

Limits of the form $\lim_{n \rightarrow \infty} \frac{p(n)}{q(n)}$ where p & q are polynomials (or polynomial-ish)

examples: $\lim_{n \rightarrow \infty} \frac{n^2 - 2n}{3n^2 - 4n + 5}$ OR $\lim_{n \rightarrow \infty} \frac{n(2n+1)}{(5n+1)^2}$

$$\lim_{n \rightarrow \infty} \frac{2n^3}{\sqrt{n^4 + 1}} \quad \text{OR} \quad \lim_{n \rightarrow \infty} \frac{n^3 + n^{3/2} + 1}{n^{5/2} - 1}$$

There are several ways to compute these limits:

① Dividing the numerator and denominator by the highest power of n in the denominator.

$$\begin{aligned} \text{ex: } \lim_{n \rightarrow \infty} \frac{n^2 - 2n}{3n^3 - 4n + 5} &= \lim_{n \rightarrow \infty} \frac{(n^2 - 2n) \cdot \frac{1}{n^3}}{(3n^3 - 4n + 5) \cdot \frac{1}{n^3}} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{n^2}{n^3} - \frac{2n}{n^3}}{\frac{3n^3}{n^3} - \frac{4n}{n^3} + \frac{5}{n^3}} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n} - \frac{2}{n^2}}{3 - \frac{4}{n^2} + \frac{5}{n^3}} = \frac{0}{3} = 0 \end{aligned}$$

$$\begin{aligned} \text{ex: } \lim_{n \rightarrow \infty} \left(\frac{n^3 + 2n}{n^2 - 3n + 1} \right) \frac{1/n^2}{1/n^2} &= \lim_{n \rightarrow \infty} \frac{n + 2/n}{1 - 3/n + 1/n^2} \\ &= \lim_{n \rightarrow \infty} \frac{n}{1} = \infty \end{aligned}$$

Notice what happens if we instead divide by n^3 , the highest power in the numerator:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{n^3 + 2n}{n^2 - 3n + 1} \right) \cdot \frac{1/n^3}{1/n^3} &= \lim_{n \rightarrow \infty} \frac{1 + 2/n^2}{1/n - 3/n^2 + 1/n^3} \\ &= \frac{1 + 0}{0} = \frac{1}{0} \\ &\quad \uparrow \\ &\text{indeterminate} \end{aligned}$$

② Using L'Hospital's Rule.

$$\begin{aligned}\text{ex: } \lim_{n \rightarrow \infty} \frac{n^2 - 2n}{3n^3 - 4n + 5} &= \lim_{x \rightarrow \infty} \frac{x^2 - 2x}{3x^3 - 4x + 5} \quad \left\{ \text{form } \frac{\infty}{\infty} \right. \\ &\stackrel{\text{LH}}{=} \lim_{x \rightarrow \infty} \frac{2x - 2}{9x^2 - 4} \quad \left\{ \text{form } \frac{\infty}{\infty} \right. \\ &\stackrel{\text{LH}}{=} \lim_{x \rightarrow \infty} \frac{2}{18x} = 0\end{aligned}$$

$$\begin{aligned}\text{ex: } \lim_{n \rightarrow \infty} \frac{n^4 + \sqrt{n}}{n^4 - \sqrt{n}} &= \lim_{x \rightarrow \infty} \frac{x^4 + \sqrt{x}}{x^4 - \sqrt{x}} \quad \left\{ \text{form } \frac{\infty}{\infty} \right. \\ &\stackrel{\text{LH}}{=} \lim_{x \rightarrow \infty} \frac{4x^3 + \frac{1}{2\sqrt{x}}}{4x^3 - \frac{1}{2\sqrt{x}}} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} \\ &= \lim_{x \rightarrow \infty} \frac{8x^3\sqrt{x} + 1}{8x^3\sqrt{x} - 1} \\ &= \lim_{x \rightarrow \infty} \frac{8x^{7/2} + 1}{8x^{7/2} - 1}\end{aligned}$$

Methods 1 & 2 can be a bit slow. If the degree is high or if the sequence involves a root, L'Hospital's Rule can be time consuming.

Methods 3 & 4 will be quicker.

③ Comparing the "degree" of the numerator and denominator.

- if $\deg(\text{numerator}) < \deg(\text{denominator})$, the limit is 0
- if $\deg(\text{numerator}) > \deg(\text{denominator})$, the limit is ∞ .
- if $\deg(\text{numerator}) = \deg(\text{denominator})$, the limit is the ratio of the leading coefficients

(Interpret degree loosely here
E.g. the "degree" of $\sqrt{n}$ is $1/2$
and the "degree" of $\sqrt{n^4 + n + 1}$ is 2)

ex: $\lim_{n \rightarrow \infty} \frac{n^2 - n}{\sqrt{n^5 + 1}} = 0$

since $\deg(\text{num}) = 2$ and
 $\deg(\text{denom}) = 5/2$ so
 $\deg(\text{num}) < \deg(\text{denom.})$

ex: $\lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)(3n+1)}{(5n+1)^3}$

$\deg(\text{num}) = 3$ and $\deg(\text{denom}) = 3$, so limit
will be the ratio of leading coefficients

the leading coefficient of $(n+1)(2n+1)(3n+1)$
will be $1 \cdot 2 \cdot 3 = 6$

the leading coefficient of $(5n+1)^3$ is $5^3 = 125$.

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)(3n+1)}{(5n+1)^3} = \frac{6}{125}$$

④ Deleting lower order terms.

A lower order term is any term whose degree is less than the highest order term

e.g. $\underbrace{n^4}_{\text{highest order term}} + \underbrace{2n^2 + n^{3/2} - \sqrt{n} + 1}_{\text{lower order terms}}$

$$\begin{aligned} \text{ex: } \lim_{n \rightarrow \infty} \frac{n^2 - \cancel{x}}{\sqrt{n^5 + \cancel{x}}} &= \lim_{n \rightarrow \infty} \frac{n^2}{\sqrt{n^5}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^{5/2}} = \lim_{n \rightarrow \infty} n^{-1/2} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{ex: } \lim_{n \rightarrow \infty} \frac{\sqrt{3n^2 + 7n^4}}{2n - n^2} &= \lim_{n \rightarrow \infty} \frac{\sqrt{7n^4}}{-n^2} = \lim_{n \rightarrow \infty} \frac{\sqrt{7} \cdot n^2}{-n^2} \\ &= \lim_{n \rightarrow \infty} -\sqrt{7} \\ &= -\sqrt{7} \end{aligned}$$