

Taylor's Inequality

$T_n(x)$ - n th degree Taylor poly. of $f(x)$
centered at a

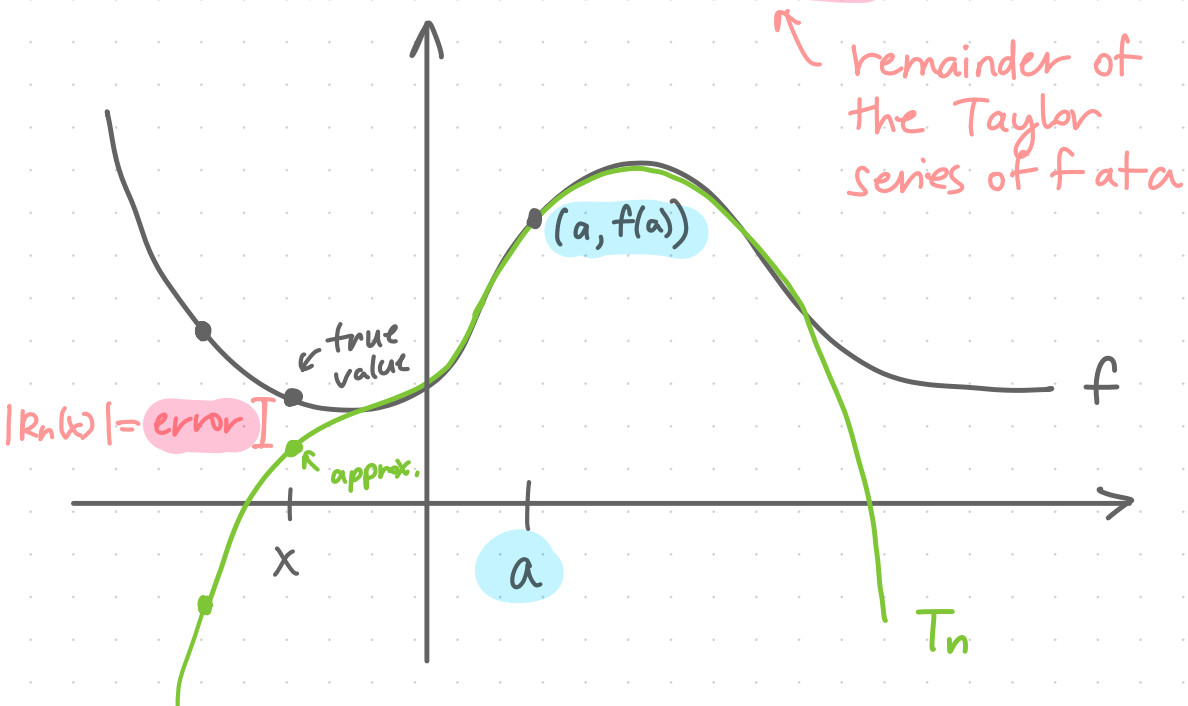
Idea: $T_n(x) \approx f(x)$ for x near a

approximation

true value

We can estimate the error in this approximation using Taylor's Inequality.

Define $R_n(x) = f(x) - T_n(x)$



Notice how far off $T_n(x)$ is from x will depend on x :

$$|R_n(x)| = \left| \begin{array}{l} \text{error in approximating } f \\ \text{using } T_n \text{ at the point } x \end{array} \right|$$

Taylor's Inequality: If $|f^{(n+1)}(x)| \leq M$ for all x satisfying $|x-a| \leq d$, then

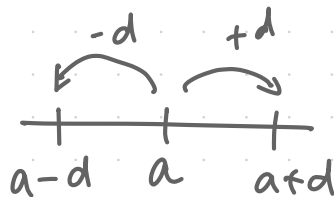
$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

for x satisfying $|x-a| \leq d$.

↑ " $|x-a| \leq d$ " means distance between x and a is less than or equal to d

$$|x-a| \leq d$$

$$\begin{array}{ccc} -d \leq x-a \leq d \\ \underline{+a} & \underline{+a} & \underline{+a} \\ a-d \leq x \leq a+d \end{array}$$



" $|x-a| \leq d$ " is equivalent to

" x is in the interval $[a-d, a+d]$ "

Taylor's Inequality: If $|f^{(n+1)}(x)| \leq M$ for all x satisfying $|x-a| \leq d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

for x satisfying $|x-a| \leq d$.

There are several parameters in Taylor's inequality:

a - the base point

n - degree of the Taylor polynomial
(can think of as the "stopping index" like in our other remainder estimates)

M - a bound for the $(n+1)$ -th derivative of f

d - distance from the base point
(tells you where the estimate is valid)

x - a point or range of points

To use Taylor's inequality, we need to determine M (unless it's given)

Depending on the question, M may depend on the other parameters (n , x or d)

Taylor's Inequality

The remainder of the Taylor series for $f(x)$ centered at a is defined as

$$R_n(x) = f(x) - T_n(x),$$

where $T_n(x)$ denotes the n th degree Taylor polynomial (also centered at a). Taylor's Inequality says that if $|f^{(n+1)}(x)| \leq M$ for all x satisfying $|x - a| \leq d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \text{ for } x \text{ satisfying } |x - a| \leq d.$$

1. Use Taylor's Inequality to determine the number of terms of the Maclaurin series for $f(x) = \sin(x)$ needed to compute $\sin(3^\circ)$ to within 0.000005.

Want: n $\swarrow$ approximate $f(3^\circ)$ using $T_n(x)$

Given: $f(x) = \sin x$

$$a = 0$$

Need to convert to radians: $3^\circ \cdot \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{60} \text{ rad.}$

$$f(x) = \sin x, \quad |f^{(n+1)}(x)| = |\sin x / \cos x| \leq 1 \text{ for all } x$$

Choose $M = 1.$

By Taylor's Inequality,

$$|R_n\left(\frac{\pi}{60}\right)| \leq \frac{1}{(n+1)!} \left|\frac{\pi}{60} - 0\right|^{n+1} = \frac{\left(\frac{\pi}{60}\right)^{n+1}}{(n+1)!} < 5 \times 10^{-6}$$

$\uparrow$ can't solve by hand

want < 0.000005

n	$\frac{(\pi/60)^{n+1}}{(n+1)!}$
1	$\frac{(\pi/60)^2}{2!} \approx 0.00137...$
2	$\frac{(\pi/60)^3}{3!} \approx 0.0000239...$
3	$\frac{(\pi/60)^4}{4!} \approx 0.000000313... \leftarrow \text{small enough!}$

$n=3$ so use $T_3(x)$ for $\sin x$ to approx.

$\sin(3^\circ)$.

Remember $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ ↖

$$T_3(x) = \frac{x}{1!} - \frac{x^3}{3!} = x - \frac{1}{6}x^3$$

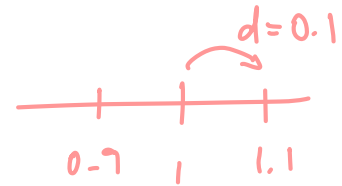
$$\begin{aligned}
 \sin(3^\circ) &= f\left(\frac{\pi}{60}\right) \approx T_3\left(\frac{\pi}{60}\right) \\
 &= \left(\frac{\pi}{60}\right) - \frac{1}{6}\left(\frac{\pi}{60}\right)^3 \\
 &= \boxed{0.052335}952... \\
 &\approx 0.052336
 \end{aligned}$$

2. (a) Find the 2nd degree Taylor polynomial, $T_2(x)$ for $f(x) = x^{-2}$ based at $a = 1$.
 (b) Use Taylor's Inequality to estimate the accuracy of the approximation using $T_2(x)$ to estimate $f(x)$ in the interval $[0.9, 1.1]$.

similar
to HW
prob #5

$n = 2$

symmetric about $a = 1$



$$(a) \quad f(x) = x^{-2}$$

$$f(1) = 1$$

$$C_0 = 1$$

$$f'(x) = -2x^{-3}$$

$$f'(1) = -2$$

$$C_1 = -2$$

$$f''(x) = 6x^{-4}$$

$$f''(1) = 6$$

$$C_2 = \frac{6}{2!} = 3$$

$$f'''(x) = -24x^{-5}$$

for
Taylor's
ineq.

$$\begin{aligned} T_2(x) &= C_0 + C_1(x-1) + C_2(x-1)^2 \\ &= 1 - 2(x-1) + 3(x-1)^2 \end{aligned}$$

(b) "estimate the accuracy" means find the max error

$$\text{Given: } d = 0.1$$

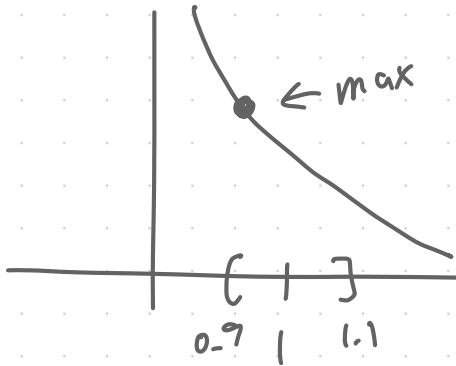
$$a = 1$$

$$n = 2$$

$$|f^{(n+1)}(x)| \leq M$$

for x in
 $[a-d, a+d]$

$$|f^{(n+1)}(x)| = |f'''(x)| = |-24x^{-5}| = \left| \frac{24}{x^5} \right|$$



↑
Find M larger
than this for
all x in
 $[0.9, 1.1]$

because $f'''(x)$ is decreasing on $[0.9, 1.1]$,
the max value is the value at the
left endpoint

$$|f'''(0.9)| = \left| \frac{24}{(0.9)^5} \right| = \frac{24}{(0.9)^5}$$

Choose $M = \frac{24}{(0.9)^5}$

Now plug in to Taylor's Ineq.

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} = \frac{(24/(0.9)^5)}{3!} |x-1|^3$$

$\swarrow$ x is in $[0.9, 1.1]$
 $\underbrace{\hspace{10em}}_{\leq d = 0.1}$

$$\leq \frac{24/(0.9)^5}{6} \cdot (0.1)^3 \approx 0.00677$$

If you're using a Taylor polynomial to approximate f at a particular value of x instead of a range of x , the following formulation of Taylor's Inequality is sometimes useful.

Alternate Statement of Taylor's Inequality

If $|f^{(n+1)}(x)| \leq M$ for all x between a and x_0 , then

$$|R_n(x_0)| \leq \frac{M}{(n+1)!} |x_0 - a|^{n+1}.$$

point where you want to approximate

3. How large should n be to guarantee that the approximation of $\ln(0.5)$ using an n th degree Taylor polynomial for $\ln(1+x)$ centered at $a=0$ is within 0.0001?

Want: n

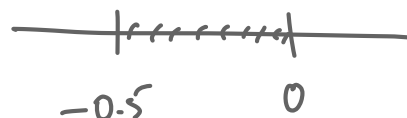
Given: $a=0$

$$f(x) = \ln(1+x)$$

$$x_0 = -0.5$$

$$\begin{aligned} \ln(0.5) &= \ln(1-0.5) \\ &= f(-0.5) \end{aligned}$$

Want to determine M .



$$|f^{(n+1)}(x)| \leq M \text{ for } x \text{ in } [-0.5, 0]$$

$$f(x) = \ln(1+x)$$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$$

$$f''(x) = -(1+x)^{-2}$$

$$f'''(x) = (-1)(-2)(1+x)^{-3} = \frac{(-1)^2 2!}{(1+x)^3}$$

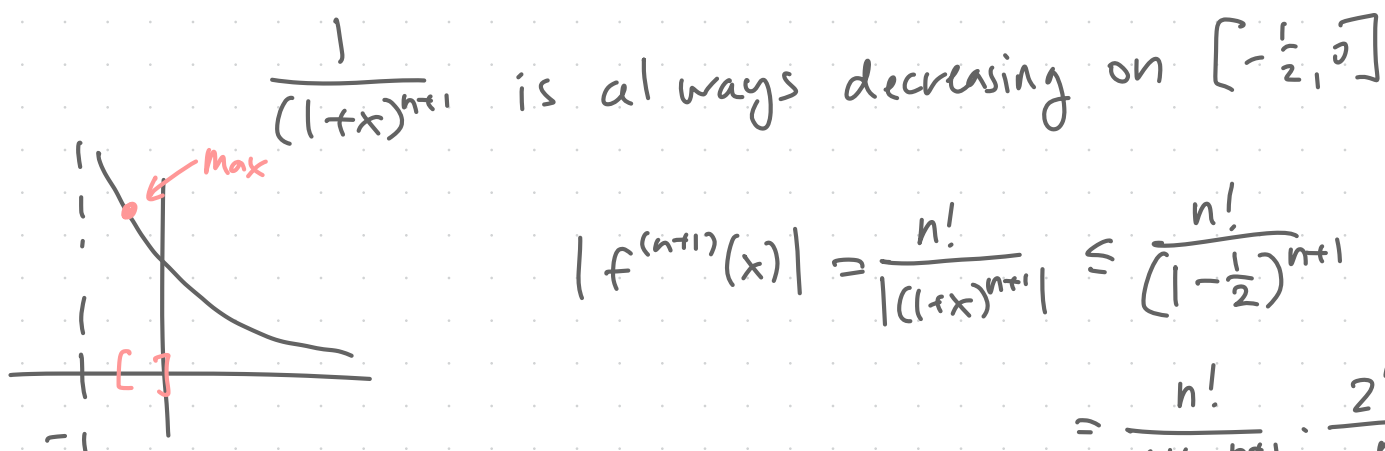
$$f^{(4)}(x) = (-1)(-2)(-3)(1+x)^{-4} = \frac{(-1)^3 3!}{(1+x)^4}$$

HW
prob
#2

$$f^{(n)}(x) = \frac{(-1)^{n-1} (n-1)!}{(1+x)^n}$$

$$|f^{(n+1)}(x)| = \left| \frac{(-1)^n n!}{(1+x)^{n+1}} \right| = \frac{n!}{|(1+x)^{n+1}|}$$

bound for
this for
x in
[-1/2, 0]



$$|f^{(n+1)}(x)| = \frac{n!}{|(1+x)^{n+1}|} \leq \frac{n!}{(1-\frac{1}{2})^{n+1}}$$

$$= \frac{n!}{(\frac{1}{2})^{n+1}} \cdot \frac{2^{n+1}}{2^{n+1}}$$

$$= n! \cdot 2^{n+1}$$

Choose $M = n! \cdot 2^{n+1}$

Plug in to Taylor's Inequality,

$$|R_n(x_0)| \leq \frac{M}{(n+1)!} |x_0 - a|^{n+1}$$

$$|R_n(-\frac{1}{2})| \leq \frac{n! \cdot 2^{n+1}}{(n+1)!} \cdot \left| -\frac{1}{2} - 0 \right|^{n+1}$$

$$= \frac{\cancel{n!} \cdot 2^{n+1}}{(n+1) \cdot \cancel{n!}} \cdot \left(\frac{1}{2} \right)^{n+1} = \frac{1}{n+1}$$

cancel

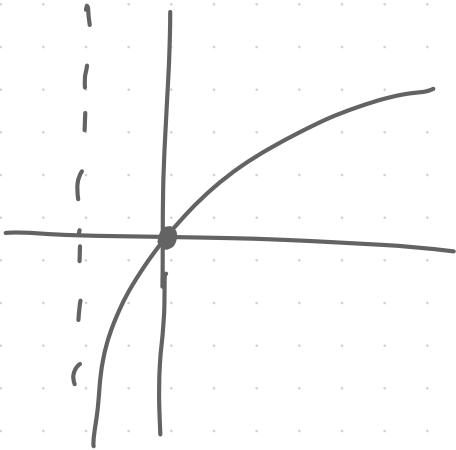
$$|R_n(-\frac{1}{2})| \leq \frac{1}{n+1} \leq 0.0001$$

$\frac{1}{n+1} \leq \frac{1}{10000}$

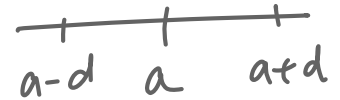
$$10000 \leq n+1$$

$$9999 \leq n$$

$$n = 9999$$



← want to find d



4. Estimate the range of values of x for which the approximation

$$\ln(x) \approx \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2$$

is accurate to within 0.01.

$f(x)$ $T_2(x)$ based at $a=2$

similar to the prob. 6

Given: $f(x) = \ln x$

Want: d

$a = 2$

$n = 2$

Determine M so that $|f^{(n+1)}(x)| \leq M$

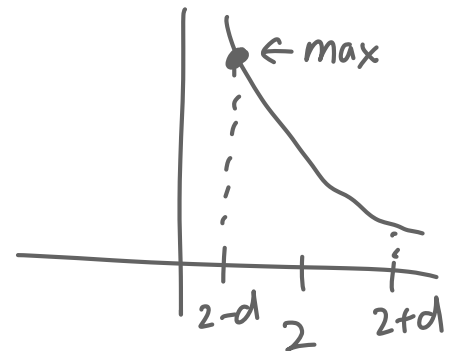
for x in $[a-d, a+d]$

$$f(x) = \ln x$$

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -x^{-2}$$

$$f'''(x) = 2x^{-3} = \frac{2}{x^3}$$



$$|f'''(x)| = \left| \frac{2}{x^3} \right| \leq \frac{2}{(2-d)^3} \leftarrow M$$

Choose $M = \frac{2}{(2-d)^3}$

Plug in to Taylor's Inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

$$a=2$$
$$n=2$$

$$|R_2(x)| \leq \frac{\left(\frac{2}{(2-d)^3}\right)}{3!} \cdot \underbrace{|x-2|^3}_{\leq d}$$

for x in

$$[2-d, 2+d]$$

$$\leq \frac{\left(\frac{2}{(2-d)^3}\right)}{6} \cdot d^3$$

$$= \frac{2d^3}{6(2-d)^3} \leq 0.01$$

← set less than
desired error

$$\frac{2d^3}{6(2-d)^3} \leq \frac{1}{100}$$

∴

$$d \leq 0.47$$

$$\left[\begin{array}{c} \text{---} \\ 1.5 \quad 2.5 \end{array} \right]$$

$$\text{Use } \boxed{d=0.47}$$

Last office hour: 11-noon Thursday