Available online at www.sciencedirect.com

ScienceDirect

IFAC Papers Online CONFERENCE PAPER ARCHIVE

IFAC PapersOnLine 53-2 (2020) 6317–6320

Divergence Conditions for Stability Study of Autonomous Nonlinear Systems ⋆

Igor Furtat * **Pavel Gushchin** *** **Artem Nekhoroshikh** *,**

* *Institute for Problems of Mechanical Engineering Russian Academy of Sciences, 61 Bolshoy ave V.O., St.-Petersburg, 199178, Russia, (e-mail: cainenash@mail.ru).*

** *ITMO University, 49 Kronverkskiy ave, Saint Petersburg, 197101, Russia*

*** *Gubkin Russian State University of Oil and Gas (National Research University), 65 Leninsky ave, Moscow, 119991, Russia*

Abstract: A novel method for stability study of autonomous dynamical systems using the flow and divergence of the vector field is proposed. Necessary and sufficient stability conditions are proposed. A relation between the method of Lyapunov functions and the proposed method is established. The examples illustrate the application of the proposed method and the existing ones.

Copyright © 2020 The Authors. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0)

Keywords: Dynamical system, stability, flow of vector field, divergence, control.

1. INTRODUCTION

The stability study methods based on the divergence of a vector field are alternative to the method of Lyapunov functions. The first fundamental results based on divergent stability conditions were proposed in Zaremba (1954); Fronteau (1965); Brauchli (1968). The important results for investigation of system stability were proposed by A. Rantzer, A.A. Shestakov, A.N. Stepanov and V.P. Zhukov. In Jukov (1978) the instability problem of nonlinear systems using the divergence of a vector field is considered. In Shestakov and Stepanov (1978); Jukov (1979) a necessary condition for the stability of nonlinear systems in the form of non-positivity of the vector field divergence is proposed. First, an auxiliary scalar function is introduced in Shestakov and Stepanov (1978); Jukov (1990) to study the instability of nonlinear systems. However, the similar scalar function is considered in Krasnoselsky et al. (1963) for stability and instability study of dynamical systems, but using the method of Lyapunov functions. In Shestakov and Stepanov (1978); Jukov (1999) stability conditions for second-order systems are obtained. Then in Rantzer and Parrilo (2000); Rantzer (2001) the convergence of almost all solutions of arbitrary order nonlinear dynamical systems is considered. As in Shestakov and Stepanov (1978); Jukov (1990, 1999) the auxiliary scalar function (density function) is used for the stability study of dynamical models. Additionally, in Rantzer and Parrilo (2000); Rantzer (2001) the synthesis of the control law based on divergence conditions is proposed. The auxiliary functions in Shestakov and Stepanov (1978); Jukov (1999); Rantzer and Parrilo (2000); Rantzer (2001) are similar except their properties at the equilibrium point. Currently, method from Rantzer and Parrilo (2000); Rantzer (2001) has been extended to various systems, see i.e. Monzon (2003); Loizou and Jadbabaie (2008); Castaneda and Robledo (2015); Karabacak at al. (2018).

However, the necessary condition is sufficiently rough in Shestakov and Stepanov (1978); Jukov (1999). The sufficient condition stability is proposed only for second-order systems in Jukov (1999). Theorem 1 in Rantzer (2001) guarantees the convergence of almost all solutions, but not all solutions. Proposition 2 in Rantzer (2001) allows to study the asymptotic stability, but proposition conditions have sufficient restriction.

In the present paper, we propose a new method for the stability study of dynamical systems using the flow and divergence of the vector field. New necessary and sufficient conditions are obtained. The relation between the method of Lyapunov functions and the proposed method is established. Numerical examples illustrate the applicability of the proposed method and the methods from Shestakov and Stepanov (1978); Jukov (1999); Rantzer and Parrilo (2000); Rantzer (2001).

The paper is organized as follows. Section 2 contains new necessary and sufficient conditions, as well as, the numerical examples and comparisons with the methods from Shestakov and Stepanov (1978); Jukov (1999); Rantzer and Parrilo (2000); Rantzer (2001). Finally, Section 3 collects some conclusions.

Notations. In the paper the following notation are used: $\text{grad}\{W(x)\} = \left[\frac{\partial W}{\partial x_1}, ..., \frac{\partial W}{\partial x_n}\right]^{\mathrm{T}}$ is the gradient of the scalar function $W(x)$, $\text{div}\{h(x)\} = \frac{\partial h_1}{\partial x_1} + ... + \frac{\partial h_n}{\partial x_n}$ is the divergence of the vector field $h(x) = [h_1(x), ..., h(x)_n]^{\mathrm{T}}$, $|\cdot|$ is the Euclidean norm of the corresponding vector. We mean that the zero equilibrium point is stable if it is Lyapunov stable (Khalil (2002)).

⋆ The results were developed under support of RSF (grant 18-79-10104) in IPME RAS.

2405-8963 Copyright © 2020 The Authors. This is an open access article under the CC BY-NC-ND license.
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2020.12.1758

2. MAUN RESULTS

Consider a dynamical system in the form

$$\dot{x} = f(x), \tag{1}$$

where $x = [x_1, ..., x_n]^{\mathrm{T}}$ is the state vector, $f = [f_1, ..., f_n]^{\mathrm{T}} : D \to \mathbb{R}^n$ is the continuously differentiable function in $D \subset \mathbb{R}^n$. The set D contains the origin and $f(0) = 0$. For simplicity, we assume that the domain of attraction D_A of the point $x = 0$ coincides with the domain D. However, all obtained results is valid if $D_A \subset D$ or $D_A = \mathbb{R}^n$. Denote by $\bar{D}$ a boundary of the domain D.

Let us formulate the necessary stability condition for system (1).

Theorem 1. Let $x = 0$ be an asymptotically stable equilibrium point of (1). Then there exists a positive definite continuously differentiable function $S(x)$ such that $S(x) \to \infty$ for $x \to \bar{D}$, $|\mathrm{grad}\{S(x)\}| \neq 0$ for any $x \in D \setminus \{0\}$ and at least one of the following conditions holds:

1) the function $\mathrm{div}\{|\mathrm{grad}\{S(x)\}|f(x)\}$ is integrable in the domain $V = \{x \in D : S(x) \leq C\} \subset D$ and $\int_V \mathrm{div}\{|\mathrm{grad}\{S(x)\}|f(x)\}dV < 0$ for all $C > 0$;

2) the function $\mathrm{div}\{|\mathrm{grad}\{S^{-1}(x)\}|f(x)\}$ is integrable in the domain $V_{inv} = \{x \in D : S^{-1}(x) \geq C\} \subset D$ and $\int_{V_{inv}} \mathrm{div}\{|\mathrm{grad}\{S^{-1}(x)\}|f(x)\}dV_{inv} > 0$ for all $C > 0$.

Before the proof of Theorem 1 consider the geometrical interpretation of two cases depending on the function $S(x)$ or $S^{-1}(x)$. Denote by F_1 the flow of the vector field $|\mathrm{grad}\{S(x)\}|f(x)$ through the surface $\Gamma = \{x \in D : S(x) = C\}$ with the unit normal vector $\frac{1}{|\mathrm{grad}\{S(x)\}|}\mathrm{grad}\{S(x)\}$, as well as, denote by F_2 the flow of the vector field $|\mathrm{grad}\{S^{-1}(x)\}|f(x)$ through the surface $\Gamma_{inv} = \{x \in D : S^{-1}(x) = C\}$ with the unit normal vector $\frac{1}{|\mathrm{grad}\{S^{-1}(x)\}|}\mathrm{grad}\{S^{-1}(x)\}$. Fig. 1 illustrates the geometrical interpretation of both cases for $x \in \mathbb{R}^2$, where the functions $S(x)$ and $S^{-1}(x)$ (see Fig. 1,*a* and Fig. 1,*b* on the left) and the flows F_1 and F_2 of the vector fields $|\mathrm{grad}\{S(x)\}|f(x)$ and $|\mathrm{grad}\{S^{-1}(x)\}|f(x)$ through the corresponding level surfaces of Γ and Γ_{inv} (see Fig. 1,*a* and Fig. 1,*b* on the right) are given. If system (1) is stable, then the flow of the vector field F_1 (F_2) through the surface Γ (Γ_{inv}) takes a negative (positive) value.

Proof 1. According to (Khalil, 2002, Theorem 4.17) if $x = 0$ is an asymptotically stable equilibrium point of system (1), then there exists a continuously differentiable positive definite function $S(x)$ such that $S(x) \to \infty$ for $x \to \bar{D}$, $\mathrm{grad}\{S(x)\}^{\mathrm{T}}f(x) < 0$ for any $x \in D \setminus \{0\}$ and $\mathrm{grad}\{S(x)\}^{\mathrm{T}}f(x)\Big|_{x=0} = 0$. If $D = \mathbb{R}^n$, then the function $S(x)$ is radially unbounded. Next, we consider two cases separately which correspond to the functions $S(x)$ and $S^{-1}(x)$.

1. If $\mathrm{grad}\{S(x)\}^{\mathrm{T}}f(x) < 0$, then $\frac{1}{|\mathrm{grad}\{S(x)\}|}\mathrm{grad}\{S(x)\}^{\mathrm{T}}|\times$ $\mathrm{grad}\{S(x)\}|f(x) < 0$. Therefore, the following expression holds

$$F_1 = \oint_{\Gamma} \frac{1}{|\mathrm{grad}\{S(x)\}|}\mathrm{grad}\{S(x)\}^{\mathrm{T}}|\mathrm{grad}\{S(x)\}|f(x)d\Gamma < 0.$$

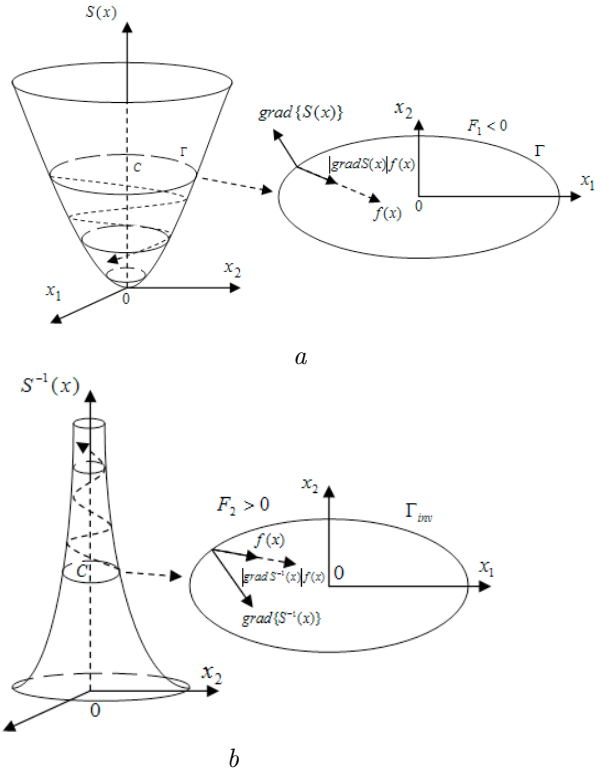


Fig. 1. The geometrical interpretation of Theorem 1.

Using Divergence theorem (or Gauss theorem), we get $F_1 = \int_V \mathrm{div}\{|\mathrm{grad}\{S(x)\}|f(x)\}dV < 0$.

2. If $\mathrm{grad}\{S(x)\}^{\mathrm{T}}f(x) < 0$, then $\mathrm{grad}\{S^{-1}(x)\}^{\mathrm{T}}f(x) = -S^{-2}(x)\mathrm{grad}\{S(x)\}^{\mathrm{T}}f(x) > 0$. On the other hand,

$$\mathrm{grad}\{S^{-1}(x)\}^{\mathrm{T}}f(x)$$
$$= \frac{1}{|\mathrm{grad}\{S^{-1}(x)\}|}\mathrm{grad}\{S^{-1}(x)\}^{\mathrm{T}}|\mathrm{grad}\{S^{-1}(x)\}|f(x).$$

Therefore, the following relation is satisfied

$$F_2 = \oint_{\Gamma_{inv}} \frac{1}{|\mathrm{grad}\{S^{-1}(x)\}|}\times$$
$$\mathrm{grad}\{S^{-1}(x)\}^{\mathrm{T}}|\mathrm{grad}\{S^{-1}(x)\}|f(x)d\Gamma_{inv} > 0.$$

According to Divergence theorem, we get

$$F_2 = \int_{V_{inv}} \mathrm{div}\{|\mathrm{grad}\{S^{-1}(x)\}|f(x)\}dV_{inv} > 0.$$

Theorem 1 is proved.

The integrals in Theorem 1 explicitly depend on the function $S(x)$ that depends on the integration surface. Let us formulate a corollary that weakens this requirement.

Corollary 1. Let $x = 0$ be the asymptotically stable equilibrium point of system (1). Then there exist positive definite continuously differentiable functions $\phi(x)$ and $S(x)$ such that $\phi(x) \to \infty$ and $S(x) \to \infty$ for $x \to \bar{D}$, $|\mathrm{grad}\{S(x)\}| \neq 0$ for any $x \in D \setminus \{0\}$ and at least one of the following conditions holds:

1) the function $\mathrm{div}\{\rho(x)f(x)\}$ is integrable in the domain $V=\{x\in D: S(x)\le C\}\subset D$ and $\int_V \mathrm{div}\{\rho(x)f(x)\}dV<0$ for all $C>0$, where $\rho(x)=\phi(x)|\mathrm{grad}\{S(x)\}|$;

2) the function $\mathrm{div}\{\rho^{-1}(x)f(x)\}$ is integrable in the domain $V_{inv}=\{x\in D: S^{-1}(x\ge C\}\subset D$ and $\int_{V_{inv}}\mathrm{div}\{\rho^{-1}(x)f(x)\}dV_{inv}>0$ for all $C>0$, where $\rho^{-1}(x)=\phi^{-1}(x)|\mathrm{grad}\{S^{-1}(x)\}|$.

Proof 2. Following the proof of Theorem 1, consider two cases.

1. If $\mathrm{grad}\{S(x)\}^{\mathrm{T}}f(x)<0$, then $\phi(x)\mathrm{grad}\{S(x)\}^{\mathrm{T}}f(x)<0$. Therefore, the further proof is similar to the proof of Theorem 1, but tacking into account the flow of the vector field $\phi(x)|\mathrm{grad}\{S(x)\}|f(x)$ through the surface Γ.

2. If $\mathrm{grad}\{S(x)\}^{\mathrm{T}}f(x)<0$, then $\phi^{-1}(x)\mathrm{grad}\{S(x)\}^{\mathrm{T}}f(x)<0$. Therefore, the further proof is similar to the proof of Theorem 1, but taking into account the flow of the vector field $\phi^{-1}(x)|\mathrm{grad}\{S^{-1}(x)\}|f(x)$ through the surface Γ_{inv}. The corollary is proved.

Remark 1. If the function $\rho(x)$ is chosen such that $\mathrm{div}\{\rho(x)f(x)\}$ and $\mathrm{div}\{\rho^{-1}(x)f(x)\}$ are integrable, as well as, $\mathrm{div}\{\rho(x)f(x)\}<0$ and $\mathrm{div}\{\rho^{-1}(x)f(x)\}>0$ for any $x\in D\setminus\{0\}$, then the corresponding conditions $\int_V\mathrm{div}\{\rho(x)f(x)\}dV<0$ and $\int_{V_{inv}}\mathrm{div}\{\rho^{-1}(x)f(x)\}dV_{inv}>0$ in Corollary 1 are satisfied. In Rantzer (2001) the integrability of $\mathrm{div}\{\rho^{-1}(x)f(x)\}$ and the condition $\mathrm{div}\{\rho^{-1}(x)\times f(x)\}>0$ are required only for convergence of almost all solutions of (1). Thus, the results of Rantzer (2001) are special case in Corollary 1.

Now let us formulate a sufficient condition for stability of (1).

Theorem 2. Let $\rho(x)$ be a positive definite continuously differentiable function in D. Then $x=0$ is stable (is asymptotically stable) if at least one of the following conditions holds:

1) $\mathrm{div}\{\rho(x)f(x)\}\le\rho(x)\mathrm{div}\{f(x)\}$ ($\mathrm{div}\{\rho(x)f(x)\}<\rho(x)\mathrm{div}\{f(x)\}$) for any $x\in D\setminus\{0\}$ and $\mathrm{div}\{\rho(x)f(x)\}\big|_{x=0}=0$;

2) $\mathrm{div}\{\rho^{-1}(x)f(x)\}\ge 0$ ($\mathrm{div}\{\rho^{-1}(x)f(x)\}>0$) and $\mathrm{div}\{f(x)\}\le 0$ for any $x\in D\setminus\{0\}$ and $\lim_{|x|\to 0}\left[\rho^2(x)\times\mathrm{div}\{\rho^{-1}(x)f(x)\}\right]=0$;

3) $\mathrm{div}\{\rho(x)f(x)\}\le\beta(x)\rho^2(x)\mathrm{div}\{\rho^{-1}(x)f(x)\}$

($\mathrm{div}\{\rho(x)f(x)\}<\beta(x)\rho^2(x)\mathrm{div}\{\rho^{-1}(x)f(x)\}$), where $\beta(x)>1$ and $\mathrm{div}\{f(x)\}\le 0$ or only $\beta(x)=1$ for any $x\in D\setminus\{0\}$, as well as, $\mathrm{div}\{\rho(x)f(x)\}\big|_{x=0}=0$ and $\lim_{|x|\to 0}\left[\rho(x)\mathrm{div}\{\rho^{-1}(x)f(x)\}\right]=0$.

Proof 3. Consider the proof for each case separately. The proof of asymptotic stability is omitted because it is similar to the proof of stability, but taking into account the sign of a strict inequality.

1. From the relation $\mathrm{div}\{\rho(x)f(x)\}=\mathrm{grad}\{\rho(x)\}^{\mathrm{T}}f(x)+\mathrm{div}\{f(x)\}\rho(x)$ implies that if

$$\mathrm{div}\{\rho(x)f(x)\}\le\mathrm{div}\{f(x)\}\rho(x),$$

then $\mathrm{grad}\{\rho(x)\}f(x)\le 0$ in the domain $D\setminus\{0\}$. Consider the condition $\rho(0)=0$. If $\mathrm{div}\{\rho(x)f(x)\}\big|_{x=0}=0$, then $\mathrm{grad}\{\rho(x)\}f(x)\big|_{x=0}=0$. Therefore, according to Lyapunov theorem (Khalil (2002)), system (1) is stable.

2. From the expression

$$\mathrm{div}\{\rho^{-1}(x)f(x)\}=\mathrm{grad}\{\rho^{-1}(x)\}^{\mathrm{T}}f(x)+\mathrm{div}\{f(x)\}\rho^{-1}(x)$$

it follows that

$$\mathrm{grad}\{\rho(x)\}^{\mathrm{T}}f(x)=\rho(x)\mathrm{div}\{f(x)\}-\rho^2(x)\mathrm{div}\{\rho^{-1}(x)f(x)\}.$$

If $\mathrm{div}\{\rho^{-1}(x)f(x)\}\ge 0$ and $\mathrm{div}\{f(x)\}\le 0$, then $\mathrm{grad}\{\rho(x)\}^{\mathrm{T}}f(x)\le 0$ in $D\setminus\{0\}$.

If $\lim_{|x|\to 0}\left[\rho^2(x)\mathrm{div}\{\rho^{-1}(x)f(x)\}\right]=0$, then

$\lim_{|x|\to 0}\left[\mathrm{grad}\{\rho(x)\}f(x)\right]=0$. Therefore, system (1) is stable.

3. Condition 3 is a combination of conditions 1 and 2. Summing $\beta(x)\mathrm{grad}\{\rho(x)\}^{\mathrm{T}}f(x)=\beta(x)\rho(x)\mathrm{div}\{f(x)\}-\beta(x)\rho^2(x)\mathrm{div}\{\rho^{-1}(x)f(x)\}$ and $\mathrm{grad}\{\rho(x)\}^{\mathrm{T}}f(x)=\mathrm{div}\{\rho(x)f(x)\}-\mathrm{div}\{f(x)\}\rho(x)$, we get $(1+\beta(x))\times\mathrm{grad}\{\rho(x)\}^{\mathrm{T}}f(x)=\mathrm{div}\{\rho(x)f(x)\}-\beta(x)\rho^2(x)\times\mathrm{div}\{\rho^{-1}(x)f(x)\}+(\beta(x)-1)\rho(x)\mathrm{div}\{f(x)\}$.

If $\mathrm{div}\{\rho(x)f(x)\}\le\beta(x)\rho^2(x)\mathrm{div}\{\rho^{-1}(x)f(x)\}$ for $\beta(x)=1$ or $\beta(x)>1$ and $\mathrm{div}\{f(x)\}\le 0$, then $\mathrm{grad}\{\rho(x)\}^{\mathrm{T}}f(x)\le 0$ in the region $D\setminus\{0\}$. If $\mathrm{div}\{\rho(x)f(x)\}\big|_{x=0}=0$ and $\lim_{|x|\to 0}\left[\rho^2(x)\mathrm{div}\{\rho^{-1}(x)f(x)\}\right]=0$, then

$\lim_{|x|\to 0}\left[\mathrm{grad}\{\rho(x)\}f(x)\right]=0$. Therefore, system (1) is stable. Theorem 2 is proved.

It is noted in Introduction that the result of Shestakov and Stepanov (1978); Jukov (1999) is applicable only to second-order systems. Next, we consider an illustration of the proposed results for third-order systems and compare the results with ones from Rantzer (2001).

Example. Consider the system

$$\begin{aligned}\dot{x}_1&=-4x_1x_2^2-x_1^3,\\ \dot{x}_2&=4x_1^2x_2-x_2^3-8x_2x_3^2,\\ \dot{x}_3&=-x_3^3+8x_2^2x_3\end{aligned}\qquad(2)$$

with equilibrium point $(0,0,0)$. The phase trajectories of (2) are shown in Fig. 2 for various initial conditions.

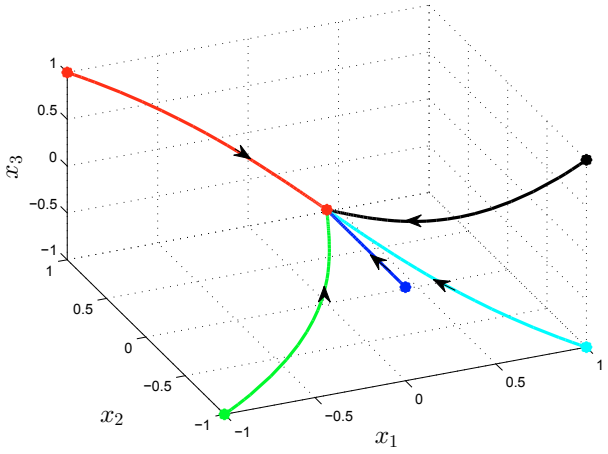


Fig. 2. Phase trajectories of system (2).

Choose $\rho(x)=|x|^{2\alpha}$, α is a positive integer and verify the conditions of Corollary 1. Considering $\mathrm{div}\{\rho(x)f(x)\}=|x|^{2\alpha-2}[(-2\alpha+1)x_1^4+(-2\alpha+1)x_2^4+(-2\alpha-11)x_3^4+2x_1^2x_2^2-$

$10x_1^2x_3^2 - 10x_2^2x_3^2]$, we get $\int_V \mathrm{div}\{\rho(x)f(x)\}dV < 0$ for any C and α. For $\mathrm{div}\{\rho^{-1}(x)f(x)\} = |x|^{-2\alpha-2}[(2\alpha+1)x_1^4 + (2\alpha+1)x_2^4 + (2\alpha-11)x_3^4 + 2x_1^2x_2^2 - 10x_1^2x_3^2 - 10x_2^2x_3^2$ the condition $\int_{V_{inv}} \mathrm{div}\{\rho^{-1}(x)f(x)\}dV_{inv} > 0$ holds for any C and $\alpha \geq 3$. Consequently, the conditions of Corollary 1 are satisfied (the conditions of Theorem 1 in Rantzer (2001) are satisfied only for $\alpha \geq 8$).

Verify the conditions of Theorem 2. The relation

$\mathrm{div}\{\rho(x)f(x)\} - \rho(x)\mathrm{div}\{f(x)\} = -2\alpha|x|^{2\alpha-2}(x_1^4 + x_2^4 + x_3^4) < 0$ holds for any α and $x \neq 0$. The function $\mathrm{div}\{f(x)\} = x_1^2 + x_2^2 - 11x_3^2$ is not positive definite. Thus, Proposition 2 in Rantzer (2001) and the second case of Theorem 2 cannot be satisfied. Condition $\mathrm{div}\{\rho(x)f(x)\} - \beta\rho^2(x)\mathrm{div}\{\rho^{-1}(x)f(x)\} < 0$ in Theorem 2 holds for $\beta = 1$ and $x \neq 0$.

As a result, the conditions of Corollary 1 and Theorem 2 are satisfied for system (2). Thus, $(0, 0, 0)$ is an asymptotically stable equilibrium point. According to Rantzer (2001), we can only conclude that almost all solutions of (2) converge to $(0, 0, 0)$, because the conditions of Proposition 2 in Rantzer (2001) are not satisfied and only the conditions of Theorem 1 in Rantzer (2001) hold.

3. CONCLUSION

A method for stability study of dynamical systems using the properties of the flow and divergence of the vector field is proposed. To study the stability, it is required the existence of a certain type of integration surface or the existence of an auxiliary scalar function. Necessary and sufficient stability conditions are proposed. All results in the present paper were proposed by I.Furtat. P. Gushchin and A. Nekhoroshikh have been participated in writing the present paper.

REFERENCES

Brauchli, H.I. (1968). *Index, divergenz und Stabilität in Autonomen equations*. Abhandlung Verlag, Zürich.

Castañeda, Á. and Robledo, G. (2015). Differentiability of Palmer's Linearization Theorem and Converse Result for Density Functions. *J. Diff. Equat.*, volume 259, 9, 4634–4650.

Fronteau, J. (1965). *Le théorèm de Liouville et le problèm général de la stabilité*. CERN, Genève.

Zhukov, V.P. (1978). On One Method for Qualitative Study of Nonlinear System Stability. *Automation and Remote Control*, volume 39, 6, 785–788.

Zhukov, V.P. (1979). On the Method of Sources for Studying the Stability of Nonlinear Systems. *Automation and Remote Control*, volume 40, 3, 330–335.

Zhukov, V.P. (1990). Necessary and Sufficient Conditions for Instability of Nonlinear Autonomous Dynamic Systems. *Automation and Remote Control*, volume 51, 12, 1652–1657.

Zhukov, V.P. (1999). On the Divergence Conditions for the Asymptotic Stability of Second-Order Nonlinear Dynamical Systems. *Automation and Remote Control*, volume 60, 7, 934–940.

Karabacak, Ö., Wisniewski, R., and Leth, J. (2018). On the Almost Global Stability of Invariant Sets. *Proc. of the 2018 Europenean Control Conference (ECC 2018), Limassol, Cyprus*, 1648–1653.

Khalil, H.K. (2002). *Nonlinear Systems*. Prentice Hall.

Krasnoselsky, M.A., Perov, A.I., Povolotsky, A.I., and Zabreiko P.P. (1963). *Vector fields on the plane*. M . Fizmatlit (in Russian).

Loizou, S.G. and Jadbabaie, A. (2008). Density Functions for Navigation-Function-Based Systems. *IEEE Transaction on Automatic Control*, volume 53, 2, 612–617.

Monzon, P. (2003). On Necessary Conditions for Almost Global Stability. *IEEE Transaction on Automatic Control*, volume 48, 4, 631–634.

Rantzer, A. and Parrilo, P.A. (2000). On Convexity in Stabilization of Nonlinear Systems. *Proc. of the 39th IEEE Conference on Decision and Control, Sydney, Australia (CDC2000)*, 2942–2946.

Rantzer, A. (2001). A Dual to Lyapunov's Stability Theorem. *Syst. & Control Lett.*, volume 42, 161–168.

Shestakov, A.A. and Stepanov, A.N.(1978). Index and divergent signs of stability of a singular point of an autonomous system of differential equations. *Differential equations*, T. 15, 4, 650–661.

Zaremba, S.K. (1954). Divergence of Vector Fields and Differential Equations. *American Journal of Mathematics*, volume LXXV, 220–234.