

Probability Densities of One-Dimensional Diffusions

CM

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1 The Simplest Case

Consider the infinitesimal generator

$$A = \frac{1}{2} \frac{d^2}{dx^2}$$

of a diffusion process X_t defined on $\mathbb{R}$. Take the domain of A to be a subset of the continuous and bounded functions on $\mathbb{R}$, denoted $C(\mathbb{R})$. Consider the KBE

$$\frac{\partial}{\partial t} p_t(x, y) = A p_t(x, y),$$

where $p_t(x, y)$ is the transition probability density function, and assume the initial condition

$$\lim_{t \rightarrow 0} p_t(x, y) = \delta_y(x).$$

Take the Laplace transform of the KBE with respect to time, t . We get

$$(\lambda - A) \tilde{p}_\lambda(x, y) = \delta_x(y)$$

where $\tilde{p}_\lambda(x, y)$ is the Laplace transform of $p_t(x, y)$ w.r.t. t . To see why, integrate by parts with $u(t) = e^{-\lambda t}$ and $dv(t) = \frac{\partial}{\partial t} p_t(x, y) dt$ and use the initial condition. We will solve for $\tilde{p}_\lambda(x, y)$ using Green's function techniques. Then we will undo the Laplace transform to recover $p_t(x, y)$.

First consider the homogeneous equation

$$(\lambda - A) \phi(x) = 0$$

which simplifies to

$$\frac{d^2}{dx^2} \phi(x) = 2\lambda \phi(x).$$

Then

$$\phi(x) = C_1 e^{\sqrt{2\lambda}x} + C_2 e^{-\sqrt{2\lambda}x}.$$

For $x < y$, $\tilde{p}_\lambda(x, y)$ must be a solution to the homogeneous equation and moreover it must vanish as $x \rightarrow -\infty$. Similarly for $x > y$, $\tilde{p}_\lambda(x, y)$ must vanish as $x \rightarrow \infty$. So we have

$$\tilde{p}_\lambda(x, y) = \begin{cases} C_2 e^{-\sqrt{2\lambda}x}, & x < y \\ C_1 e^{\sqrt{2\lambda}x}, & x > y. \end{cases}$$

Continuity at $x = y$ is assumed, and thus we have the relation

$$C_1 e^{\sqrt{2\lambda}y} = C_2 e^{-\sqrt{2\lambda}y}.$$

Further, there is a jump in the derivative of $\tilde{p}_\lambda(x, y)$ implied by the delta function in $(\lambda - A) \tilde{p}_\lambda(x, y) = \delta_x(y)$:

$$\frac{\partial}{\partial x} \tilde{p}_\lambda(x, y)|_{x=y+} - \frac{\partial}{\partial x} \tilde{p}_\lambda(x, y)|_{x=y-} = 1.$$

This becomes

$$C_1 \sqrt{2\lambda} e^{\sqrt{2\lambda}y} - (-C_2 \sqrt{2\lambda} e^{-\sqrt{2\lambda}y}) = 1.$$

Then some algebraic manipulation gives

$$C_1 = \frac{1}{2\sqrt{2\lambda}} e^{-\sqrt{2\lambda}y}, \quad C_2 = \frac{1}{2\sqrt{2\lambda}} e^{\sqrt{2\lambda}y}.$$

Then the Green's function is

$$\tilde{p}_\lambda(x, y) = \begin{cases} \frac{1}{2\sqrt{2\lambda}} e^{\sqrt{2\lambda}(y-x)}, & x < y \\ \frac{1}{2\sqrt{2\lambda}} e^{\sqrt{2\lambda}(x-y)}, & x > y. \end{cases}$$

Now consider the identity

$$\mathcal{L} \left[\frac{1}{\sqrt{t}} e^{-a/t} \right] (\lambda) = \frac{\sqrt{\pi}}{\sqrt{\lambda}} e^{-2\sqrt{a\lambda}}$$

which can be found in Roberts and Kaufman's Table of Laplace Transforms p.22. We assume that this gives the inverse identity

$$\mathcal{L}^{-1} \left[\frac{\sqrt{\pi}}{\sqrt{\lambda}} e^{-2\sqrt{a\lambda}} \right] (t) = \frac{1}{\sqrt{t}} e^{-a/t}.$$

Taking $a = (x - y)^2/2$ and multiplying on both sides of the identity by $(2\sqrt{2\pi})^{-1}$, we get

$$p_t(x, y) = \frac{1}{2\sqrt{2\pi t}} e^{-(x-y)^2/2t}.$$

In the language of the more general case, the boundary points for the functions in the domain of the operator A , $-\infty$ and ∞ , are inaccessible. This condition is in logical correspondence with A being a contractive semi-group in $C([-\infty, \infty])$, and this in turn is in logical correspondence with the equation $(\lambda - A)\tilde{p}_\lambda(x, y) = \delta_x(y)$ having a unique solution in the domain of A .

2 The equation $\lambda F - D_m D_p^+ F = f$ has a unique solution

These notes follow the work done by Mandl, Feller, DT94. We need some definitions first.

A transition function $P(t, x; B)$ will be called Fellerian if for $f \in C(I)$

$$T_t f(x) = \int f(y) P(t, x; dy) \in C(I)$$

and if the semi group $\{T_t\}$ associated with $P(t, x; B)$ is strongly continuous in $C(I)$.

If $\{T_t\}$ is a strongly continuous semi group and A is defined by

$$Af = \lim_{t \rightarrow 0} \frac{T_t f - f}{t}$$

where $D(A)$ contains all functions for which the limit exists, then A is the infinitesimal operator of the semi group $\{T_t\}$. For those $f \in D(A)$ we have, for $t \geq 0$, $T_t f \in D(A)$ and

$$\frac{d}{dt} T_t f = A T_t f = T_t A f.$$

A strongly continuous semi group in C' (a closed linear subspace of $C(I)$) is contractive if $\|T_t\| \leq 1$ and if $f(x) \geq 0$ for $x \in I$, then $T_t f(x) \geq 0$ for $x \in I$. There is a one to one correspondence between Fellerian transition functions and contractive semi groups in $C(I)$.

A linear operator A is the infinitesimal operator of a contractive semi group in C' if and only if

1. $D(A)$ is dense in C' .

2. For $f \in C'$ and $\lambda > 0$, the equation $\lambda F - AF = f$ has a unique solution.
3. If $f(x) \geq 0$ for $x \in I$, then $F(x) \geq 0$ for $x \in I$.
4. $\lambda \|F\| \leq \|f\|$.

The Hille-Yosida Theorem is critical for guaranteeing uniqueness of solutions.

Theorem 1 (Hille-Yosida). Let $\{T_t\}$ be a contractive semi-group with range dense in X . The infinitesimal generator A is an additive operator whose domain is dense in X and such that to each $x \in X$ and to each $\lambda > 0$ there exists a unique $y_\lambda \in D(A)$ with

$$\lambda y_\lambda - Ay_\lambda = x, \quad \lambda \|y_\lambda\| \leq \|x\|$$

and $y_\lambda \geq 0$ whenever $x \geq 0$.

We now consider the infinitesimal generator

$$A = \frac{1}{2} \frac{d^2}{dx^2} + k \frac{d}{dx}$$

of a diffusion process X_t with drift, and further, we insist that X_t is defined on the interval $[r_0, r_1]$ where $-\infty \leq r_0 < r_1 \leq \infty$. The domain of A is a subset of the continuous and bounded functions on the interval $[r_0, r_1] \subset \mathbb{R}$, $C([r_0, r_1])$. As before we transform the KBE to

$$(\lambda - A)\tilde{p}_\lambda(x, y) = \delta_x(y)$$

and solve this equation for the Laplace transform of $p_t(x, y)$ and then for $p_t(x, y)$ itself.

Now we define the generalized differential operator $D_m D_p^+$ and show that A can be written as a special case of $D_m D_p^+$. Let $p(x)$ be a strictly increasing function on the interior of I . Let $m(x)$ be a continuous on the right and strictly increasing function on the interior of I . Let x be in the interior of I and take a function real valued f defined in a neighborhood of x . Then

$$D_p^+ f(x) = \lim_{y \rightarrow x^+} \frac{f(y) - f(x)}{p(y) - p(x)} \text{ and } D_m f(x) = \lim_{y \rightarrow x} \frac{f(y) - f(x)}{m(y) - m(x)}$$

provided that the limits exist. A function $f \in C(I)$ belongs to the domain of the operator $D_m D_p^+$ if $D_m D_p^+ f(x)$ exists and is continuous in the interior of I and if the limits $D_m D_p^+ f(r_i)$ exist for $i = 0, 1$, i.e., $D_m D_p^+ f \in C(I)$.

We can formally write the differential operator $a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}$ as a particular case of $D_m D_p^+$. Take

$$B(x) = \int^x b(y) a(y)^{-1} dy,$$

$$dm(x) = a(x)^{-1} e^{B(x)} dx, \text{ and } dp(x) = e^{-B(x)} dx,$$

where $D_m = \frac{d}{dm(x)}$ and $D_p = \frac{d}{dp(x)}$. We will continue with a more general situation before specializing to $A = 1/2 \frac{d^2}{dx^2} + k \frac{d}{dx}$ in section 5.

The boundary points r_0, r_1 of the interval I will be called the boundaries of the operator $D_m D_p^+$. Their classification depends on two values:

$$u^1(x) = \int_0^x m(s) dp(s) \text{ and } v^1(x) = \int_0^x p(s) dm(s).$$

We give conditions for the equation

$$(\lambda - D_m D_p^+) F = f$$

to have unique solutions. We write A in the form $D_m D_p^+$ and we show that in order to maintain the operator $D_m D_p^+$'s (A 's) status as the infinitesimal operator of a contractive semi group (and thus maintaining unique solutions to $(\lambda - D_m D_p^+) F = f$), we must impose boundary conditions on the functions in the domain of $D_m D_p^+$. Moreover we will show that the imposed boundary conditions depend on the accessibility of the boundary points of the interval $[r_0, r_1]$.

r_i is accessible if $u^1(r_i) < \infty$	r_i is inaccessible if $u^1(r_i) = \infty$
regular if $v^1(r_i) < \infty$	natural if $v^1(r_i) < \infty$
exit boundary if $v^1(r_i) = \infty$	entrance boundary if $v^1(r_i) = \infty$

Theorem 2. $D_m D_p^+$ is the infinitesimal operator of a Fellerian transition function (a contractive semi groups in $C(I)$) if and only if the boundary points r_0 and r_1 are both inaccessible.

Proof. Keeping in mind that contractive semi groups are in one to one correspondence with Fellerian transition functions, we show that the four conditions above hold if and only if r_0 and r_1 are inaccessible boundaries of $D_m D_p^+$.

1. Claim: $D(D_m D_p^+)$ is dense in $C(I)$. Proof: By a change of variables, $y = p(x)$, it suffices to show that $D_m D_x^+$ has dense domain in $C(I)$. Indeed, make a change of variables $y = p(x)$ that maps $f(x) \in C(I)$ into

$$\hat{f}(y) = f(p^{-1}(y)) \in C([p(r_0), p(r_1)]),$$

and $D_m D_p^+$ into $D_{\hat{m}} D_y^+$ where $\hat{m}(y) = m(p^{-1}(y))$. Thus it suffices to prove that $D_m D_x^+$ has dense domain in $C(I)$. Next, we use the density of the AC functions in $C(I)$: For $f \in C(I)$ and $\epsilon > 0$, there is a g ,

$$g(x) = g(0) + \int_0^x g'(y) dy$$

with $\|f - g\| < \epsilon$. Next there is an h ,

$$h(x) = \int_0^x k(y) dm(y), \quad k \in C(I)$$

for which

$$\int |g'(x) - h(x)| dx < \epsilon.$$

See page 24 of Mandl for an argument for the existence of such an h . Set

$$g_0(x) = g(0) + \int_0^x \int_{r_0}^y k(s) dm(s) dy.$$

Then $g_0 \in D(D_m D_x^+)$ with $D_m D_x^+ g_0 = k$. Finally,

$$\|f - g_0\| \leq \|f - g\| + \int |h(y) - g'(y)| dy < 2\epsilon.$$

2. Claim: For $f \in C(I)$ and $\lambda > 0$, the equation $\lambda F - D_m D_p^+ F = f$ has a unique solution iff r_0 and r_1 are inaccessible. Proof: The general form of a solution of the equation $\lambda F - D_m D_p^+ F = f$ is

$$F(x) = \int R^0(x, y) f(y) dm(y) + c_+ u_+(x) + c_- u_-(x).$$

If r_0 is accessible, then $u_+(x)$ is in $C(I)$ and the solution of $\lambda F - D_m D_p^+ F = f$ is not unique. Similarly for r_1 being accessible, we get that $u_-(x) \in C(I)$ and again we lose the uniqueness of the solution. Next, when both boundaries are inaccessible, we have $u_+(r_0) = u_-(r_1) = \infty$ and $c_+ = c_- = 0$, and then

$$F(x) = \int R^0(x, y) f(y) dm(y)$$

is the unique solution of $\lambda F - D_m D_p^+ F = f$.

3. Claim: $f(x) \geq 0$ for $x \in I$ implies $F(x) \geq 0$ for $x \in I$, if and only if r_0 and r_1 are inaccessible. Proof: Since $R^0(x, y) \geq 0$, and since we know the unique solution must be of the form just written, the result follows.
4. Claim: $\lambda \|F\| \leq \|f\|$. Proof: Write the unique solution of $\lambda F - D_m D_p^+ F = f$ when $f = 1$, $F(x) = \lambda^{-1}$. The inequality is satisfied for this pair; see the remark on page 5 of Mandl to see that this suffices to prove the inequality for any $f \in C(I)$.

□

Thus the equation $\lambda F - D_m D_p^+ F = f$ has a unique solution in $C([-\infty, \infty])$.

3 Solutions of $\lambda F - D_m D_p^+ F = f$ on $[-\infty, \infty]$

We use an iterative method to find solutions of $\lambda w - D_m D_p^+ w = 0$. Define

$$u(x, \lambda) = \sum_{n=0}^{\infty} \lambda^n u^n(x)$$

where

$$u^0(x) = 1 \text{ and } u^{n+1}(x) = \int_0^x \int_0^y u^n(s) dm(s) dp(y).$$

Proposition 3.1. $u(x, y)$ converges for all $\lambda > 0$.

Proof. First, the inequality

$$u^k(x) \leq u^j(x) \frac{u^1(x)^{k-j}}{(k-j)!}$$

holds for $j \leq k$. This follows by induction on k . For $k = j$ the inequality is trivial. Supposing that the inequality holds for some k , then

$$\begin{aligned} u^{k+1}(x) &= \int_0^x \int_0^y u^k(s) dm(s) dp(y) \\ &\leq \int_0^x \int_0^y u^j(x) \frac{u^1(x)^{k-j}}{(k-j)!} \\ &= u^j(x) \int_0^x \frac{u^1(x)^{k-j}}{(k-j)!} \int_0^y dm(s) dp(y) \\ &= u^j(x) \int_0^x \frac{u^1(x)^{k-j}}{(k-j)!} du^1(y) \\ &= u^j(x) \frac{u^1(x)^{k+1-j}}{(k+1-j)!} \end{aligned}$$

where we used the fact that $du^1(y) = dm(y)dp(y)$. Thus the inequality holds for $j \leq k$ and thus for any $n \geq 0$,

$$u^n(x) \leq \frac{u^1(x)^n}{n!}.$$

Hence,

$$\begin{aligned} u(x, \lambda) &= \sum_{n=0}^{\infty} \lambda^n u^n(x) \\ &\leq \sum_{n=0}^{\infty} \lambda^n \frac{u^1(x)^n}{n!} \\ &= e^{\lambda u^1(x)}. \end{aligned}$$

The exponential series converges for all $\lambda > 0$ so the result follows. □

Proposition 3.2. The function $u(x, \lambda)$ is a solution of $\lambda w - D_m D_p^+ w = 0$.

Proof. We have the relation $D_m D_p^+ u^{n+1}(x) = u^n(x)$. Then

$$\begin{aligned}
\lambda u(x, \lambda) - D_m D_p^+ u(x, \lambda) &= \sum_{n=0}^{\infty} \lambda^{n+1} u^n(x) - \sum_{n=0}^{\infty} \lambda^n D_m D_p^+ u^n(x) \\
&= \sum_{n=0}^{\infty} \lambda^{n+1} u^n(x) - \sum_{n=0}^{\infty} \lambda^n u^{n-1}(x) \\
&= \sum_{n=1}^{\infty} \lambda^n u^{n-1}(x) - \sum_{n=0}^{\infty} \lambda^n u^{n-1}(x) \\
&= \sum_{n=1}^{\infty} \lambda^n u^{n-1}(x) - \left(1 \cdot u^{-1}(x) + \sum_{n=1}^{\infty} \lambda^n u^{n-1}(x) \right) \\
&= -u^{-1}(x) \\
&= 0
\end{aligned}$$

where the last inequality holds because $0 = D_m D_p^+ u^0(x) = u^{-1}(x)$. $\square$

Let $w_1(x)$ and $w_2(x)$ be two solutions of $\lambda w - D_m D_p^+ w = 0$. Write the Wronskian

$$W(w_1, w_2)(x) = (D_p^+ w_1(x))w_2(x) - (D_p^+ w_2(x))w_1(x)$$

and note that it is independent of x (which confirms the linear independence of w_1, w_2). Indeed the derivative of the Wronskian is zero, implying that it is constant:

$$\begin{aligned}
\frac{d}{dx} W(w_1, w_2)(x) &= \frac{d}{dx} [(D_p^+ w_1(x))w_2(x) - (D_p^+ w_2(x))w_1(x)] \\
&= \frac{d}{dx} (D_p^+ w_1(x))w_2(x) + D_p^+ w_1(x) \frac{d}{dx} w_2(x) \\
&\quad - \frac{d}{dx} (D_p^+ w_2(x))w_1(x) - D_p^+ w_2(x) \frac{d}{dx} w_1(x) \\
&= \lambda w_1(x)w_2(x)m(x) + D_p^+ w_1(x)\lambda w_2(x) - \lambda w_2(x)w_1(x)m(x) - D_p^+ w_2(x)\lambda w_1(x) \\
&= 0.
\end{aligned}$$

Let w, w_1, w_2 be solutions of $\lambda w - D_m D_p^+ w = 0$ and let the Wronskian be nonzero. Then w can be written $w = c_1 w_1 + c_2 w_2$ where c_1 and c_2 are constants. For $\lambda > 0$, define

$$u_+(x, \lambda) = u(x, \lambda) \int_x^{r_1} u(y, \lambda)^{-2} dp(y)$$

and

$$u_-(x, \lambda) = u(x, \lambda) \int_{r_0}^x u(y, \lambda)^{-2} dp(y).$$

We say more about these functions (solutions) in the next section. $u(0, \lambda) = 1$ and $D_p^+ u(0, \lambda) = 0$ so we have $W(u, u_+) = -D_p^+ u_+(0) = 1$. Since the Wronskian is independent of x , a solution to $\lambda w - D_m D_p^+ w = 0$ must be of the form $w(x) = c_1 u(x) + c_2 u_+(x)$.

Now we focus on the non-homogeneous equation. Take $W = W(u_-, u_+) > 0$. Define the kernel

$$R^0(x, y, \lambda) = \begin{cases} W^{-1} u_-(x) u_+(y), & r_0 < x \leq y < r_1 \\ W^{-1} u_-(y) u_+(x), & r_0 < y \leq x < r_1. \end{cases}$$

For $f \in C(I^\circ)$ (continuous and bounded functions on the interior of I), we have

Proposition 3.3.

$$F_0(x) = \int R^0(x, y) f(y) dm(y)$$

satisfies the non-homogeneous equation $\lambda F_0 - D_m D_p^+ F_0 = f$.

Proof. We have

$$\begin{aligned} W \int R^0(x, y) f(y) dm(y) &= u_+(x) \int_{r_0}^x u_-(y) f(y) dm(y) + u_-(x) \int_x^{r_1} u_+(y) f(y) dm(y) \\ &= \lambda^{-1} u_+(x) \int_{r_0}^x f(y) dD_p^+ u_-(y) + \lambda^{-1} u_-(x) \int_x^{r_1} f(y) dD_p^+ u_+(y). \end{aligned}$$

Since $D_p^+ u_-(y)$ and $D_p^+ u_+(y)$ are of bounded variation on $[r_0, x]$ and $[x, r_1]$ respectively, $F_0(x)$ is well-defined in (r_0, r_1) . Further,

$$\begin{aligned} W(F_0(x_1) - F_0(x_0)) &= W \int_{x_0}^{x_1} dF_0(x) \\ &= \int_{x_0}^{x_1} \int_{r_0}^x u_-(y) f(y) dm(y) du_+(x) + \int_{x_0}^{x_1} u_+(x) u_-(x) f(x) dm(x) \\ &\quad + \int_{x_0}^{x_1} \int_x^{r_1} u_+(y) f(y) dm(y) du_-(x) - \int_{x_0}^{x_1} u_+(x) u_-(x) f(x) dm(x) \\ &= \int_{r_0}^{x_1} \left(\int_{r_0}^x u_-(y) f(y) dm(y) \right) D_p^+ u_+(x) dp(x) \\ &\quad + \int_{x_0}^{x_1} \left(\int_x^{r_1} u_+(y) f(y) dm(y) \right) D_p^+ u_-(x) dp(x) \end{aligned}$$

Thus we have

$$W D_p^+ F_0(x) = D_p^+ u_+(x) \int_{r_0}^x u_-(y) f(y) dm(y) + D_p^+ u_-(x) \int_x^{r_1} u_+(y) f(y) dm(y).$$

Next apply D_m and since the Wronskian is independent of x we have

$$W D_m D_p^+ F_0(x) = \lambda W F_0(x) - W f(x).$$

□

Further, $f(r_i) = \lim_{x \rightarrow r_i} f(x)$ exists $\implies F_0(r_i) = \lim_{x \rightarrow r_i} F_0(x)$ exists.

Proposition 3.4. Next, $R^0(x, y)$ solves

$$(\lambda - D_m D_p^+) R^0(x, y) = \delta_y(x) \left(\frac{dm(y)}{(dy)} \right)^{-1}$$

with the boundary conditions

1. $R^0(x, y)$ is continuous at $x = y$.
2. $(D_p^+ R^0(x, y)|_{x=y+} - D_p^+ R^0(x, y)|_{x=y+}) = -1$.
3. $R^0(x, y) = 0$ at $x = r_0$.
4. $D_p^+ R^0(x, y) = 0$ at $x = r_1$.
5. $R^0(x, y)$ is increasing (decreasing) in x for $x < y$ ($x > y$).

Proof. Let $x \leq y$. Then $R^0(x, y, \lambda) = W^{-1} u_-(x) u_+(y)$ and we have

$$\begin{aligned} \lambda R^0(x, y, \lambda) - D_m D_p^+ R^0(x, y, \lambda) &= \lambda W^{-1} u_-(x) u_+(y) - D_m D_p^+ [W^{-1} u_-(x) u_+(y)] \\ &= W^{-1} u_+(y) [\lambda u_-(x) - D_m D_p^+ u_-(x)]. \end{aligned}$$

Given that $u_-(x)$ satisfies the homogeneous equation $\lambda u_-(x) - D_m D_p^+ u_-(x) = 0$, we have

$$\lambda R^0(x, y, \lambda) - D_m D_p^+ R^0(x, y, \lambda) = W^{-1} u_+(y) \cdot 0 = 0.$$

The case for $x > y$ is the same. When $x = y$, we check

$$(\lambda - D_m D_p^+) W^{-1} u_-(y) u_+(y) = \left(\frac{dm(y)}{dy} \right)^{-1}.$$

This is not done. □

Finally, r_0 inaccessible and r_1 inaccessible implies that $F_0(x)$ is the only solution of $\lambda F_0 - D_m D_p^+ F_0 = f$. This will follow from the work done in the next section where we show that the most general solution for the non-homogeneous equation is of a form that requires accessibility of the boundaries in order for certain terms of the solution to exist.

4 Solutions of $\lambda F - D_m D_p^+ F = f$ on $[r_0, r_1]$

Now we give some specialized solutions depending on the accessibility of the boundaries r_0 and r_1 .

For $\lambda > 0$, define

$$u_+(x, \lambda) = u(x, \lambda) \int_x^{r_1} u(y, \lambda)^{-2} dp(y)$$

and

$$u_-(x, \lambda) = u(x, \lambda) \int_{r_0}^x u(y, \lambda)^{-2} dp(y).$$

The functions $u_+(x, \lambda)$, and $u_-(x, \lambda)$ are well-defined for $\lambda > 0$ and they are solutions of $\lambda w - D_m D_p^+ w = 0$. We take this and the following claims about these functions as fact, but they can be found in Mandl. We organize the characteristics of u_+ and u_- into a table.

$u_+(x, \lambda)$	$u_-(x, \lambda)$
decreasing function of x	increasing function of x
r_1 not an entrance boundary $\implies u_+(r_1, \lambda) = 0$	r_0 not an entrance boundary $\implies u_-(r_0, \lambda) = 0$
$D_p^+ u_+(x, \lambda)$ is increasing in x	$D_p^+ u_-(x, \lambda)$ is decreasing in x
r_1 inaccessible $\implies D_p^+ u_+(r_1, \lambda) = 0$.	r_0 inaccessible $\implies D_p^+ u_-(r_0, \lambda) = 0$.
r_1 natural $\implies u_+(r_1, \lambda) = 0 \wedge D_p^+ u_+(r_1, \lambda) = 0$	r_0 natural $\implies u_-(r_0, \lambda) = 0 \wedge D_p^+ u_-(r_0, \lambda) = 0$

Define

$$u_0(x, \lambda) = u_+(r_0, \lambda)^{-1} u_+(x, \lambda)$$

for r_0 accessible. Further, for r_0 accessible, $u_0 \in C(I)$. Similarly define

$$u_1(x, \lambda) = u_-(r_1, \lambda)^{-1} u_-(x, \lambda)$$

for r_1 accessible, and note that for r_1 accessible, $u_1 \in C(I)$.

From the work done in the last section, for $f \in C(I)$ the general solution of $\lambda F_0 - D_m D_p^+ F_0 = f$ is of the form

$$F(x) = F_0(x) + c_0 u_0(x) + c_1 u_1(x).$$

When r_0 accessible and r_1 inaccessible, $u_- \notin C(I)$ and

$$F(x) = F_0(x) + c u_0(x).$$

When r_0 and r_1 accessible, $u_0, u_1 \in C(I)$ and

$$F(x) = F_0(x) + c_0 u_0(x) + c_1 u_1(x).$$

When r_0 inaccessible and r_1 inaccessible, $F_0(x)$ is the only solution of $\lambda F_0 - D_m D_p^+ F_0 = f$ as we saw in the previous section.

For r_i a regular and accessible boundary point of $D_m D_p^+$, we see next that we must introduce boundary conditions on the functions in the domain of $D_m D_p^+$. These boundary conditions allow us to solve for the constants c, c_0, c_1 in the solutions above.

Theorem 3. Take A to be the infinitesimal operator of a Fellerian transition function $P(t, x; B)$ with $D(A) \subset D(D_m D_p^+)$. Assume A agrees with $D_m D_p^+$ within its domain: For $F \in D(A)$, $AF = D_m D_p^+ F$. Let r_i be a regular and accessible boundary point of $D_m D_p^+$. Then all $F \in D(A)$ satisfy a non trivial boundary condition

$$\phi_i(F) = \chi_i F(r_i) + \int \frac{F(r_i) - F(x)}{|p_i(r_i) - p_i(x)|} dq_i(x) + \sigma_i D_m D_p^+ F(r_i) = 0$$

where χ_i and σ_i are non negative, $q_i(x)$ is a non decreasing and bounded function on I , $p_i(x)$ is a non decreasing and continuous function on I . $p_i(x)$ is equal to $p(x)$ in a neighborhood of r_i .

Further, it must be the case that $\sigma_i > 0$ or $\int \frac{dq_i(x)}{|p_i(r_i) - p_i(x)|} = \infty$.

Proof. In the case when $\sigma_i = 0$ and $\int \frac{dq_i(x)}{|p_i(r_i) - p_i(x)|} < \infty$, then the boundary condition

$$\phi_i(F) = \chi_i F(r_i) + \int \frac{F(r_i) - F(x)}{|p_i(r_i) - p_i(x)|} dq_i(x) = 0$$

defines a proper and closed linear subspace of $C(I)$. This contradicts the fact that $D(A)$ is dense in $C(I)$. $\square$

This then implies that

$$\frac{F(r_i) - F(x)}{|p_i(r_i) - p_i(x)|} \Big|_{x=r_i} = (-1)^{i+1} D_p^+ F(r_i).$$

Here is the proof of the theorem.

Proof. From the relation $D(A) \subset D(D_m D_p^+)$ we indeed have $AF = D_m D_p^+ F$. For $F \in D(A)$,

$$\begin{aligned} D_m D_p^+ F(r_i) &= AF(r_i) = \lim_{t \rightarrow 0} \frac{T_t F(r_i) - F(r_i)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\int F(y) P(t, r_i; dy) - F(r_i)}{t} \\ &= \lim_{t \rightarrow 0} \frac{-\int (F(r_i) - F(y)) P(t, r_i; dy) - r(t) F(r_i)}{t} \end{aligned}$$

where $r(t) = 1 - P(t, r_i; I)$. Let $p_i(x)$ be a function equal to $p(x)$ in a neighborhood of r_i , non-decreasing and continuous on I . Define

$$\begin{aligned} q(t) &= \int |p_i(r_i) - p_i(x)| P(t, r_i; dx) \text{ and} \\ \int_{[r_0, x]} |p_i(r_i) - p_i(y)| P(t, r_i; dy) &= q(t) \chi(x, t) \end{aligned}$$

for $r_0 \leq x \leq r_1$. Then we have

$$\begin{aligned} D_m D_p^+ F(r_i) &= \lim_{t \rightarrow 0} \frac{-\int (F(r_i) - F(y)) P(t, r_i; dy) - r(t) F(r_i)}{t} \\ &\iff \\ \lim_{t \rightarrow 0} \left[\int \frac{F(r_i) - F(x)}{|p_i(r_i) - p_i(x)|} \chi(t, dx) \frac{q(t)}{t} + F(r_i) \frac{r(t)}{t} + D_m D_p^+ F(r_i) \right] &= 0. \end{aligned}$$

Now take $\tau = \{t_k\} \subset \mathbb{R}$ where $\lim_{k \rightarrow \infty} t_k = 0$ and such that the following limits exist, but may not be finite:

$$\lim_{t \in \tau} \frac{q(t)}{t} = Q, \quad \lim_{t \in \tau} \frac{r(t)}{t} = R,$$

and

$$\lim_{t \in \tau} \frac{q(t)}{r(t)} = S.$$

Further, we insist that the sequence $\{\chi(x, t) : t \in \tau\}$ converges to a distribution function $k(x)$ in every continuity point of this function (essential convergence). Such a sequence $\{\chi(x, t) : t \in \tau\}$ exists due to Helly's Theorem. Thus,

$$\lim_{t \in \tau} \int f(x) \chi(t, x) = \int f(x) dk(x)$$

for $f \in C(I)$. We have

$$\frac{F(r_i) - F(x)}{|p_i(r_i) - p_i(x)|} \in C(I)$$

for $F \in D(D_m D_p^+)$. Suppose that $Q + R < \infty$. Then from the limit

$$\lim_{t \rightarrow 0} \left[\int \frac{F(r_i) - F(x)}{|p_i(r_i) - p_i(x)|} \chi(t, dx) \frac{q(t)}{t} + F(r_i) \frac{r(t)}{t} + D_m D_p^+ F(r_i) \right] = 0$$

we have

$$Q \int \frac{F(r_i) - F(x)}{|p_i(r_i) - p_i(x)|} dk(x) + R F(r_i) + D_m D_p^+ F(r_i) = 0.$$

If $Q + R = \infty$ then divide the quantity in the limit by $\frac{q(t) + r(t)}{t}$. We get

$$\frac{S}{S+1} \int \frac{F(r_i) - F(x)}{|p_i(r_i) - p_i(x)|} dk(x) + F(r_i) \frac{1}{1+S} = 0.$$

These last two displayed equations are of the form of the boundary condition in the statement of the theorem. $\square$

Proposition 4.1. For r_0 regular and r_1 inaccessible, and with a boundary condition of the form above, (write $\phi_0(F) = 0$), the non homogeneous equation $\lambda F - D_m D_p^+ F = f$ has a unique solution.

Proof. We have the general form of the solution as

$$F(x) = \int R^0(x, y) f(y) dm(y) + c u_0(x)$$

since r_1 is inaccessible. We can evaluate $\phi_0(F) = 0$ since for any $g \in D(D_m D_p^+)$, $\phi_0(g) < \infty$ and $\phi_0(u_0) > 0$. Thus we find the relationship that uniquely defines the constant c :

$$c = -\phi_0(F_0) \phi_0(u_0)^{-1}.$$

$\square$

5 The Half Line Case

The above theory supports finding pdf's for a variety of boundary conditions. We left out further analysis of the r_0, r_1 both accessible case, but it is similar to the r_0, r_1 accessible, inaccessible case except that one must find two unique constants determined by the boundary condition instead of one. We also skipped the analysis for the cases when r_0 and r_1 are exit or entrance boundaries since regular boundaries are of more interest. Note that this theory does not seem to be sufficient to study one-dimensional diffusions that have a "boundary" in the middle of their domain, e.g., a sticky diffusion on all of $\mathbb{R}$ with a sticky point at 0.

We now consider the special case of an infinitesimal generator

$$A = \frac{1}{2} \frac{d^2}{dx^2} + k \frac{d}{dx}$$

of a diffusion process X_t defined on $[0, \infty)$. The goal is still to get the probability density function, so, as before we solve

$$(\lambda - A)\tilde{p}_\lambda(x, y) = \delta_x(y).$$

We take $dm(x) = 2e^{2kx}dx$ and $dp(x) = e^{-2kx}dx$ so that we can write $A = D_m D_p^+$. Further we have that $r_0 = 0$ is accessible and regular, and, $r_1 = \infty$ is inaccessible and natural. From the work above we must have a solution of the form

$$F(x) = F_0(x) - \frac{\phi_0(F_0)}{\phi_0(u_0)} \cdot \frac{u_+(x, \lambda)}{u_+(r_0, \lambda)}$$

where

$$\int R^0(x, y)f(y)dm(y) \text{ and } R^0(x, y, \lambda) = \begin{cases} W^{-1}u_-(x)u_+(y), & r_0 < x \leq y < r_1 \\ W^{-1}u_-(y)u_+(x), & r_0 < y \leq x < r_1. \end{cases}$$

We will solve for $R^0(x, y)$. From the work above, $R^0(x, y)$ solves

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} - k\frac{d}{dx} + \lambda\right)R^0(x, y) = \frac{1}{2}\delta_y(x)e^{-2ky}$$

with the boundary conditions listed above.

For $x \neq y$, the general solution is of the form

$$R^0(x, y) = \begin{cases} Ae^{-kx+\sqrt{k^2+2\lambda}x} + Be^{-kx-\sqrt{k^2+2\lambda}x}, & \text{for } x < y, \\ Ce^{-kx+\sqrt{k^2+2\lambda}x} + De^{-kx-\sqrt{k^2+2\lambda}x}, & \text{for } x > y. \end{cases}$$

Continuity at $x = y$ implies

$$Ae^{-ky+\sqrt{k^2+2\lambda}y} + Be^{-ky-\sqrt{k^2+2\lambda}y} = Ce^{-ky+\sqrt{k^2+2\lambda}y} + De^{-ky-\sqrt{k^2+2\lambda}y}.$$

The discontinuity in the derivative is

$$\begin{aligned} & e^{2ky} \left(C(-k + \sqrt{k^2 + 2\lambda})e^{-ky+\sqrt{k^2+2\lambda}y} + D(-k - \sqrt{k^2 + 2\lambda})e^{-ky-\sqrt{k^2+2\lambda}y} \right) \\ & - e^{2ky} \left(A(-k + \sqrt{k^2 + 2\lambda})e^{-ky+\sqrt{k^2+2\lambda}y} + B(-k - \sqrt{k^2 + 2\lambda})e^{-ky-\sqrt{k^2+2\lambda}y} \right) = -1. \end{aligned}$$

This cleans up to

$$\left((-k + \sqrt{k^2 + 2\lambda})(C - A)e^{ky+\sqrt{k^2+2\lambda}y} - (-k - \sqrt{k^2 + 2\lambda})(D - B)e^{ky-\sqrt{k^2+2\lambda}y}\right) = -1.$$

Next, $R^0(0, y) = 0$ implies $A = -B$ and $e^{2kx}\frac{d}{dx}R^0(x, y) = 0$ as $x \rightarrow \infty$ implies $C = -D$ and thus $D = -B$. Thus we have

$$R^0(x, y) = \begin{cases} B \left(e^{-kx-\sqrt{k^2+2\lambda}x} - e^{-kx+\sqrt{k^2+2\lambda}x} \right), & \text{for } x < y, \\ B \left(e^{-ky+\sqrt{k^2+2\lambda}y} - e^{-ky-\sqrt{k^2+2\lambda}y} \right), & \text{for } x > y. \end{cases}$$

Then the jump condition in the derivative allows us to solve for B :

$$B = -\frac{1}{2\sqrt{k^2 + 2\lambda}}e^{-ky-\sqrt{k^2+2\lambda}y}.$$

Thus we have

$$R^0(x, y) = \begin{cases} \frac{e^{-ky-\sqrt{k^2+2\lambda}y}}{2\sqrt{k^2+2\lambda}} \left(e^{-kx+\sqrt{k^2+2\lambda}x} - e^{-kx-\sqrt{k^2+2\lambda}x} \right), & \text{for } x \leq y, \\ \frac{e^{-kx-\sqrt{k^2+2\lambda}x}}{2\sqrt{k^2+2\lambda}} \left(e^{-ky+\sqrt{k^2+2\lambda}y} - e^{-ky-\sqrt{k^2+2\lambda}y} \right), & \text{for } x \geq y. \end{cases}$$

This is where we would move straight to the step of inverting the Laplace transform if we had both boundaries being inaccessible. Since we have r_0 accessible (and regular) we now move to finding the constant that uniquely determines the solution.

Now we recall that

$$R^0(x, y, \lambda) = \begin{cases} W^{-1}u_-(x)u_+(y), & r_0 < x \leq y < r_1 \\ W^{-1}u_-(y)u_+(x), & r_0 < y \leq x < r_1. \end{cases}$$

With this representation of R^0 we can find u_+ and u_- . Up to a constant $\alpha \geq 0$,

$$u_- = \frac{1}{\alpha\sqrt{k^2+2\lambda}} \left(e^{-kx+\sqrt{k^2+2\lambda}x} - e^{-kx-\sqrt{k^2+2\lambda}x} \right), \text{ and } u_+ = \alpha e^{-kx-\sqrt{k^2+2\lambda}x}.$$

Now we find the next component of the solution $-\frac{\phi_0(F_0)}{\phi_0(u_0)}$. We have

$$-\phi_0(F_0) = \pi_0 \int_0^\infty e^{-ky-\sqrt{k^2+2\lambda}y} f(y) 2e^{2ky} dy + \sigma_0 f(0)$$

and

$$\phi_0(u_0) = \chi_0 + \pi_0(k + \sqrt{k^2+2\lambda}) + \lambda\sigma_0.$$

These come from direct calculations like this:

For $x \leq y$,

$$F_0(x) = \int_x^{r_1} \frac{e^{-ky-\sqrt{k^2+2\lambda}y}}{2\sqrt{k^2+2\lambda}} \left(e^{-kx+\sqrt{k^2+2\lambda}x} - e^{-kx-\sqrt{k^2+2\lambda}x} \right) f(y) 2e^{2ky} dy.$$

We apply the operator $D_p^+ = \frac{d}{dp(x)} = e^{2kx} \frac{d}{dx}$.

$$\begin{aligned} D_p^+ F_0(x) &= e^{2kx} \frac{d}{dx} \left[\int_x^{r_1} \frac{e^{-ky-\sqrt{k^2+2\lambda}y}}{2\sqrt{k^2+2\lambda}} \left(e^{-kx+\sqrt{k^2+2\lambda}x} - e^{-kx-\sqrt{k^2+2\lambda}x} \right) f(y) 2e^{2ky} dy \right] \\ &= \int_x^{r_1} e^{2kx} \cdot \frac{e^{-ky-\sqrt{k^2+2\lambda}y}}{2\sqrt{k^2+2\lambda}} \left[\left(-k + \sqrt{k^2+2\lambda} \right) \cdot e^{-kx+\sqrt{k^2+2\lambda}x} + \dots \right. \\ &\quad \left. \dots + \left(k + \sqrt{k^2+2\lambda} \right) \cdot e^{-kx-\sqrt{k^2+2\lambda}x} \right] f(y) \cdot 2e^{2ky} dy. \end{aligned}$$

Evaluating at $x = 0$ and setting $r_1 = \infty$ we get

$$\begin{aligned} D_p^+ F_0(0) &= \int_0^{r_1} \left(2\sqrt{k^2+2\lambda} \cdot \frac{e^{-ky-\sqrt{k^2+2\lambda}y}}{2\sqrt{k^2+2\lambda}} \right) f(y) 2e^{2ky} dy \\ &= \int_0^{r_1} e^{-ky-\sqrt{k^2+2\lambda}y} f(y) 2e^{2ky} dy. \end{aligned}$$

The calculations $F_0(0) = 0$ and $D_m D_p^+ F_0(0) = f(0)$ together with the one just computed give the claimed form of $\phi_0(F_0)$ above. $\phi_0(u_0)$ follows by similar calculations.

Now we have all of the pieces to write

$$F(x) = F_0(x) - \frac{\phi_0(F_0)}{\phi_0(u_0)} \cdot \frac{u_+(x, \lambda)}{u_+(r_0, \lambda)},$$

and once we recall that the original equation to solve was

$$(\lambda - A)\tilde{p}_\lambda(x, y) = \delta_x(y),$$

we see that we have

$$\begin{aligned} \tilde{p}_\lambda(x, y) &= \left(\frac{e^{-kx+\sqrt{k^2+2\lambda}x}}{\sqrt{k^2+2\lambda}} - \frac{e^{-kx-\sqrt{k^2+2\lambda}x}}{\sqrt{k^2+2\lambda}} \right) e^{ky-\sqrt{k^2+2\lambda}y} \\ &\quad + \frac{2\pi_0 e^{ky-\sqrt{k^2+2\lambda}y} e^{-kx-\sqrt{k^2+2\lambda}x} + \sigma_0 \delta_y(0) e^{-kx-\sqrt{k^2+2\lambda}x}}{\chi_0 + \pi_0(k + \sqrt{k^2+2\lambda}) + \lambda\sigma_0} \end{aligned}$$

for $x < y$ and

$$\begin{aligned}\tilde{p}_\lambda(x, y) &= \left(\frac{e^{ky + \sqrt{k^2 + 2\lambda}y}}{\sqrt{k^2 + 2\lambda}} - \frac{e^{ky - \sqrt{k^2 + 2\lambda}y}}{\sqrt{k^2 + 2\lambda}} \right) e^{-kx - \sqrt{k^2 + 2\lambda}x} \\ &+ \frac{2\pi_0 e^{ky - \sqrt{k^2 + 2\lambda}y} e^{-kx - \sqrt{k^2 + 2\lambda}x} + \sigma_0 \delta_y(0) e^{-kx - \sqrt{k^2 + 2\lambda}x}}{\chi_0 + \pi_0(k + \sqrt{k^2 + 2\lambda}) + \lambda\sigma_0}\end{aligned}$$

for $x > y$.

Now we recover $p_t(x, y)$ by recalling that $\tilde{p}_\lambda(x, y)$ is the Laplace transform w.r.t time of $p_t(x, y)$. Write

$$\chi_0 + \pi_0(k + \sqrt{k^2 + 2\lambda}) + \lambda\sigma_0 = \sigma_0(p^{1/2} + \alpha)(p^{1/2} + \beta)$$

where $p = (k^2 + 2\lambda)/2$ and where

$$\alpha = \frac{\pi_0}{\sqrt{2}\sigma_0} + \sqrt{\frac{\pi_0^2}{2\sigma_0^2} - \left(\frac{\chi_0 + k\pi_0 - k^2/2\sigma_0}{\sigma_0} \right)} \text{ and } \beta = \frac{\pi_0}{\sqrt{2}\sigma_0} - \sqrt{\frac{\pi_0^2}{2\sigma_0^2} - \left(\frac{\chi_0 + k\pi_0 - k^2/2\sigma_0}{\sigma_0} \right)}.$$

Then by the method of partial fractions and using the identity

$$\begin{aligned}\mathcal{L}_p^{-1} \left(\frac{\exp(-\sigma p^{1/2})}{1 + \gamma^{1/2} p^{1/2}} \right) (t) &= \left(1 + \gamma^{1/2} \frac{\partial}{\partial \sigma} \right) \mathcal{L}_p^{-1} \left((1 - \gamma p)^{-1} \exp(-\sigma p^{1/2}) \right) (t) \\ &= \frac{1}{\pi^{1/2} \gamma^{1/2} t^{1/2}} \exp \left(-\frac{\sigma^2}{4t} \right) \\ &- \frac{1}{\gamma} \exp \left(\frac{\sigma}{\gamma^{1/2}} + \frac{t}{\gamma} \right) \text{Erfc} \left(\frac{\sigma}{2t^{1/2}} + \frac{t^{1/2}}{\gamma^{1/2}} \right), \quad t > 0\end{aligned}$$

where the complement to the error function is defined

$$\text{Erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-u^2} du,$$

we can recover $p_t(x, y)$.

$$\begin{aligned}p_t(x, y) &= \frac{1}{\sqrt{2\pi t}} e^{k(y-x) - k^2 t/2} \left(e^{-(y-x)^2/2t} - e^{-(y+x)^2/2t} \right) \\ &+ \frac{2\pi_0 \alpha}{(\alpha - \beta)\sigma_0} e^{k(y-x) - k^2 t/2} e^{\alpha^2 t + \sqrt{2}\alpha(x+y)} \text{Erfc} \left[\frac{x+y}{\sqrt{2t}} + \alpha\sqrt{t} \right] \\ &- \frac{2\pi_0 \beta}{(\alpha - \beta)\sigma_0} e^{k(y-x) - k^2 t/2} e^{\beta^2 t + \sqrt{2}\beta(x+y)} \text{Erfc} \left[\frac{x+y}{\sqrt{2t}} + \beta\sqrt{t} \right] \\ &+ \frac{\delta_y(0)\alpha}{(\alpha - \beta)\sigma_0} e^{-kx - k^2 t/2} e^{\alpha^2 t + \sqrt{2}\alpha x} \text{Erfc} \left[\frac{x}{\sqrt{2t}} + \alpha\sqrt{t} \right] \\ &- \frac{\delta_y(0)\beta}{(\alpha - \beta)\sigma_0} e^{-kx - k^2 t/2} e^{\beta^2 t + \sqrt{2}\beta x} \text{Erfc} \left[\frac{x}{\sqrt{2t}} + \beta\sqrt{t} \right].\end{aligned}$$

Next we present the graph of the probability density function for a diffusion with drift $k = 0.5$ that starts at $x = 1$ and has a sticky boundary at 0 with a stickiness coefficient of 1. We can see how the density dissipates over time and is pushed to the right by the positive drift. Moreover, the spike at 0 represents the stickiness at 0, and the fact that there is always a nonzero chance that the particle can be found stuck at 0.

