

1.1 Systems of Linear Equations

A **linear equation** is an equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

where $a_1, a_2, \dots, a_n, b$ are constants (real numbers) and $x_1, x_2, \dots, x_n$ are variables. The constants $a_1, a_2, \dots, a_n$ are called the **coefficients** of the linear equation.

Examples (and non-examples):

$$2x_1 + 3x_2 + 5x_3 = 1 \quad \text{linear}$$

$$5x_1^2 - 3x_3 = 10 \quad \text{not linear}$$

$$x + 1 = 4y - 6z \quad \text{linear}$$

$$x_1x_2 + 2x_3 - \sqrt{x_4} = 5 \quad \text{not linear}$$

$$x^2 + y^2 = 4 \quad \text{not linear}$$

$$\ln(2)x_1 + \sqrt{7}x_2 + \cos(3)x_3 = 0 \quad \text{linear}$$

A **system of linear equations** (or linear system) is a collection of linear equations.

Example: $\begin{cases} x_1 - x_2 + x_3 = 0 \\ 5x_1 + 2x_2 = 8 \end{cases}$ is a 2×3 linear system because there are 2 equations and 3 variables.

You're very used to solving 2×2 systems but you may have already experienced an unwieldy 3×3 system (and they only get worse with more variables). Part of what you'll learn in this class is very **efficient** ways to solve linear systems of equations, or be able to quickly tell whether a solution exists.

We say an ordered list $(s_1, s_2, \dots, s_n)$ is a **solution** of a system if plugging in s_1 for x_1 , s_2 for x_2 , etc. satisfies each equation in the system.

For example, $(1, \frac{3}{2}, \frac{1}{2})$ is a solution of the system $\begin{cases} x_1 - x_2 + x_3 = 0 \\ 5x_1 + 2x_2 = 8 \end{cases}$

More definitions:

- We say that a linear system is **consistent** if it has at least one solution. We say that the system is **inconsistent** if it has no solutions.
- If a system has exactly one solution, we say the solution is **unique**.
- The set of all possible solutions of a linear system is called the **solution set**.
- We say that two linear systems are **equivalent** if they have the same solution set.

Fact: Two lines in $\mathbb{R}^2$ either:

- intersect at exactly one point
- never intersect (means they're parallel)
- intersect at every point (means they're the same line)

Exercises:

1. Consider the following situations involving systems of equations in three unknowns. For each situation, either sketch an example or state a reason why it can't happen. You can use Desmos 3D or index cards to help you visualize the planes.

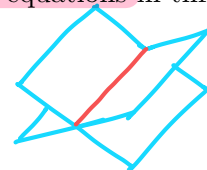
- (a) Is it possible that there are **no solutions** to a system of **two linear equations** in three unknowns?

yes, two parallel planes



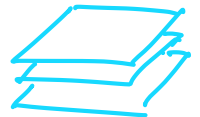
- (b) Is it possible there is **exactly one solution** to a system of **two linear equations** in three unknowns?

No, if two planes in $\mathbb{R}^3$ intersect, they must intersect in a line



- (c) Is it possible that there are **no solutions** to a system of **three linear equations** in three unknowns?

yes, e.g. if each pair of planes in a line that lies outside the 3rd plane (or three parallel planes)



- (d) Is it possible that the solution of a system of **four linear equations** in three unknowns **form a line**?

yes, e.g. if they all intersect along the same line.



2. Write a statement similar to the fact given above about two lines in $\mathbb{R}^2$, but this time for two planes in $\mathbb{R}^3$.

Two planes in $\mathbb{R}^3$ either:

- never intersect (parallel)
- intersect in a line
- intersect at every point (same plane)

3. Write a statement similar to the fact given above about two lines in $\mathbb{R}^2$, but this time for three planes in $\mathbb{R}^3$.

Three planes in $\mathbb{R}^3$ either:

- never intersect
- intersect at exactly one point
- intersect in a line
- intersect in a plane

