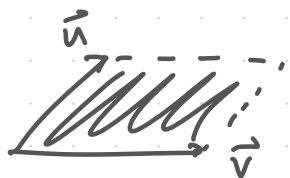


15.5 - Surface Area

Recall: for $\vec{u}, \vec{v}$, $|\vec{u} \times \vec{v}| =$ area of the parallelogram formed by $\vec{u}, \vec{v}$

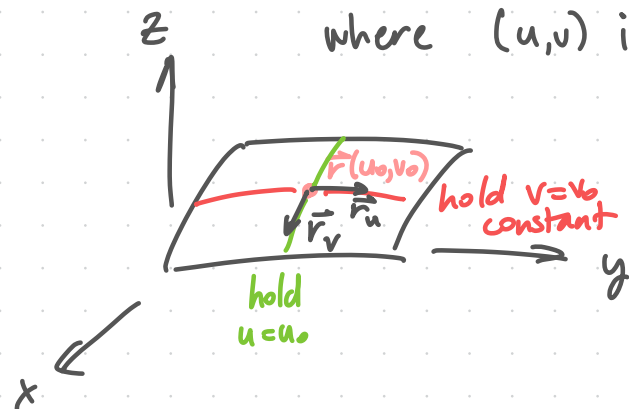


$$(|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta)$$

Consider a surface param. by

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

where (u, v) in a domain D .



$\vec{r}_u(u_0, v_0)$ - tangent vector to the curve we get by holding $v=v_0$ constant

$\vec{r}_v(u_0, v_0)$ - tangent vector ...



$|\vec{r}_u \times \vec{r}_v| =$ area of a small parallelogram approx. surf. area.

If ΔS a tiny bit of surf. area, then

$$\Delta S = |(\Delta u) \vec{r}_u \times (\Delta v) \vec{r}_v|$$

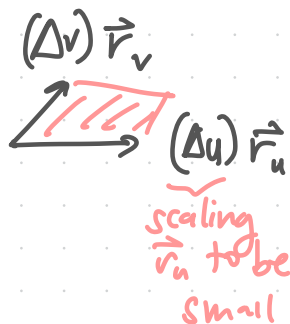
$$= \Delta u \Delta v |\vec{r}_u \times \vec{r}_v|$$

$$= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

↓ limit $\Delta u, \Delta v \rightarrow 0$

$$dS = |\vec{r}_u \times \vec{r}_v| dA$$

$du dv$ or $dv du$

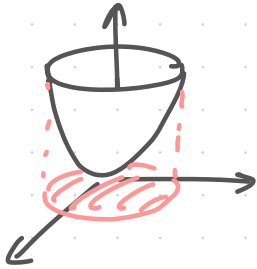


Surf. Area of a Parametric Surface

$\vec{r}(u,v)$ where $(u,v) \in D$

$$\iint_D dS = \iint_D |\vec{r}_u \times \vec{r}_v| dA$$

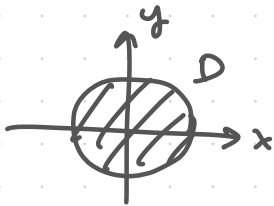
ex: Find the area of the part of $z = x^2 + y^2$ below $z = 9$.



Param: $\begin{cases} x = x \\ y = y \\ z = x^2 + y^2 \end{cases}$

Let $x = u, y = v$:

$$\vec{r}(u, v) = \langle u, v, u^2 + v^2 \rangle$$



Domain? circle of radius 3
 $x^2 + y^2 \leq 9$

Now compute $dS = |\vec{r}_u \times \vec{r}_v| dA$:

$$\vec{r}_u = \langle 1, 0, 2u \rangle$$

$$\vec{r}_v = \langle 0, 1, 2v \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = \langle -2u, -2v, 1 \rangle$$

$$\begin{aligned} |\vec{r}_u \times \vec{r}_v| &= \sqrt{(-2u)^2 + (-2v)^2 + 1} \\ &= \sqrt{4u^2 + 4v^2 + 1} \end{aligned}$$

$$SA = \iint_{u^2+v^2 \leq 9} \sqrt{4u^2+4v^2+1} \, dA_{uv}$$

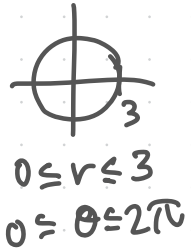
convert
to polar
($u = r \cos \theta$
 $v = r \sin \theta$)

$$= \int_0^{2\pi} \int_0^3 \sqrt{4r^2+1} \cdot r \, dr \, d\theta$$

$$\begin{aligned} u &= 4r^2+1 \\ du &= 8r \, dr \end{aligned}$$

$$r=0 \Rightarrow u=1$$

$$r=3 \Rightarrow u=37$$



$$= \int_0^{2\pi} \left(\int_1^{37} \sqrt{u} \cdot \frac{1}{8} \, du \right) d\theta$$

← doesn't depend on θ

$$= \int_0^{2\pi} 1 \, d\theta \cdot \int_1^{37} \frac{1}{8} u^{1/2} \, du$$

$$= 2\pi \cdot \frac{1}{8} \left(\frac{2}{3} u^{3/2} \right) \Big|_1^{37}$$

$$= \frac{2\pi \cdot 2}{8 \cdot 3} \cdot (37^{3/2} - 1)$$

$$= \frac{\pi}{6} (37^{3/2} - 1)$$

We could have also param. in cylindrical

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = r^2 \end{cases}$$

Let $r = u$, $\theta = v$:

$$\vec{r}(u, v) = \langle u \cos v, u \sin v, u^2 \rangle$$

$$\vec{r}_u = \langle \cos v, \sin v, 2u \rangle$$

$$\vec{r}_v = \langle -u \sin v, u \cos v, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix}$$

$$= \langle -2u^2 \cos v, -2u^2 \sin v, u \cos^2 v + u \sin^2 v \rangle$$

$$= \langle -2u^2 \cos v, -2u^2 \sin v, u \rangle$$

$$\begin{aligned}
 |\vec{r}_u \times \vec{r}_v| &= \sqrt{(-2u^2 \cos v)^2 + (-2u^2 \sin v)^2 + u^2} \\
 &= \sqrt{4u^4 \cos^2 v + 4u^4 \sin^2 v + u^2} \\
 &= \sqrt{4u^4 + u^2} \\
 &= \sqrt{u^2(4u^2 + 1)} \\
 &= u \sqrt{4u^2 + 1}
 \end{aligned}$$

Remember, we let $u=r$

$$|\vec{r}_u \times \vec{r}_v| = r \sqrt{4r^2 + 1}$$

$$\iint_D |\vec{r}_u \times \vec{r}_v| dA = \iint_D r \sqrt{4r^2 + 1} \underbrace{dA}_{\substack{dr d\theta \\ \text{or} \\ d\theta dr}}$$



$$\Delta u \Delta v \xrightarrow[\text{limit to } 0]{} du dv$$

$$\sum_{i,j} |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v \rightarrow \iint_D |\vec{r}_u \times \vec{r}_v| du dv$$

If the surface is the graph of a function $z = f(x, y)$ on a domain D then we can do the graph param.

$$\begin{cases} x = u \\ y = v \\ z = f(u, v) \end{cases}$$

$$\vec{r}(u, v) = \langle u, v, f(u, v) \rangle$$

$$\vec{r}_u = \langle 1, 0, f_u \rangle$$

$$\vec{r}_v = \langle 0, 1, f_v \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & f_u \\ f_v & 0 & 1 \end{vmatrix}$$

$$= \langle -f_u, -f_v, 1 \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(-f_u)^2 + (-f_v)^2 + 1}$$

$$\Rightarrow dS = \sqrt{(f_u)^2 + (f_v)^2 + 1} dA$$

$$SA = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1} dA$$

for surfaces, written as
 $z = f(x, y)$