

BOUNDARY BEHAVIOR FOR STICKY BROWNIAN MOTION

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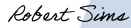
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Abstract

We study a reflecting Brownian motion on state space $[0, \infty)$ by viewing it as a diffusion process associated with the infinitesimal generator A given by

$$(Af)(x) = \frac{1}{2}f''(x) \quad (0.1)$$

and equipped with the boundary condition

$$\frac{m(\{0\})}{2}f''(0) = f'(0+). \quad (0.2)$$

This condition is referred to as a sticky boundary or a slowly reflecting boundary and we will call $m(\{0\})$ the stickiness parameter. We will show, by means of a local time process, that the time spent at the boundary is in fact positive for such a process. We introduce the singular measure, the so called speed measure m , in order to investigate the occupation time at the boundary. Moreover, we will show that the sticky boundary condition (0.2) arises as a domain condition for the infinitesimal generator A of the reflecting Brownian motion.

1 Introduction

A Brownian motion historically refers to the random movement of particles suspended in a medium. A more modern description of the dynamics of such a system involves the construction of a stochastic process with independent and normally distributed increments [3, Ch. VI, Sec. 5]. Such a process can be further characterized as a diffusion process which can then be associated with second order differential equations by means of the infinitesimal generator of the process [5, Ch. 15, Sec. 3]. Diffusion processes are used to model phenomena across a wide variety of sciences. The boundary behavior of a diffusion process is particularly important for certain applications, such as in the field of biology where one may, for instance, study the dynamics of a molecule near a cell membrane (see [1]). For one-dimensional diffusion processes, which we study here, boundary behavior properties are well studied and can be explicitly calculated. In this paper, we study “sticky” boundaries where the occupation time of the diffusion process, that is, the time spent by the Brownian motion at a single point, is nonzero at the boundary.

A reflecting Brownian motion, denoted $|B_t|$, on state space $[0, \infty)$ is a fundamental example of a diffusion process having a so-called sticky boundary condition. In this paper, we will classify a reflecting Brownian motion as a diffusion process and review the sticky boundary condition

$$\frac{m(\{0\})}{2}f''(0) = f'(0+). \quad (1.1)$$

This type of boundary condition was first introduced by Feller as a lateral condition in [2], and it has been the point of interest for much work since (see [8] for sticky Bessel processes). In order to gain insight into (1.1), and in particular to understand what the stickiness parameter $m(\{0\})$ is and what it represents, we study the domain of the infinitesimal generator of $|B_t|$ near the boundary 0. We will show that (1.1) arises as a domain condition for the infinitesimal generator

$$(Af)(x) = \frac{1}{2}f''(x). \quad (1.2)$$

of $|B_t|$ by means of the speed measure m .

The paper is organized as follows. In section 2, we review the definition of standard Brownian motion and its basic properties, and we classify B_t as a diffusion process with transition density satisfying the Kolmogoroff backward equation. In section 3, we derive the infinitesimal generator (1.2) for B_t on the state space $(0, \infty)$. In section 4, we show that

$$T_t = t + \frac{m(\{0\})}{2}L_t \quad (1.3)$$

provides a random time-change for $|B_t|$ where

$$L_t = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{(-\epsilon, \epsilon)}(X_s) ds \quad (1.4)$$

is the local time process at 0 [5, p. 316-7]. We show that the occupation time at 0 for the time-changed process, $B_{T_t^{-1}}$, is positive, thus showing that 0 is a sticky boundary for $B_{T_t^{-1}}$. Heuristically, the time-change (1.3) will stretch time out while the boundary is occupied by the stochastic process, and the local time process (1.4) can be thought of as (the limit of) the ratio of the occupation time of a process near a boundary to the speed at which the time runs when the process is near the boundary. A representation of the local time process is derived in the appendix. In section 5, we use Green's functions techniques to derive the sticky boundary condition (1.1).

2 Brownian Motion as a Diffusion Process

Throughout this paper we use $\mathcal{B}$ to denote the Borel σ -algebra on $\mathbb{R}$ and to denote the space of Borel functions, with the meaning being clear from context in each case. We write $b\mathcal{B}$ for the space of bounded Borel functions and $\mathcal{B}_+$ for positive Borel functions. We work in a probability space $(\Omega, \mathcal{F}, Pr)$.

We begin with the definition of Brownian motion taken from [6, p. 2].

Definition 1. A real-valued stochastic process $B_t = (B_t)_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, Pr)$ is a *Brownian motion* if it satisfies

- i) *a.s.* $B_0 = 0$,
- ii) $\forall 0 \leq t_1 \leq \dots \leq t_n$ the increments

$$B_{t_n} - B_{t_{n-1}}, \dots, B_{t_2} - B_{t_1} \quad (2.1)$$

are independent random variables,

- iii) if $0 \leq s < t$, then $B_t - B_s$ is a Gaussian random variable with 0 mean and variance $t - s$,
- iv) the sample paths of B_t are *a.s.* continuous¹.

With $E_i \in \mathcal{B}$, and taking the *a.s.* continuity of the sample paths for granted, we can equivalently define Brownian motion B_t as a real-valued stochastic process whose finite-dimensional marginal distributions²

$$Pr\{B_{t_1} \in E_1, \dots, B_{t_k} \in E_k\} = \int_{E_1 \times \dots \times E_k} p(t_1, x, x_1) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k) dx_1 \cdots dx_k \quad (2.2)$$

are given in terms of Gaussian transition densities

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x - y|^2}{2t}\right). \quad (2.3)$$

Such a process is called a Gaussian process with *a.s.* continuous sample paths, and its finite-dimensional marginal distributions are determined by its mean $M_X(t) = E[X_t]$ and covariance $\Gamma_X(s, t) = \text{cov}(X_s, X_t)$ functions. Conversely, an arbitrary mean function $M : \mathbb{R}_+ \rightarrow \mathbb{R}$ together with a symmetric and nonnegative

¹It can easily be shown by appealing to the Kolmogoroff continuity Theorem that a version of Brownian motion exists which has *a.s.* continuous sample paths (see any reasonable text on Brownian motion.)

²We assume the reader is familiar with the compatibility notions that must be addressed when defining stochastic processes. See [7, p. 10-1] for the Kolmogoroff extension theorem, and see [3, Ch. 1] for a more in-depth discussion.

definite covariance function $\Gamma : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$, determine a Gaussian process [6, p. 222]. We have for $t \leq s$,

$$\begin{aligned}
\text{cov}(X_s, X_t) &= E[B_t B_s] \\
&= E[B_t(B_t + (B_s - B_t))] \\
&= E[B_t^2] + E[(B_s - B_t)B_t] \\
&= E[B_t^2] + E[B_t] E[B_s - B_t] \\
&= E[B_t^2] \\
&= \int_{\mathbb{R}} x^2 p(t, x, y) dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2t}} dx \\
&= t \\
&= \min(s, t),
\end{aligned}$$

and since

$$\min(s, t) = \int_0^\infty \mathbf{1}_{[0, s]}(r) \mathbf{1}_{[0, t]}(r) dr,$$

we can write

$$\sum_{i, j=1}^n a_i a_j \min(t_i, t_j) = \int_0^\infty \left(\sum_{i=1}^n a_i \mathbf{1}_{[0, t_i]}(r) \right)^2 dr \geq 0$$

for all $n \in \mathbb{N}$ and all $a_i \in \mathbb{R}$ with $i = 1, \dots, n$ so that B_t can indeed be viewed as a Gaussian process with *a.s.* continuous sample paths.

Recall that a filtration $\mathcal{F}_t = (\mathcal{F}_t)_{t \geq 0}$ is an increasing collection of sub σ -fields of $\mathcal{F}$, and a process is adapted with respect to a filtration $\mathcal{F}_t$ if X_t is $\mathcal{F}_t$ measurable for all $t \geq 0$. Further, the natural filtration for Brownian motion is the σ -field $\mathcal{F}_t$ generated by the history of the state space, that is, $(B_s)_{0 \leq s \leq t}$, along with the P -null sets of $\mathcal{F}$ (see [6, p. 9-10]).

Definition 2. An adapted process X_t is a *Markov process* with respect to a filtration $\mathcal{F}_t$ if, for any $s \geq 0$, $t \geq 0$, and any $f \in b\mathcal{B}$,

$$E[f(X_{s+t}) | \mathcal{F}_s] = E[f(X_{s+t}) | X_s] \text{ a.s.} \quad (2.4)$$

From [6, Theorem 1.6.1] we have the following.

Theorem 1. For any $s \geq 0$, $t \geq 0$, and any $f \in b\mathcal{B}$, we have

$$E[f(B_{s+t}) | \mathcal{F}_s] = P_t f(B_s),$$

where

$$P_t f(x) = \int_{\mathbb{R}} f(y) p(t, x, y) dy$$

and $p(t, x, y)$ is given in (2.3).

Thus, B_t is a Markov process with respect to its natural filtration.

The precise definition of a diffusion process depends on the text that one reads (see for instance [7, p. 115], [9, p. 294], or [4, p. 282]). We Follow Karlin and Taylor, [5, p. 157-9].

Definition 3. A continuous time parameter stochastic process X_t with *a.s.* continuous sample paths and which possesses the (strong)³ Markov property is a *diffusion process*. A diffusion process can be characterized in terms of its infinitesimal parameters

$$\mu(x, t) = \lim_{h \rightarrow 0} \frac{E^x [X_{t+h} - X_t]}{h} \quad (2.5)$$

³One may define a stopping time τ , for which a filtration $\mathcal{F}_\tau = \{A \in \mathcal{F} | \{\tau \leq t\} \cap A \in \mathcal{F}_t\}$ can be defined. A process can be characterized as having the strong Markov property if it obeys the Markov property (2.4) for any stopping time τ .

and

$$\sigma^2(x, t) = \lim_{h \rightarrow 0} \frac{E^x [(X_{t+h} - X_t)^2]}{h} \quad (2.6)$$

respectively called the drift and diffusion parameters.

We are concerned with the standard Brownian motion for which there is no drift, $\mu(x, t) = 0$, and the diffusion is constant $\sigma^2(x, t) = 1$. We may also characterize a diffusion process X_t through its transition probabilities (see [5, Ch. 15, Sec. 5]) as

$$P(t, x, A) = Pr\{X_t \in A | X_s = x\}. \quad (2.7)$$

The transition distribution of X_t can be written as

$$P(t, x, (-\infty, y]) = Pr\{X_t \in (-\infty, y] | X_0 = x\}$$

with the initial distribution $P(0, x, (-\infty, y]) = \mathbf{1}_{(-\infty, y]}(x)$. We assume that the transition distributions are derived from continuous transition densities

$$\frac{dP(t, x, (-\infty, y])}{dy} = p(t, x, y). \quad (2.8)$$

Formally,

$$dP(t, x, (-\infty, y]) = p(t, x, y)dy$$

and then

$$P(t, x, (-\infty, y]) = \int_{\mathbb{R}} p(t, x, y)dy.$$

We can determine a differential equation for the equation $P(t, x, y) = E^x [g(X_t)]$ where $g \in b\mathbb{B}$ and g is piecewise continuous on $(0, \infty)$. Taking advantage of the work done in [5, p. 215-7], we have

$$\frac{\partial P(t, x, y)}{\partial t} = \frac{\sigma^2(x)}{2} \frac{\partial^2 P(t, x, y)}{\partial x^2} + \mu(x) \frac{\partial P(t, x, y)}{\partial x} \quad (2.9)$$

for $t > 0$ and $0 < x, y < \infty$ and in terms of the densities

$$\frac{\partial p}{\partial t} = \frac{\sigma^2(x)}{2} \frac{\partial^2 p}{\partial x^2} + \mu(x) \frac{\partial p}{\partial x}. \quad (2.10)$$

Equation (2.10) is referred to as the Kolmogoroff backward equation (see also [9, p. 282]).

For a standard Brownian motion we have

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2}, \quad (2.11)$$

i.e., the density (2.3) solves the heat equation.

3 Infinitesimal Generator of Brownian Motion

We take the definition of the infinitesimal generator of a diffusion process from [4, p. 281]. Throughout this section, C_0 will refer to continuous functions on $\mathbb{R}$ that vanish at infinity, i.e., for each $f \in C_0$, f is continuous and

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 0.$$

We write $C_0(\cdot)$ when the state space of the diffusion process must be specified. We write C_0^2 for functions in C_0 that also posses continuous second partial derivatives.

Definition 4. The *infinitesimal generator* A of a diffusion process X_t is given by

$$Af(x) = \lim_{t \rightarrow 0} \frac{E^x[f(X_t)] - f(x)}{t}. \quad (3.1)$$

A function $f \in C_0$ is said to belong to the *domain* of A , $D(A)$, if the limit (3.1) exists in C_0 .

Theorem 2. The *infinitesimal generator* of B_t on the state space $(0, \infty)$ is

$$(Af)(x) = \frac{1}{2}f''(x), \quad (3.2)$$

and $D(A) = C_0^2(0, \infty)$.

Proof. We assume at first that $f \in C_0^2(0, \infty)$. Note that $f(B_t)$ can be considered as an Itô process [see Appendix] since B_t itself can be written as an Itô process in integral form

$$B_t = B_0 + \int_0^t dB_s \quad (3.3)$$

and by an application of the Itô formula (6.10) we have

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2} \int_0^t f''(B_s)ds. \quad (3.4)$$

The stochastic integral will disappear under expectation value, we take $B_0 = x$, and thus in (3.1) we are left with

$$\lim_{t \rightarrow 0} \frac{E^x[f(B_t)] - f(x)}{t} = \frac{1}{2} \lim_{t \rightarrow 0} \frac{E^x[\int_0^t f''(B_s)ds]}{t} \quad (3.5)$$

$$= \frac{1}{2} E^x \left[\lim_{t \rightarrow 0} \frac{\int_0^t f''(B_s)ds}{t} \right] \quad (3.6)$$

$$= \frac{1}{2} E^x \left[\lim_{t \rightarrow 0} \frac{\frac{\partial}{\partial t} \int_0^t f''(B_s)ds}{1} \right] \quad (3.7)$$

$$= \frac{1}{2} E \left[\lim_{t \rightarrow 0} f''(B_t) \mid B_0 = x \right] \quad (3.8)$$

$$= \frac{1}{2} f''(x) \quad (3.9)$$

and thus $f \in D(A)$. The passage of the limit in (3.6) is justified by the dominated convergence theorem for Lebesgue integrals since by nature of $f \in C_0^2$, there is an $M > 0$ such that $|f''| \leq M$ and moreover,

$$\left| \frac{1}{t} \int_0^t f''(B_s)ds \right| \leq \frac{1}{t} \int_0^t Mds = M.$$

The steps in (3.7) and (3.8) are justified by L'Hopital's rule and the Lebesgue's differentiation theorem, respectively.

Next, we claim that the domain of A is in fact equal to $C_0^2(0, \infty)$. For, if $f \in D(A)$ then (3.2) holds and moreover $(Af)(x) \in C_0(0, \infty)$ for all $x \in (0, \infty)$. Thus $f \in C_0^2(0, \infty)$ and $D(A) = C_0^2(0, \infty)$. $\square$

4 Existence of a Sticky Boundary

We use the notion of a random time-change in order to investigate the boundary behavior of a reflected Brownian motion. We follow the work done in [5, Ch. 15, Sec. 8].

Definition 5. A process $A_t(\omega)$ is called an *additive functional* associated with a diffusion process X_t if

$$A_{t+\tau}(\omega) = A_t(\omega) + A_\tau(\omega_t^+) \quad (4.1)$$

where ω_t^+ represents the sample path that has been translated by t time units.

We formally define the inverse of such a process as

$$A^{-1}(s) = \inf\{t | A_t > s\} \quad (4.2)$$

and further, due to [5, p. 253-4], it is sufficient that the process be right continuous and increasing in order for its inverse to be right continuous and increasing. Such a process will be used as a random time-change in the following theorem.

Theorem 3. A reflecting Brownian motion $X_t = |B_t|$ with state space $[0, \infty)$ has a sticky boundary at 0 under the random time-change $T_t = t + \frac{m(\{0\})}{2} L_t$.

Proof. Take $X_t = |B_t|$ to be reflecting Brownian motion where B_t is a standard Brownian motion. Take

$$L_t = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{(-\epsilon, \epsilon)}(X_s) ds \quad (4.3)$$

to be the local time process at the origin [5, p. 257] (cf. (6.14)). The local time process (1.4) determines an increasing additive functional since the occupation time of the process, that is, the integral in (1.4), determines an increasing additive functional [4, p.201-3]. Further,

$$T_t = t + \frac{m(\{0\})}{2} L_t,$$

for $m(\{0\}) > 0$ fixed, is an additive process since

$$\begin{aligned} T_{t+\tau}(0) &= t + \tau + \frac{m(\{0\})}{2} L_{t+\tau}(0) \\ &= t + \frac{m(\{0\})}{2} L_t(0) + \tau + \frac{m(\{0\})}{2} L_\tau(0_t^+) \text{ since } L_t \text{ is additive} \\ &= T_t(0) + T_\tau(0_t^+). \end{aligned}$$

Moreover, T_t increases continuously for a fixed sample path as L_t does. We take T_s^{-1} as the random time-change for the reflected Brownian motion process X_t by constructing a new process

$$Y_s = X_{T_s^{-1}}. \quad (4.4)$$

Note that Y_s has the same state space as X_t , that is $[0, \infty)$. We evaluate the occupation time process for Y_s at $\{0\}$, that is, we calculate

$$\int_0^t \mathbf{1}_{\{0\}}(Y_s) ds = \int_0^t \mathbf{1}_{\{0\}}(X_{T_s^{-1}}) ds. \quad (4.5)$$

Next, make the change of variable for each sample path, $s = T_\tau$ and $\tau = T_s^{-1}$ so that the integral on the right becomes

$$\int_0^{T_t^{-1}} \mathbf{1}_{\{0\}}(X_\tau) d\tau + \frac{m(\{0\})}{2} \int_0^{T_t^{-1}} \mathbf{1}_{\{0\}}(X_\tau) dL_\tau. \quad (4.6)$$

We claim that

$$\int_0^{T_t^{-1}} \mathbf{1}_{\{0\}}(X_\tau) d\tau \equiv 0. \quad (4.7)$$

Indeed, since this is the occupation time of the set $\{0\}$ for the Brownian motion process evaluated at time T_t^{-1} , and the occupation time of a point (as measured by the Lebesgue integral) is 0, (4.7) holds. Moving

to the second integral, note that the (1.3) can differ from t only at the time points where a sample path contacts the origin. Formally,

$$dT_t = \begin{cases} 1 + \frac{m(\{0\})}{2} dL_t & B_t = 0 \\ 1 & B_t \neq 0. \end{cases} \quad (4.8)$$

Thus, L_t increases only when the process B_t contacts the origin and thus $dL_\tau > 0$ only where $\mathbf{1}_{\{0\}}(B_\tau) = 1$. Thus, the second integral in (4.6) becomes

$$\frac{m(\{0\})}{2} \int_0^{T_t^{-1}} dL_\tau = \frac{m(\{0\})}{2} L_{T_\tau^{-1}}.$$

Since the local time process $L_{T_\tau^{-1}}$ is strictly positive, and $m(\{0\})$ is as well, the occupation time (4.5) of the process Y_t is positive for all $t \geq 0$. Thus, the barrier $\{0\}$ is a sticky boundary for the process $Y_t = X_{T_s^{-1}}$. $\square$

5 Deriving the Sticky Boundary Condition

From [5, Ch. 15, Sec. 3] we construct differential equations from functionals such as (5.1) below.

Let X_t be a diffusion process and consider, for $g \in b\mathcal{B}$ and continuous, the equation

$$w(x) = E^x \left[\int_0^{T_a \wedge T_b} g(X_s) ds \right] \quad (5.1)$$

where $T_a \wedge T_b$ is the first time the process reaches a or b . The integral is well-defined since the sample paths of a diffusion process are continuous. For the case of a standard Brownian motion, it can be shown that $w(x)$ solves the equation

$$-g(x) = \frac{1}{2} \frac{d^2 w(x)}{dx^2}, \quad 0 < a < x < b \quad (5.2)$$

when $w(a) = w(b) = 0$.

Integrating twice and applying the boundary conditions yields

$$w(x) = \int_a^b G(x, y) g(y) dy \quad (5.3)$$

where

$$G_{[a,b]}(x, y) = \begin{cases} \frac{(x-a)(b-y)}{b-a} & 0 < a \leq x \leq y \leq b \\ \frac{(y-a)(b-x)}{b-a} & 0 < a \leq y \leq x \leq b \\ 0 & \text{else.} \end{cases} \quad (5.4)$$

In order to study equation (5.2) for the case when $a = 0$, we introduce the notion of a speed measure, m . From [9, p. 301-5] we have the following.

Theorem 4. *There is a unique Radon measure m on $\mathbb{R}$ such that for any open sub-interval $I = (a, b)$ of $\mathbb{R}$,*

$$m_I(x) = \int G_I(x, y) m(dy). \quad (5.5)$$

Definition 6. This measure m is the *speed measure*⁴ of the diffusion process X_t in (5.1).

⁴Note that for a diffusion, this is the same speed measure as defined in [5, p. 194-5].

Intertwined with the notion of a speed measure is the notion of scaling. We work with standard Brownian motion, however, and we take what is called the scale function in [9] (or the scale density in [5]) to be $s(x) = x$. The definition of the function G can be extended to allow for $a = 0$ with the following equation.

$$G_{[0,b]}(x, y) = G_{[-b,b]}(x, y) + G_{[-b,b]}(x, -y) \quad (5.6)$$

$$= \begin{cases} b - y, & -b \leq x \leq y \leq b \\ b - x, & -b \leq y \leq x \leq b. \end{cases} \quad (5.7)$$

Further, by [9, Prop. 3.10],

$$E^x \left[\int_0^{T_b} g(X_s) ds \right] = \int_{[0,b]} G_{[0,b]}(x, y) g(y) m(dy), \quad (5.8)$$

i.e., we can take $m(\{0\})$ to be the mean value of T_b for any $x \in [0, b)$. We can now state the main result.

Theorem 5. *We have*

$$m(\{0\})Af(0) = f'(0+) \quad (5.9)$$

where A is the infinitesimal generator for a reflecting Brownian motion $X_t = |B_t|$.

Proof. Take T to be a stopping time with finite mean as in the proof of [9, Theorem 3.12]. Then we have what is sometimes referred to as Dynkin's formula (cf. [7, p. 127])

$$E^x[f(X_T)] - f(x) = E^x \left[\int_0^T Af(X_s) ds \right]. \quad (5.10)$$

Let $T = T_b$ and from (5.8) and (5.6) above, and by noting that $X_{T_b} = b$, we have

$$f(b) - f(x) = \int_0^x (b - x)Af(y)m(dy) + \int_x^b (b - y)Af(y)dy \quad (5.11)$$

$$\implies \frac{f(b) - f(x)}{b - x} = \int_0^x Af(y)m(dy) + \int_x^b \frac{b - y}{b - x} Af(y)dy \quad (5.12)$$

Next, take the limit as $b \rightarrow x$, which zeros out the Lebesgue integral on the right side so that

$$\frac{df}{dx}(x+) = \int_0^x Af(y)m(dy). \quad (5.13)$$

Then, we take the limit as $x \rightarrow 0$ and we have

$$\frac{df}{dx}(0+) = \int \mathbf{1}_{\{0\}}(y)Af(y)m(dy) \quad (5.14)$$

and by Lebesgue's differentiation theorem,

$$\frac{df}{dx}(0+) = Af(0)m(\{0\}). \quad (5.15)$$

Since $Af(x)$ is continuous at 0,

$$\frac{df}{dx}(0+) = \frac{1}{2}f''(0)m(\{0\}). \quad (5.16)$$

Thus, the stickiness parameter, $m(\{0\})$, in (1.1) is the speed measure of the Brownian process as it occupies the boundary 0. $\square$

6 Appendix

The definition of a stochastic integral can be motivated by the fact that

$$\int_0^t X_t(\omega) dB_t(\omega)$$

can not be well-defined as a path-wise integral. This is essentially due to the fact that $\int_0^t |dB_t|$, which is the total variation of the measure associated with $B_t(\omega)$, is not well-defined. More to the point, the quantity

$$\sum_{j=0}^{n-1} |B_{t_{j+1}} - B_{t_j}|$$

increases without bound as $n \rightarrow \infty$. This means that we can not define integration with respect to $dB_t(\omega)$ in the usual way. We follow the Itô integral construction as in [6, Ch. 2].

Definition 7. Take a process $X_t(\omega) : [0, \infty) \times \Omega \rightarrow \mathbb{R}$ on the probability space $(\Omega, \mathcal{F}, Pr)$ such that

- i) the map $(t, \omega) \rightarrow X_t(\omega)$ is $\mathcal{B}([0, \infty)) \times \mathcal{F}$ -measurable,
- ii) $X_t(\omega)$ is $\mathcal{F}_t$ -adapted, and
- iii)

$$E \left[\int_0^\infty X_t(\omega)^2 dt \right] < \infty. \quad (6.1)$$

Write $X_t(\omega) = X_t$. We define the *Itô integral of X_t* as

$$\int_0^\infty X_t dB_t = \lim_{n \rightarrow \infty} \int_0^\infty \phi_n dB_t, \quad (6.2)$$

where the limit is in $L^2(P)$ and for all $n \geq 0$, ϕ_n is a simple process such that

$$E \left[\int_0^\infty (X_t - \phi_n)^2 dt \right] \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (6.3)$$

We have the following isometry property.

Proposition 1. (*Itô Isometry*) For any simple process ϕ ,

$$E \left[\left(\int_0^\infty \phi_t dB_t \right)^2 \right] = E \left[\int_0^\infty \phi_t^2 dt \right] \quad (6.4)$$

Example 1. We show by example that the Itô integral does not obey the classical chain rule. Take a Brownian motion B_t . We compute, for $T > 0$,

$$\int_0^T B_t dB_t = \frac{1}{2} B_T^2 - \frac{1}{2} T. \quad (6.5)$$

Proof. B_t is continuous in $L^2(\Omega)$. Thus, an approximating sequence of simple processes for B_t is

$$\sum_{j=0}^{n-1} B_{t_j} \mathbf{1}_{(t_j, t_{j+1}]}(t)$$

where $t_j = jT/n$. Note that

$$B_{t_{j+1}}^2 - B_{t_j}^2 = (B_{t_{j+1}} - B_{t_j})^2 + 2B_{t_j}(B_{t_{j+1}} - B_{t_j})$$

and thus

$$\begin{aligned}
\int_0^T B_t dB_t &= \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} B_{t_j} (B_{t_{j+1}} - B_{t_j}) \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} (B_{t_{j+1}}^2 - B_{t_j}^2) - \frac{1}{2} \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})^2 \\
&= \frac{1}{2} B_T^2 - \frac{1}{2} T,
\end{aligned}$$

where the convergence holds in $L^2(\Omega)$. $\square$

Denote $L_{loc}^p(\mathcal{P})$ for $p = 1, 2$, to be the space of progressively measurable processes $v(t)$ such that for all $t > 0$,

$$Pr \left(\int_0^t |v(s)|^p ds < \infty \right) = 1. \quad (6.6)$$

Definition 8. A continuous and adapted stochastic process X_t is called an *Itô process* if

$$X_t = X_0 + \int_0^t u(s) dB_s + \int_0^t v(s) ds, \quad (6.7)$$

where $u \in L_{loc}^2(\mathcal{P})$, $v \in L_{loc}^1(\mathcal{P})$ and X_0 is an $\mathcal{F}_0$ -measurable random variable. Note that $\mathcal{F}_0$ -measurable random variables are constant a.s.

Theorem 6. (*Itô's Formula*) Let $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a function of class $C^{1,2}$, i.e., it is C^1 in its first parameter and C^2 in the second. Suppose that X_t is an Itô process as defined above. Then the process $Y_t = f(t, X_t)$ is also an Itô process with representation

$$Y_t = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) \cdot u(s) dB_s \quad (6.8)$$

$$+ \int_0^t \frac{\partial f}{\partial x}(s, X_s) \cdot v(s) ds + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s) \cdot u(s)^2 ds, \quad (6.9)$$

which holds a.s. for all $t \geq 0$.

Note that the processes u, v and the function f were chosen so that the integrals above are all well-defined. When X_t is a standard Brownian motion, Itô's formula can be written as

$$f(t, X_t) = f(0, 0) + \int_0^t \frac{\partial f}{\partial x}(s, B_s) dB_s + \int_0^t \frac{\partial f}{\partial t}(s, B_s) ds + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s) ds \quad (6.10)$$

by taking $u(t) = 1, v(t) = 0, X_0 = 0$.

Moreover, if we fix time, then for $f : \mathbb{R} \rightarrow \mathbb{R} \in C^2$, then

$$f(B_t) - f(B_0) = \int_0^t \frac{\partial f}{\partial x}(s, B_s) dB_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s) ds. \quad (6.11)$$

If we apply Itô's formula for a standard Brownian motion to the function

$$f_\epsilon(x) = \begin{cases} x & |x| \geq \epsilon \\ \epsilon + \frac{x^2}{2} & |x| < \epsilon, \end{cases} \quad (6.12)$$

we will find a representation of the local time process. For, by applying (6.10) we have

$$f_\epsilon(X_t) = f_\epsilon(B_0) + \int_0^t \frac{B_s}{\epsilon} \mathbf{1}_{(-\epsilon, \epsilon)}(B_s) dB_s + \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{(-\epsilon, \epsilon)}(B_s) ds. \quad (6.13)$$

By applying (6.4) to the central integral we have

$$\begin{aligned} E \left[\left(\int_0^t \frac{B_s}{\epsilon} \mathbf{1}_{(-\epsilon, \epsilon)}(B_s) dB_s \right)^2 \right] &= E \left[\int_0^t \frac{B_s^2}{\epsilon^2} \mathbf{1}_{(-\epsilon, \epsilon)}(B_s) ds \right] \\ &\leq E \left[\int_0^t \mathbf{1}_{(-\epsilon, \epsilon)}(B_s) ds \right] \end{aligned}$$

Taking the limit as $\epsilon \rightarrow 0$ sends the occupation time of the process $B_t \in (-\epsilon, \epsilon) \rightarrow 0$. Thus, by passage of the limit under the expectation, the central term in (6.13) tends to 0 in the L^2 sense. Taking the limit of the remaining terms in (6.13), we are left with Tanaka's formula

$$|B_t| = |B_0| + \int_0^t \text{sgn}(B_s) dB_s + L_t \tag{6.14}$$

where $\text{sgn}(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0. \end{cases}$

7 References

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