

*Definition 1.* A transition function  $P(t, x; B)$  is called **Fellerian** if for  $f \in C(\overline{\mathbb{R}})$

$$T_t f(x) = \int f(y) P(t, x; dy) \in C(\overline{\mathbb{R}})$$

and if the semi group  $\{T_t\}$  associated with  $P(t, x; B)$  is strongly continuous in  $C(I)$ .

*Definition 2.* Let  $p(x)$  be a strictly increasing function on  $\mathbb{R}$ . Let  $m(x)$  be a continuous on the right and strictly increasing function on  $\mathbb{R}$ . Let  $x \in \mathbb{R}$  and  $f : U \rightarrow \mathbb{R}$  where  $U$  is a neighborhood of  $x$ . Define the operators  $\mathbf{D}_p^+ : C(\overline{\mathbb{R}}) \rightarrow C(\overline{\mathbb{R}})$  and  $\mathbf{D}_m : C(\overline{\mathbb{R}}) \rightarrow C(\overline{\mathbb{R}})$  by

$$D_p^+ f(x) = \lim_{y \rightarrow x+} \frac{f(y) - f(x)}{p(y) - p(x)} \text{ and } D_m f(x) = \lim_{y \rightarrow x} \frac{f(y) - f(x)}{m(y) - m(x)}$$

provided that the limits exist. Denote  $C(\overline{\mathbb{R}})$  to be the set of all bounded and continuous functions on  $(-\infty, \infty)$  and  $C(\mathbb{R})$  to be the set of all bounded and continuous functions on  $[-\infty, \infty]$ . A function  $f \in C(\overline{\mathbb{R}})$  belongs to the domain of the operator  $D_m D_p^+$  if  $D_m D_p^+ f \in C(\overline{\mathbb{R}})$ .

*Definition 3.* Let  $x \in (a, b) \subset \overline{\mathbb{R}}$

$$u^1(x) = \int_a^b m(s) dp(s) \text{ and } v^1(x) = \int_a^b p(s) dm(s).$$

$r$  is called accessible if  $u^1(r) < \infty$ . Further,  $r$  is called regular if  $v^1(r) < \infty$ . The point  $r$  is called a **sticky interior point** if it is regular, accessible, and the local time  $L_t^r > 0$ .

*Theorem 1.* Let  $A$  be the infinitesimal operator of a Fellerian transition function  $P(t, x; B)$  with  $D(A) \subset D(D_m D_p^+)$ . We assume for  $F \in D(A)$  that  $AF = D_m D_p^+ F$ . Let  $r \in \mathbb{R}$  be a sticky point of  $D_m D_p^+$ . Then all  $F \in D(A)$  satisfy a non trivial condition at  $r$ :

$$\phi_r(F) = \chi F(r) + \pi_- D_p^+ F(r-) - \pi_+ D_p^+ F(r+) + \sigma D_m D_p^+ F(r) = 0$$

where  $\chi, \sigma, \pi_-, \pi_+ \geq 0$ .

*Proof.* Let  $p_-(x), p_+(x)$  be non decreasing and continuous functions on  $\overline{\mathbb{R}}$  such that there exists an  $0 < \epsilon_1 < |a|$  and  $0 < \epsilon_2 < |b|$  with  $p_-(x) = p(x)$  for  $x \in (r - \epsilon_1, r]$  and  $p_+(x) = p(x)$  for  $x \in [r, r + \epsilon_2)$ . Let  $q_-(x), q_+(x)$  be non decreasing and bounded functions on  $\mathbb{R}$  with  $\int dq_-(x) = \pi_-$  and  $\int dq_+(x) = \pi_+$ . One of  $\sigma > 0$  or

$$\int \frac{dq_-(x)}{|p_-(r) - p_-(x)|} - \int \frac{dq_+(x)}{|p_+(r) - p_+(x)|} = \infty$$

must hold. In the case when  $\sigma = 0$  and  $\int \frac{dq_-(x)}{|p_-(r) - p_-(x)|} - \int \frac{dq_+(x)}{|p_+(r) - p_+(x)|} < \infty$ , the condition

$$\phi_r(F) = \chi F(r) + \int \frac{F(r-) - F(x)}{|p_-(r) - p_-(x)|} dq_-(x) - \int \frac{F(r+) - F(x)}{|p_+(r) - p_+(x)|} dq_+(x) = 0$$

defines a proper and closed linear subspace of  $C(\overline{\mathbb{R}})$ . This contradicts the fact that  $D(A)$  is dense in  $C(\overline{\mathbb{R}})$ , and hence contradicts the definition of  $A$  as the infinitesimal operator of a Fellerian transition function. Thus the limits

$$\lim_{x \rightarrow r-} \lim_{y \rightarrow x+} \frac{F(y) - F(x)}{|p_-(y) - p_-(x)|} = \lim_{x \rightarrow r-} D_p^+ F(x) = D_p^+ F(r-)$$

and

$$\lim_{x \rightarrow r+} \lim_{y \rightarrow x+} \frac{F(y) - F(x)}{|p_+(y) - p_+(x)|} = - \lim_{x \rightarrow r+} D_p^+ F(x) = D_p^+ F(r+)$$

are well-defined and thus we can write

$$\int \frac{F(r-) - F(x)}{|p_-(r) - p_-(x)|} dq_-(x) = \pi_- D_p^+ F(r-) \text{ and } \int \frac{F(r+) - F(x)}{|p_+(r) - p_+(x)|} dq_+(x) = -\pi_+ D_p^+ F(r+).$$

For  $F \in D(A)$ ,

$$\begin{aligned}
AF(r) &= D_m D_p^+ F(r) = \lim_{t \rightarrow 0} \frac{T_t F(r) - F(r)}{t} \\
&= \lim_{t \rightarrow 0} \frac{\int F(y) P(t, r; dy) - F(r)}{t} \\
&= \lim_{t \rightarrow 0} \frac{-\int (F(r) - F(y)) P(t, r; dy) - r(t) F(r)}{t} \\
&= \lim_{t \rightarrow 0} \frac{-\int \left( \left( \frac{F(r+) - F(r-)}{2} \right) - F(y) \right) P(t, r; dy) - r(t) F(r)}{t} \\
&= \lim_{t \rightarrow 0} \frac{\int \frac{F(y) - F(r+)}{2} P(t, r; dy) + \int \frac{F(y) - F(r-)}{2} P(t, r; dy) - r(t) F(r)}{t}
\end{aligned}$$

where  $r(t) = 1 - P(t, r; I)$ . Define

$$q(t) = \int |p_-(r) - p_-(x)| P(t, r; dx) = \int |p_+(r) - p_+(x)| P(t, r; dx)$$

and

$$q(t) \chi(t, x) = \int_{[-\infty, x]} |p_-(r) - p_-(y)| P(t, r; dy) = \int_{[-\infty, x]} |p_+(r) - p_+(y)| P(t, r; dy)$$

for  $-\infty \leq x \leq \infty$ . Then the limit equation above is equivalent to

$$0 = \lim_{t \rightarrow 0} \left[ \int \frac{F(r+) - F(x)}{|p_+(r) - p_+(x)|} \chi(t, dx) \frac{q(t)}{2t} + \int \frac{F(r-) - F(x)}{|p_-(r) - p_-(x)|} \chi(t, dx) \frac{q(t)}{2t} + F(r) \frac{r(t)}{t} + D_m D_p^+ F(r) \right].$$

Now take a sequence  $\tau = \{t_k\}$  such that

1.  $t_k \geq 0$  for all  $k \geq 0$ ,
2.  $\lim_{k \rightarrow \infty} t_k = 0$ ,
3. the following limits exist:

$$\lim_{t \in \tau} \frac{q(t)}{t} = Q, \quad \lim_{t \in \tau} \frac{r(t)}{t} = R, \quad \lim_{t \in \tau} \frac{q(t)}{r(t)} = S, \quad \text{and,}$$

4.  $\{\chi(t, x), t \in \tau\}$  converges to a distribution function  $k(x)$  for all points of continuity of  $k$ .

This last statement is due to the Helly-Bray theorem which states that for  $F$  and  $F_1, F_2, \dots$  cumulative distribution functions on the real line, if  $F_n$  converges weakly to  $F$ , then

$$\int_{\mathbb{R}} g(x) dF_n(x) \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}} g(x) dF(x)$$

for each bounded, continuous function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , where the integrals are Riemann-Stieltjes integrals. Thus we have

$$\lim_{t \in \tau} \int f(x) \chi(t, dx) = \int f(x) dk(x), \quad \text{for } f \in C(\overline{\mathbb{R}})$$

and in particular, for  $F \in D(D_m D_p^+)$  we have both

$$\frac{F(r+) - F(x)}{|p_+(r) - p(x)|}, \quad \frac{F(r-) - F(x)}{|p_-(r) - p(x)|} \in C(\overline{\mathbb{R}}).$$

Thus for  $Q + R < \infty$ ,

$$\begin{aligned} 0 &= \lim_{t \rightarrow 0} \left[ \int \frac{F(r+) - F(x)}{|p_+(r) - p_+(x)|} \chi(t, dx) \frac{q(t)}{2t} + \int \frac{F(r-) - F(x)}{|p_-(r) - p_-(x)|} \chi(t, dx) \frac{q(t)}{2t} + F(r) \frac{r(t)}{t} + D_m D_p^+ F(r) \right] \\ &= \frac{Q}{2} \int \frac{F(r+) - F(x)}{|p_+(r) - p_+(x)|} dk(x) + \frac{Q}{2} \int \frac{F(r-) - F(x)}{|p_-(r) - p_-(x)|} dk(x) + RF(r) + D_m D_p^+ F(r). \end{aligned}$$

For  $Q + R = \infty$ ,

$$\begin{aligned} 0 &= \lim_{t \rightarrow 0} \left[ \int \frac{F(r+) - F(x)}{|p_+(r) - p_+(x)|} \chi(t, dx) \frac{q(t)}{2t} + \int \frac{F(r-) - F(x)}{|p_-(r) - p_-(x)|} \chi(t, dx) \frac{q(t)}{2t} + F(r) \frac{r(t)}{t} + D_m D_p^+ F(r) \right] \\ &\implies \\ 0 &= \lim_{t \rightarrow 0} \frac{\left[ \int \frac{F(r+) - F(x)}{|p_+(r) - p_+(x)|} \chi(t, dx) \frac{q(t)}{2t} + \int \frac{F(r-) - F(x)}{|p_-(r) - p_-(x)|} \chi(t, dx) \frac{q(t)}{2t} + F(r) \frac{r(t)}{t} + D_m D_p^+ F(r) \right]}{\frac{q(t) + r(t)}{t}} \\ &= \frac{S}{1 + S} \int \frac{F(r+) - F(x)}{|p_+(r) - p_+(x)|} dk(x) + \frac{S}{1 + S} \int \frac{F(r-) - F(x)}{|p_-(r) - p_-(x)|} dk(x) + F(r) \frac{1}{1 + S}. \end{aligned}$$

These last two displayed equations are of the form of the condition in the statement of the theorem.  $\square$

HERE IS AI'S VERSION OF MY WORK

**Definition 4.** A transition function  $P(t, x; B)$  is called **Fellerian** if for  $f \in C(\overline{\mathbb{R}})$  we have

$$T_t f(x) = \int f(y) P(t, x; dy) \in C(\overline{\mathbb{R}})$$

and the semigroup  $\{T_t\}$  associated with  $P(t, x; B)$  is strongly continuous in  $C(\overline{\mathbb{R}})$ .

**Definition 5.** Let  $p(x)$  be a strictly increasing function on  $\mathbb{R}$  and  $m(x)$  be a right-continuous and strictly increasing function on  $\mathbb{R}$ . Let  $x \in \mathbb{R}$  and  $f : U \rightarrow \mathbb{R}$  where  $U$  is a neighborhood of  $x$ . Define the operators  $\mathbf{D}_p^+ : C(\mathbb{R}) \rightarrow C(\mathbb{R})$  and  $\mathbf{D}_m : C(\mathbb{R}) \rightarrow C(\mathbb{R})$  by

$$D_p^+ f(x) = \lim_{y \rightarrow x+} \frac{f(y) - f(x)}{p(y) - p(x)}, \quad D_m f(x) = \lim_{y \rightarrow x} \frac{f(y) - f(x)}{m(y) - m(x)}$$

provided the limits exist. Denote  $C(\mathbb{R})$  as the set of all bounded continuous functions on  $(-\infty, \infty)$ , and  $C(\overline{\mathbb{R}})$  as the set of all bounded continuous functions on  $[-\infty, \infty]$ . A function  $f \in C(\overline{\mathbb{R}})$  belongs to the domain of the operator  $D_m D_p^+$  if  $D_m D_p^+ f \in C(\overline{\mathbb{R}})$ .

**Definition 6.** Let  $x \in (a, b) \subset \overline{\mathbb{R}}$ . Define

$$u^1(x) = \int_a^b m(s) dp(s), \quad v^1(x) = \int_a^b p(s) dm(s).$$

A point  $r$  is called *accessible* if  $u^1(r) < \infty$ , and *regular* if  $v^1(r) < \infty$ . The point  $r$  is called a **sticky interior point** if it is accessible and regular, and if the speed measure  $m$  satisfies  $m(\{r\}) > 0$ , i.e., the process accumulates local time at  $r$ .

**Theorem 2.** Let  $A$  be the infinitesimal operator of a Fellerian transition function  $P(t, x; B)$  with  $D(A) \subset D(D_m D_p^+)$ , and assume that for  $F \in D(A)$  we have  $AF = D_m D_p^+ F$  pointwise away from any singularities. Let  $r \in \mathbb{R}$  be a sticky interior point. Then all  $F \in D(A)$  satisfy a nontrivial boundary condition at  $r$ :

$$\phi_r(F) = \chi F(r) + \pi_- D_p^+ F(r-) - \pi_+ D_p^+ F(r+) + \sigma D_m D_p^+ F(r) = 0,$$

where  $\chi, \sigma, \pi_-, \pi_+ \geq 0$ .

*Proof.* Let  $p_-(x), p_+(x)$  be nondecreasing and continuous functions on  $\overline{\mathbb{R}}$  such that  $p_-(x) = p(x)$  for  $x \in (r - \epsilon_1, r]$  and  $p_+(x) = p(x)$  for  $x \in [r, r + \epsilon_2)$  for some small  $\epsilon_1, \epsilon_2 > 0$ . Let  $q_-, q_+$  be nondecreasing and bounded functions on  $\overline{\mathbb{R}}$  with  $\int dq_-(x) = \pi_-$  and  $\int dq_+(x) = \pi_+$ . One of the following must hold:

- $\sigma > 0$ , or
- the difference of singular integrals diverges:

$$\int \frac{dq_-(x)}{|p_-(r) - p_-(x)|} - \int \frac{dq_+(x)}{|p_+(r) - p_+(x)|} = \infty.$$

If instead  $\sigma = 0$  and the difference of integrals is finite, then the condition

$$\phi_r(F) = \chi F(r) + \int \frac{F(r-) - F(x)}{|p_-(r) - p_-(x)|} dq_-(x) - \int \frac{F(r+) - F(x)}{|p_+(r) - p_+(x)|} dq_+(x) = 0$$

defines a proper closed subspace of  $C(\overline{\mathbb{R}})$ , contradicting the density of  $D(A)$ . Hence, both

$$D_p^+ F(r-) = \lim_{x \rightarrow r-} D_p^+ F(x), \quad D_p^+ F(r+) = \lim_{x \rightarrow r+} D_p^+ F(x)$$

must exist. Similarly, we can write:

$$\int \frac{F(r-) - F(x)}{|p_-(r) - p_-(x)|} dq_-(x) = \pi_- D_p^+ F(r-), \quad \int \frac{F(r+) - F(x)}{|p_+(r) - p_+(x)|} dq_+(x) = -\pi_+ D_p^+ F(r+).$$

For  $F \in D(A)$ ,

$$\begin{aligned} AF(r) &= D_m D_p^+ F(r) = \lim_{t \rightarrow 0} \frac{T_t F(r) - F(r)}{t} \\ &= \lim_{t \rightarrow 0} \frac{-\int (F(r) - F(y)) P(t, r; dy) - r(t) F(r)}{t}, \end{aligned}$$

where  $r(t) = 1 - P(t, r; I)$  is the mass lost (e.g., via delay or killing). Define

$$q(t) = \int |p(r) - p(x)| P(t, r; dx), \quad q(t) \chi(t, x) = \int_{[-\infty, x]} |p(r) - p(y)| P(t, r; dy).$$

Let  $\tau = \{t_k\}$  be a sequence such that

1.  $t_k \geq 0$  and  $t_k \rightarrow 0$ ,
2.  $\lim_{t \in \tau} \frac{q(t)}{t} = Q$ ,  $\lim_{t \in \tau} \frac{r(t)}{t} = R$ ,  $\lim_{t \in \tau} \frac{q(t)}{r(t)} = S$ ,
3.  $\chi(t, x)$  converges pointwise to a distribution function  $k(x)$ .

Then by the Helly–Bray theorem, for  $f \in C(\overline{\mathbb{R}})$ ,

$$\lim_{t \in \tau} \int f(x) \chi(t, dx) = \int f(x) dk(x).$$

Therefore, for  $Q + R < \infty$ ,

$$0 = \frac{Q}{2} \int \frac{F(r+) - F(x)}{|p(r) - p(x)|} dk(x) + \frac{Q}{2} \int \frac{F(r-) - F(x)}{|p(r) - p(x)|} dk(x) + RF(r) + D_m D_p^+ F(r).$$

For  $Q + R = \infty$ ,

$$0 = \frac{S}{1+S} \int \frac{F(r+) - F(x)}{|p(r) - p(x)|} dk(x) + \frac{S}{1+S} \int \frac{F(r-) - F(x)}{|p(r) - p(x)|} dk(x) + \frac{1}{1+S} F(r),$$

which is again of the form claimed in the theorem.  $\square$