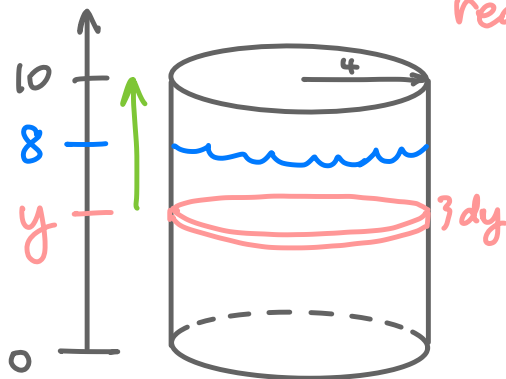
A 4 meter piece of this cable has

$$\text{mass} = \frac{3}{2} \cdot 4 = 6 \text{ kg}$$

It weighs $6 \cdot (9.8) = 58.8 \text{ N}$

Example 4: A cylindrical tank with radius 4 m and height 10 m is filled with water to a level of 8 m. Find the work required to pump the water out of the top of the cylindrical tank.

↑ Need to lift all the water to the top of the tank. Water at the bottom travels further (so requires more work) $\Rightarrow$ need an integral!



Strategy: think of thin (horizontal) slices of water & compute the work needed to raise a slice to top

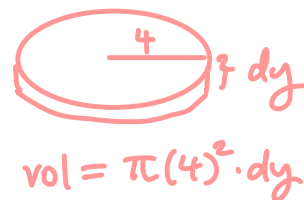
(density of water is $\rho = 1000 \text{ kg/m}^3$)

Let dW = work needed to raise a slice at height y

$$= \underbrace{\left(\text{weight of slice} \right)}_{\text{?}} \cdot \underbrace{\left(\text{distance to top} \right)}_{10-y}$$

$$\text{weight} = \underbrace{\text{mass}}_{\rho \cdot \text{volume}} \cdot g = \rho \cdot (\text{volume}) \cdot g = \rho \cdot (16\pi dy) \cdot g$$

our slices are squatty cylinders



$$\Rightarrow dW = \rho \cdot 16\pi dy \cdot g \cdot (10-y)$$

$$= 16\pi \cdot \rho \cdot g (10-y) dy$$

$$= 16\pi (1000)(9.8)(10-y) dy$$

← plug in $\rho = 1000$ and $g = 9.8$

Now integrate (add up the work needed for each slice):

$$W = \int_0^8 dW \quad \leftarrow \text{where the water is!}$$

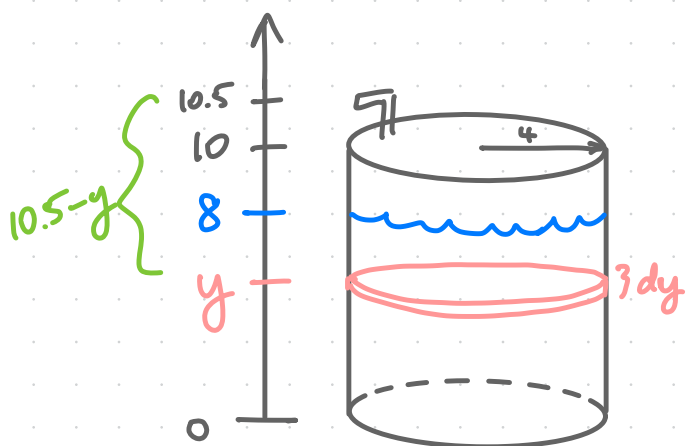
$$= \int_0^8 16\pi (1000) (9.8) (10-y) dy$$

$$= 16\pi (9800) \int_0^8 10-y dy$$

$$= 16\pi (9800) \left[10y - \frac{1}{2} y^2 \right]_0^8$$

$$= 16\pi (9800) \left(80 - \frac{64}{2} - 0 \right) \approx 23,632,896 \text{ J}$$

example 4.5: Same tank and water level as ex. 4 but we want to pump the water out of a 0.5m spout at the top



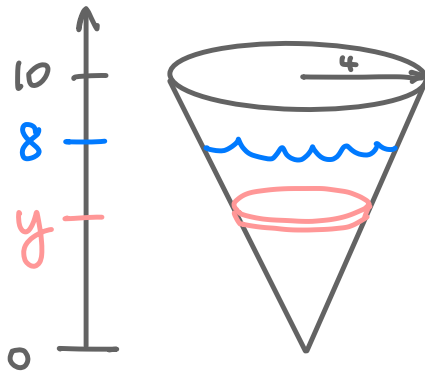
same tank $\Rightarrow$ same slices so same weight

same water level $\Rightarrow$ same bounds

distance traveled is the only thing that changes!

$$W = \int_0^8 16\pi \rho g (10.5-y) dy$$

Example 5: A conical tank with base radius 4 m and height 10 m is filled with water to a level of 8 m. Find the work required to pump the water out of the top of the ~~cylindrical~~ tank.



Now the volume of a slice depends on what height you slice at

$$W = F \cdot d = m \cdot g \cdot d = \rho \cdot \text{vol} \cdot g \cdot d$$

Need to find these for each water pumping prob.

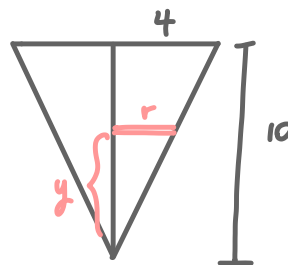
vol = volume of a slice
d = distance from the slice to top/spout

Our slices look like disks again:



$$\text{vol} = \pi r^2 dy$$

But what is r in terms of y ?



by similar triangles,

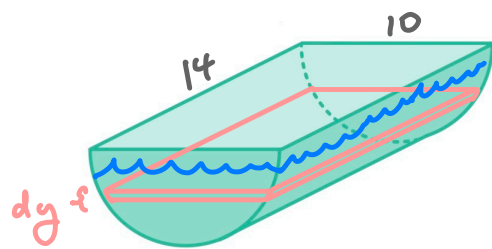
$$\frac{r}{y} = \frac{4}{10}$$

$$r = \frac{2}{5}y$$

Also $d = 10 - y$ so $dW = \rho \cdot \pi \left(\frac{2}{5}y\right)^2 dy \cdot g \cdot (10 - y)$

$$\begin{aligned} \Rightarrow W &= \int_0^8 \rho \cdot g \cdot \pi \left(\frac{2}{5}y\right)^2 (10 - y) dy \\ &= \int_0^8 1000 \cdot (9.8) \pi \left(\frac{2}{5}y\right)^2 (10 - y) dy \end{aligned}$$

Example 6: A tank is formed by taking half of a cylinder of radius 5 m and height 14 m and laying it on its side. It is filled with water to a level of 4 m. Find the work required to pump the water out of the top of the cylindrical tank.

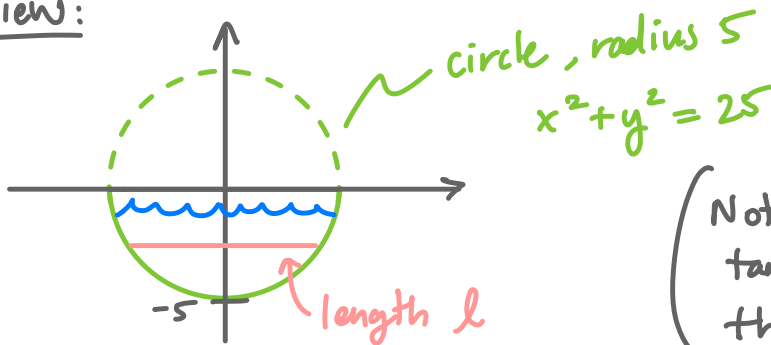


$$W = F \cdot d = m \cdot g \cdot d = \rho \cdot \text{vol} \cdot g \cdot d$$

Slices are rectangles:

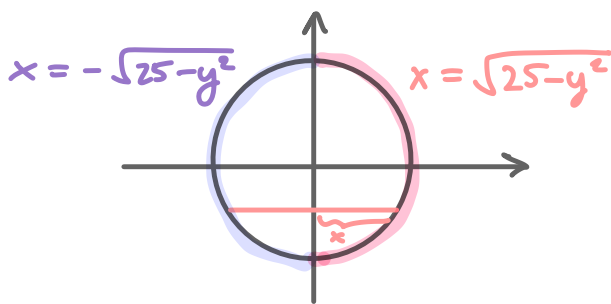
$$l \left\{ \begin{array}{l} \text{vol} = 14 l dy \\ 14 \end{array} \right. \quad \text{width } dy$$

Side View:



(Notice the bottom of the tank is at $y = -5$ and the top is at $y = 0$.)

We need l in terms of y :

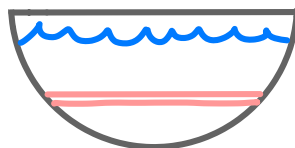


$$l = 2x = 2\sqrt{25 - y^2}$$

$$\Rightarrow \text{vol} = 14 (2\sqrt{25 - y^2}) dy$$

distance?

$$d = 0 - y \left\{ \begin{array}{l} 0 \\ -1 \\ y \\ -5 \end{array} \right.$$



$$d = -y$$

(remember $-5 \leq y \leq -1$
so d will be positive)

$$dW = \rho \cdot \text{vol} \cdot g \cdot d$$

$$= \rho \cdot (14 \cdot 2 \sqrt{25-y^2} \cdot dy) \cdot g \cdot (-y)$$

$$W = \int_{-5}^{-1} -\rho \cdot g \cdot 28y \sqrt{25-y^2} dy$$

$$= \int_{-5}^{-1} -(1000)(9.8) 28y \sqrt{25-y^2} dy$$