INCOMPRESSIBLE SURFACES

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3-Manifolds Seminar

Departamento de Matemática - ICEx - UFMG

01/02/2021

Contents

Notation

- M denotes a compact orientable manifold with (possibly empty) boundary.

- $S \subset M$ denotes a properly embedded compact orientable surface.

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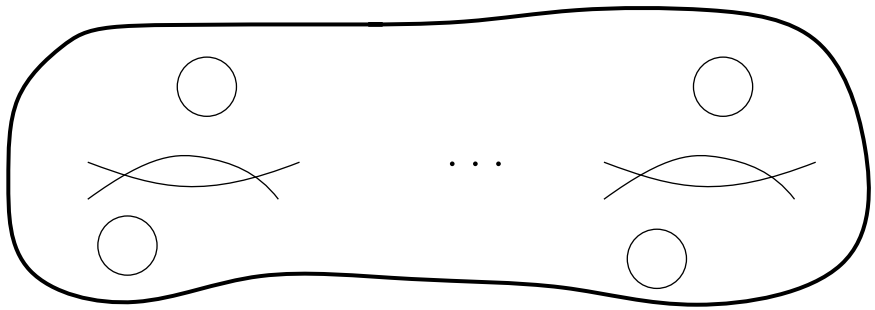
- M denotes a compact orientable manifold with (possibly empty) boundary.
- $S \subset M$ denotes a properly embedded compact orientable surface.
- Properly embedded means $\partial S = S \cap \partial M$.

Compact Orientable Surfaces

- Closed surfaces: g-torus.
- Non-closed surfaces: g-torus with k holes.



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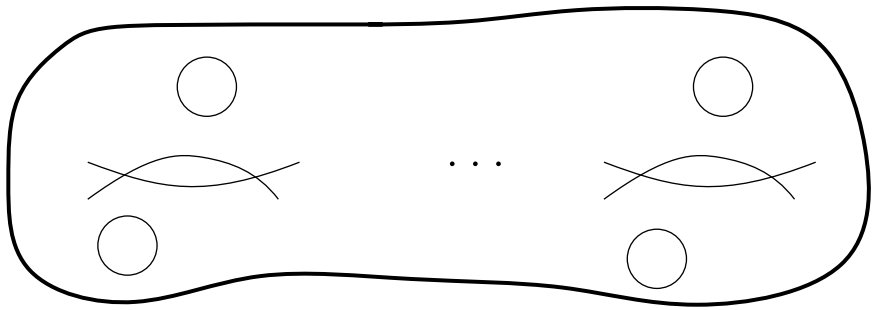
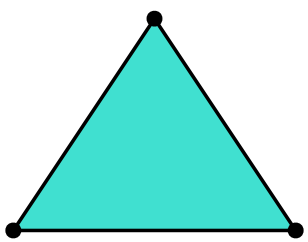


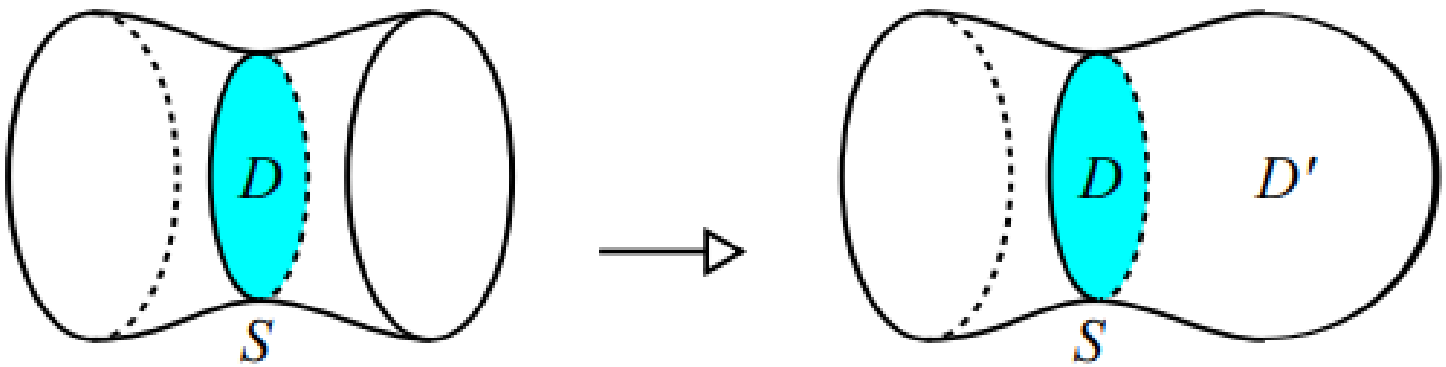
- ∂S is a disjoint union of k loops.
- $\chi(S) = 2 - 2g - k$.



Definition I

Definition

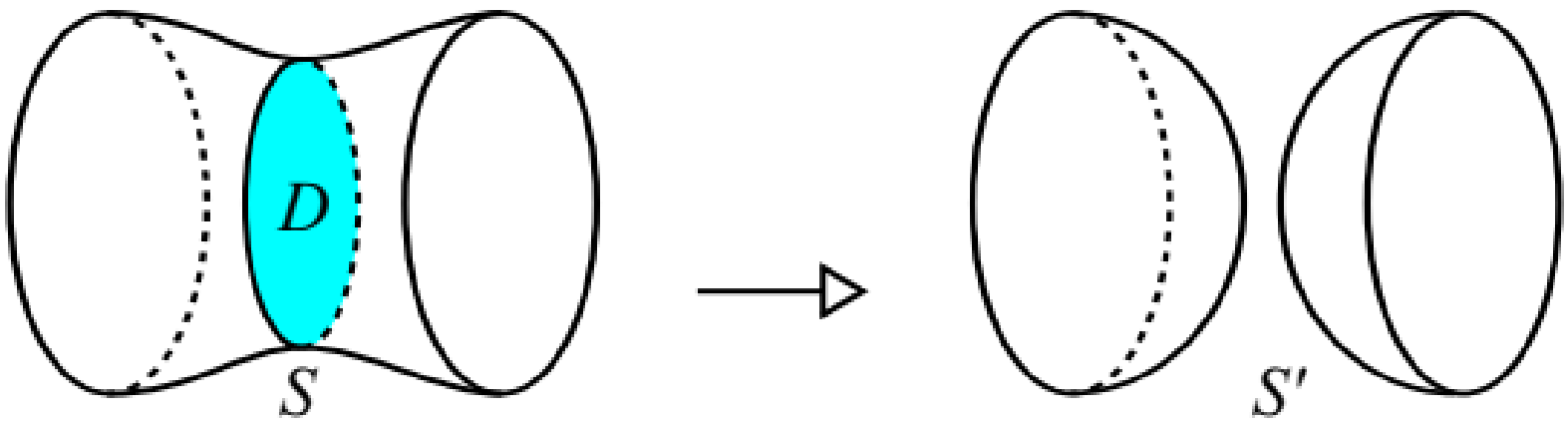
A compressing disc for S is disc $D \subset M$ with $\partial D = D \cap S$, such that ∂D does not bound a disc in S.



Definition II

Definition

Let $D \subset M$ a compressing disc for $S \subset M$.. A compression on S consists of removing an annular neighborhood of ∂D and adding two parallel copies of D. The result is a new surface S'.



About the surface S'

Proposition

The surface S' may have one or two components S'_i, and $\chi(S'_i) > \chi(S)$ for each component.

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Proof.

- $\chi(S') = \chi(S) + 2$ (Addiction of two faces).
- If S' has one component, ok!
- $S' = S_1 \sqcup S_2$.
- $S_i \neq S^2$, because D is a compressing disc.

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- If S' has one component, ok!
- $S' = S_1 \sqcup S_2$.
- $S_i \neq S^2$, because D is a compressing disc.
- S_i is a compact surface not equal to a sphere, then $\chi(S_i) = 2 - 2g - k \leq 1$.
- $\chi(S') = \chi(S_1) + \chi(S_2)$.
- If $\chi(S) \geq \chi(S_i)$, then $\chi(S_j) \geq 2$.
- Contradiction.

Incompressible Surfaces

In particular, $\chi(S) \leq 0$.

Definition

Let $S \subset M$ with $\chi(S) \leq 0$. Then

- *S is compressible if has a compressing disc.*
- *Otherwise, S is incompressible.*

Remark

Every disc in a sphere or disc properly embedded in a 3-manifold is not compressing.

A Corollary

Corollary

Let $S \subset M$. After compressing it a finite number of times it transforms into a disjoint union of discs, spheres, and incompressible surfaces.

Proof.

- $S \longrightarrow S_1 \sqcup S_2 \longrightarrow \ldots \longrightarrow \sqcup_{i,j} S_{ij}$.
- $\chi(S) < \chi(S_{ij}) \leq 1$.

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- $S \longrightarrow S_1 \sqcup S_2 \longrightarrow \ldots \longrightarrow \sqcup_{i,j} S_{ij}$.
- $\chi(S) < \chi(S_{ij}) \leq 1$.
- If S_{ij} is incompressible, S_{ij} decomposes as two components isotopic to S_{ij} and S^2.
- Compressing spheres results in spheres.
- Compressing discs results in discs and spheres.

Criterion of Incompressibility

Proposition

Let γ be a closed simple curve in S_g. If γ is homotopically trivial, $\gamma = \partial D$ for some disc in $D \subset S_g$.

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- Let D be a compressing disc.
- If $\partial D = 1 \in \pi_1(S)$, then ∂D bounds a disc in S_g embedded in S. Contradiction.

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Proof.

- Let D be a compressing disc.
- If $\partial D = 1 \in \pi_1(S)$, then ∂D bounds a disc in S_g embedded in S. Contradiction.
- Then ∂D is not trivial in S.
- But ∂D is trivial in M. Contradicion.

The Components of ∂S

Proposition

If $S \subset M$ is incompressible , every component of ∂S is non-trivial in M.

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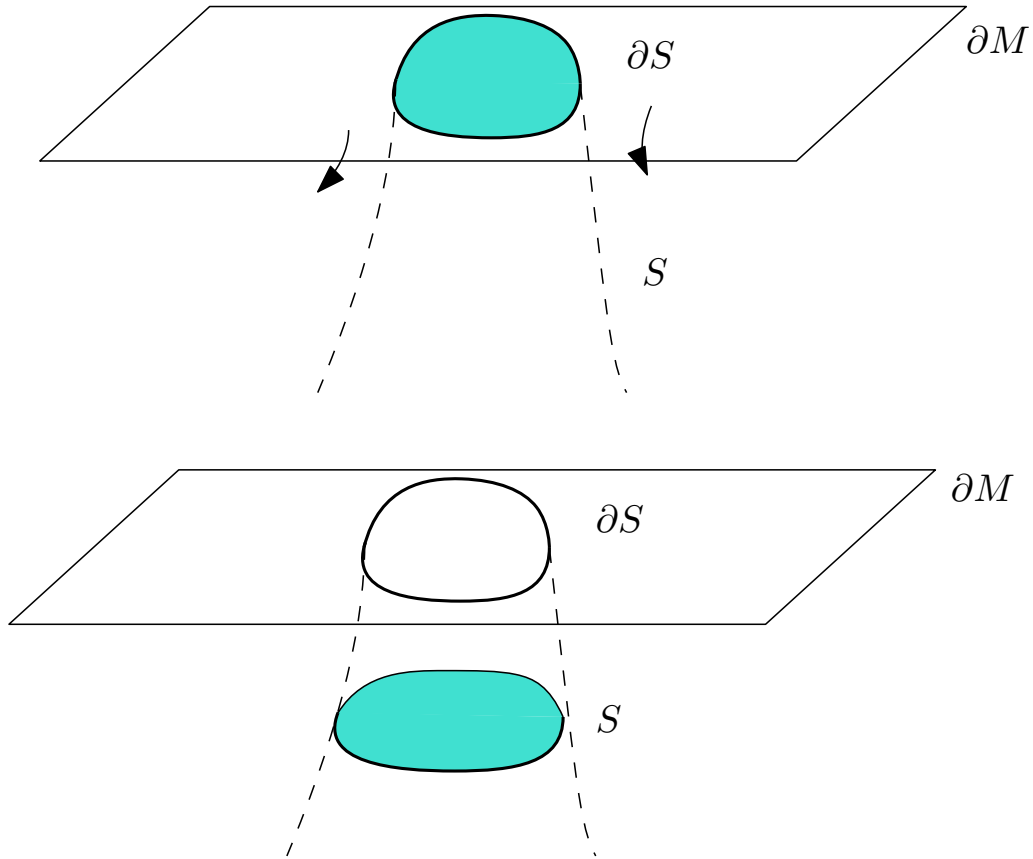
Proof.

- Suppose that a component of ∂S is trivial in ∂M, then it bounds a disc D in ∂M.
- We may suppose $D \cap S = \partial D$.
- Pushing D inside M, D is a compressing disc.

□

Continuation



Contents

Tori

Proposition

Let $T \subset M$ be a torus in an irreducible 3-manifold. One of the following holds:

1. *T is incompressible.*
2. *T bounds a solid torus.*
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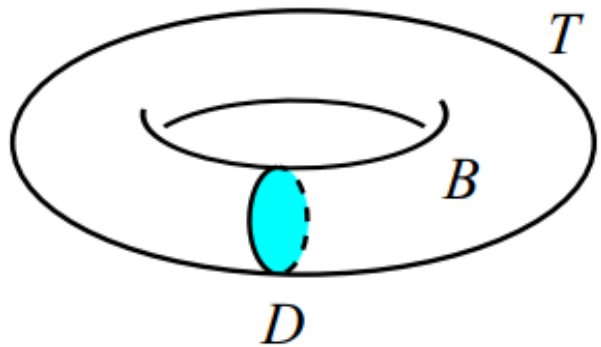

Continuation

- Since M is irreducible, $S^2 = \partial B$ for some $B \subset M$.

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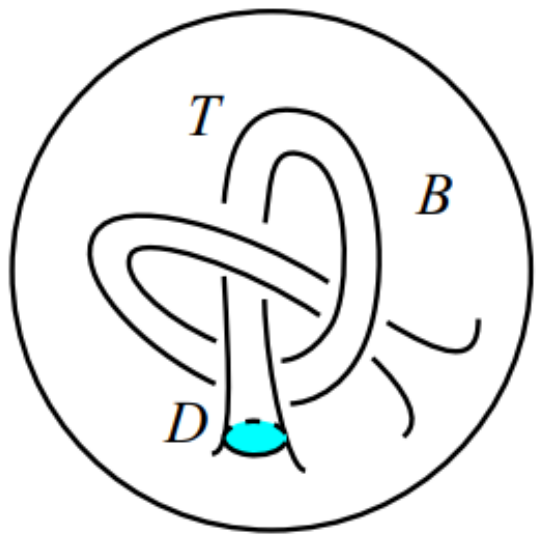
- If $B \cap T = \emptyset$, then $T = \partial H_1$.
- Otherwise, T is embedded in B.

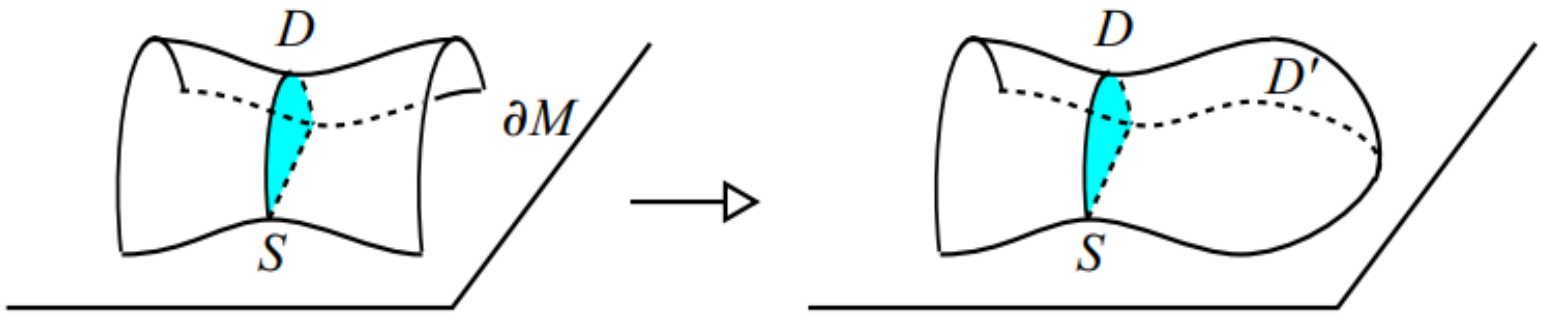


Contents

Introduction I

Definition

A ∂-compressing disc for $S \subset M$ is a disc D with $D = \alpha \cup \beta$, where $\alpha \subset S$, $\beta \subset \partial M$. Moreover, α does not bound a disc in S, i.e there is no sub-disc $D' \subset S$ with $\partial D' = \alpha \cup \beta'$ with $\beta' \subset \partial S$.



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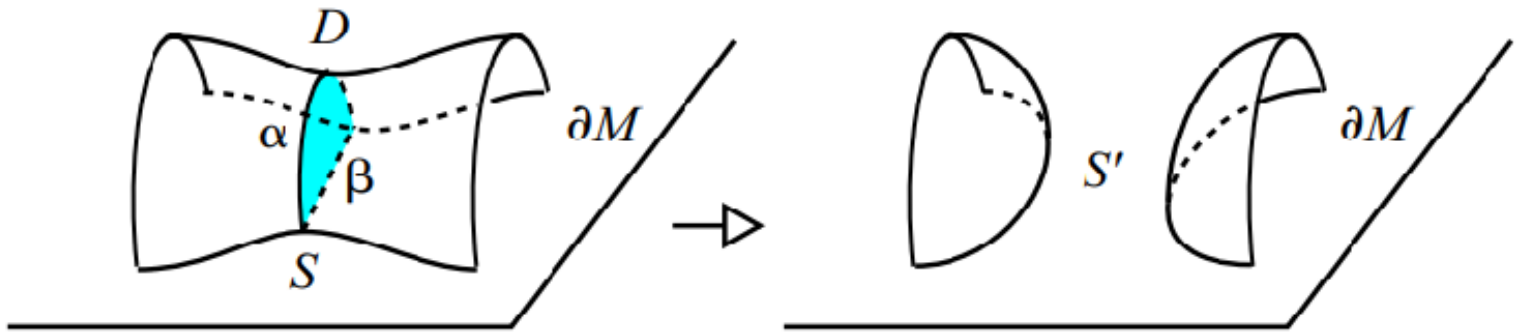
Definition

Let $D \subset M$ with $\partial D = \alpha \sqcup \beta$ with $\alpha \subset S$ and $\beta \subset \partial M$. A ∂-compression on S consists of removing an annular neighborhood of ∂D and touching the boundary in a segment. The result is a new surface S'.



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The surface S' may have one or two components S'_i, and $\chi(S'_i) > \chi(S)$ for each component.

Proof.

- $\chi(S') = \chi(S) + 1$ (Addiction of one boundary component).
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- $S_i \neq D$, because D is a ∂-compressing disc.
- S_i is a compact surface not equal to a disc, then $g \geq 1$ and $k \geq 1$.
- Hence $\chi(S_i) = 2 - 2g - k \leq 0$.
- If $\chi(S) \geq \chi(S_i)$, then $\chi(S_j) \geq 1$.
- Contradiction.

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Definition

Let $S \subset M$ with $\chi(S) \leq 0$. Then

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Corollary

Let $S \subset M$. After compressing it a finite number of times it transforms into a disjoint union of discs, spheres, and incompressible surfaces.

Contents

∂-irredutibility

Definition

A properly embedded surface $S \subset M$ is ∂-parallel if it is obtained by slightly pushing inside M the interior of a compact surface $S' \subset \partial M$, possibily with boundary.

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Definition

A disc $D \subset M$ is essential if it is not ∂-parallel.

Definition

A manifold M is ∂-irreducible if it does not contain essential discs.

Annuli

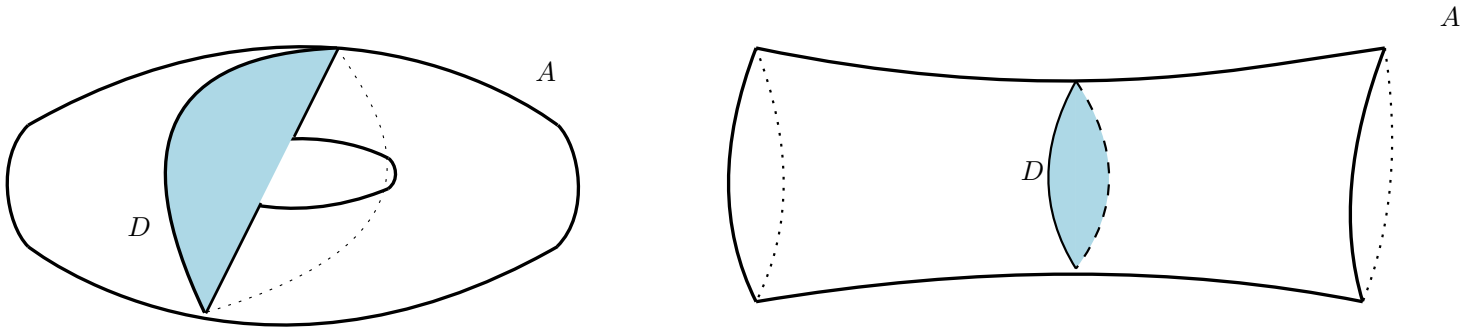
Proposition

Let $A \subset M$ be a properly embedded annulus in an irreducible and ∂-irreducible 3-manifold. One of the following holds:

1. *A is incompressible and ∂-incompressible.*
2. *A bounds a tube.*
3. *A is parallel to an annulus in ∂M.*
4. *A is contained in a ball B intersecting ∂M in a disc.*

Proof

- If A compresses along a disc D, it transforms into two discs D_1 and D_2.



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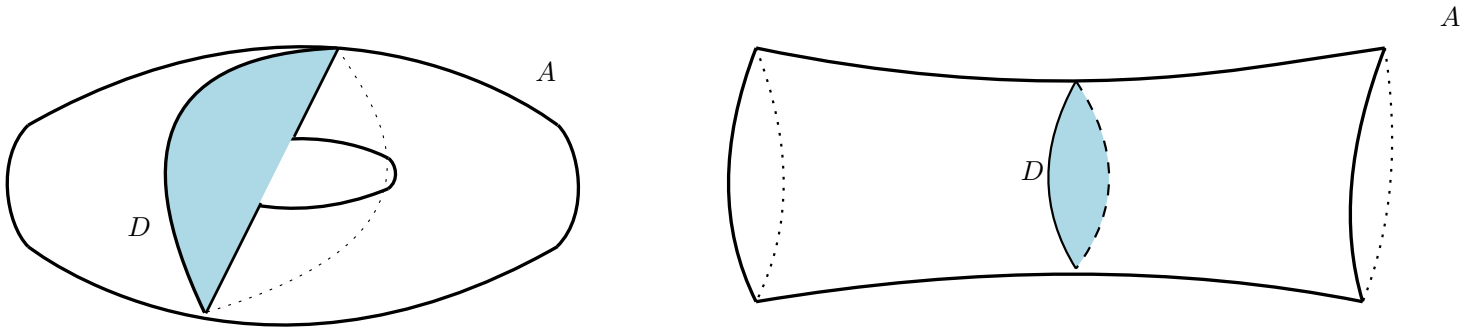
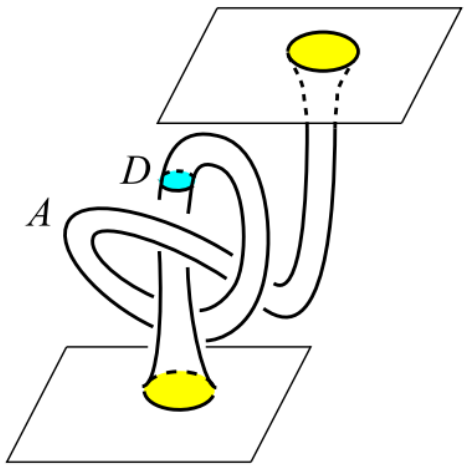
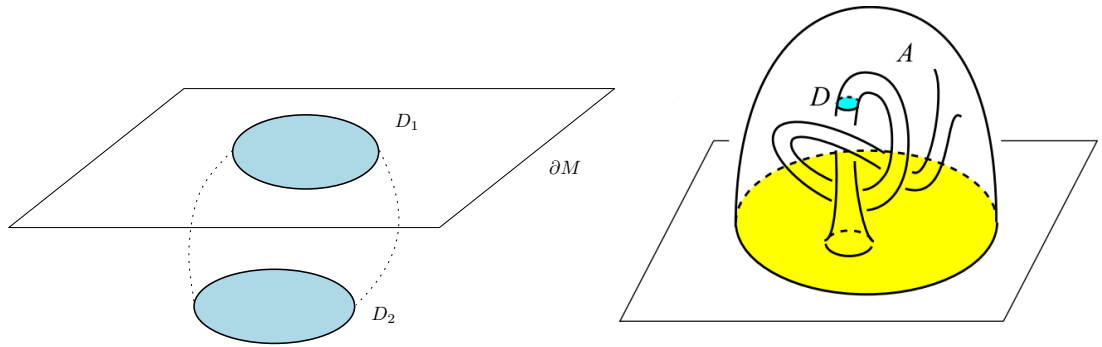


- Since M is ∂-irreducible, $D_1, D_2 \subset \partial M$.
- If $D_1 \cap D_2 = \emptyset$:

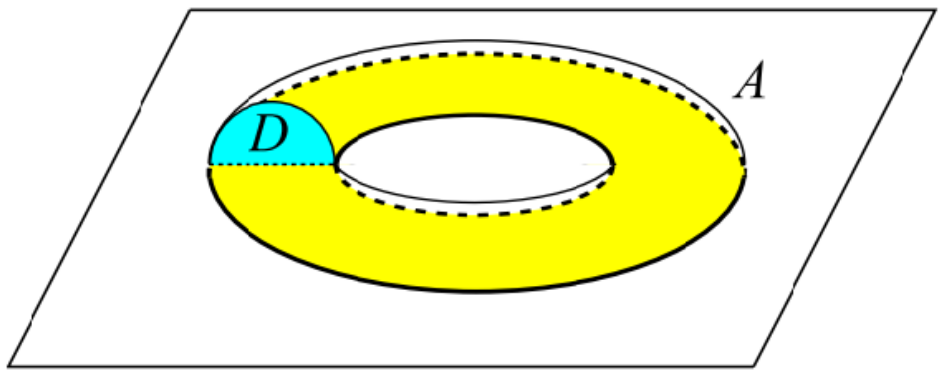


Continuation

- If $D_1 \cap D_2 \neq \emptyset$:



- If A ∂-compresses along a disc D:



- D' is ∂-parallel.

Contents

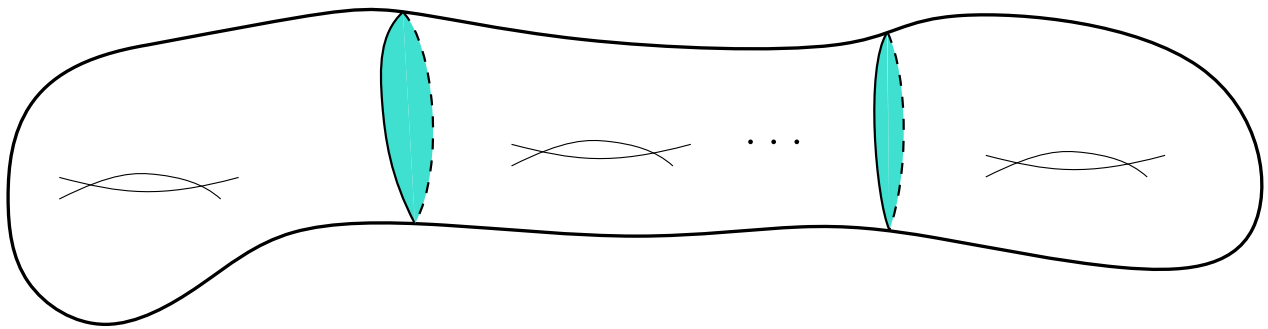
Euclidean Space $\mathbb{R}^3$

Proposition

There are no closed incompressible surfaces in $\mathbb{R}^3$.

Proposition

A closed surface S with $\chi(S) \leq 0$ is compressible.



The ball and the sphere

Corollary

There are no incompressible surfaces in S^3.

Proof.

- $\partial S = \emptyset$.
- $S \subset S^3 \simeq \mathbb{R}^3 \cup \{\infty\}$.

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- Contradiction with the previous result.

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Corollary

There are no incompressible surfaces in the ball B.

Proof.

- If $\partial S = \emptyset$, $S \subset \text{int}(B) \simeq \mathbb{R}^3$. Contradiction.
- The components of ∂S are non-trivial in $\partial B = S^2$.

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Proof.

- If $\partial S = \emptyset$, $S \subset \text{int}(B) \simeq \mathbb{R}^3$. Contradiction.
- The components of ∂S are non-trivial in $\partial B = S^2$.
- S^2 is simply connected. Contradiction.

□

Torus in S^3

Corollary

Every torus in S^3 bounds a solid torus.

Proof.

- S^3 is irreducible.
- T incompressible is a contradiction.

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Proof.

- S^3 is irreducible.
- T incompressible is a contradiction.
- A torus $T \subset S^3$ divides S^3 into two solid torus ([3, p.47]).
- By a previous Corollary, T bounds a solid torus in S^3.

□

Handlebodies

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The handlebodie H_g of genus g contains no incompressible and *∂-incompressible surfaces.*

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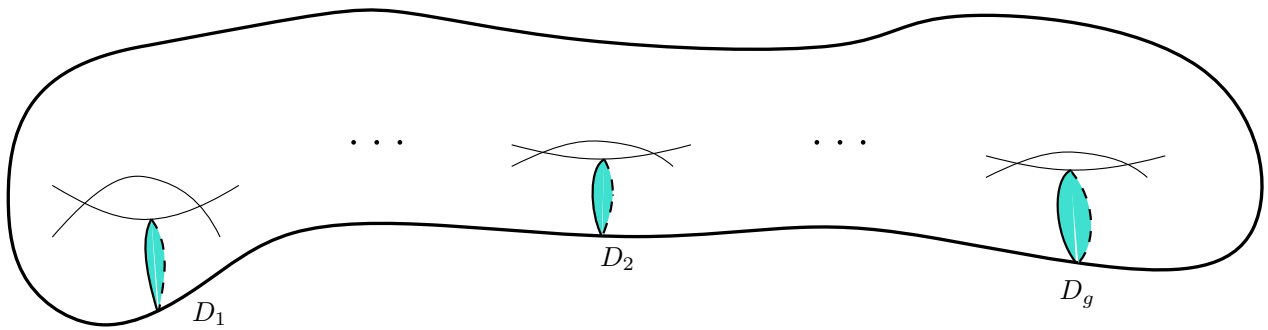
Proof:

- Let S be incompressible and ∂-incompressible.
- Pick disjoint essential discs $D_1, ..., D_g$ that cut H_g into a ball B.



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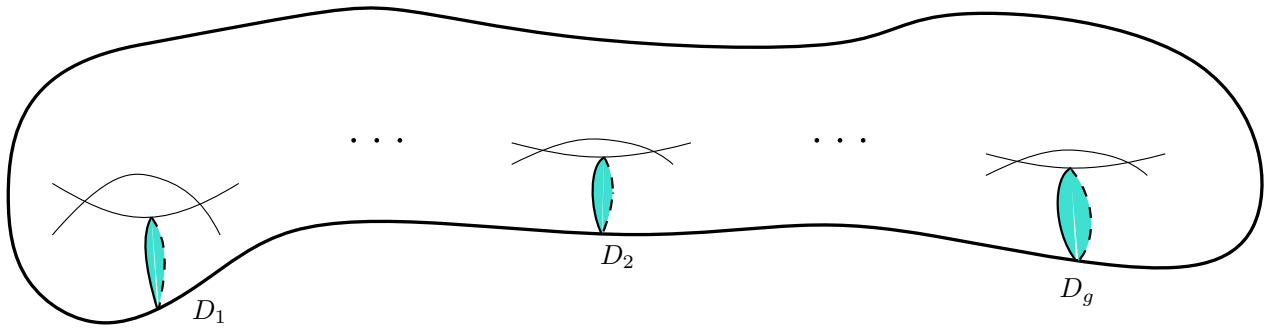

- Put S transverse to $\sqcup D_i$.

First Case

- Objective: remove intersections of S with $\sqcup D_i$.

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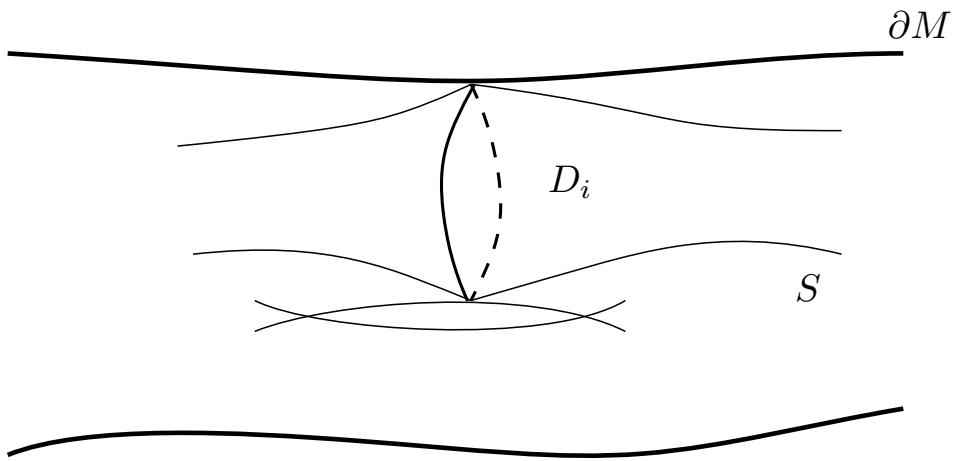
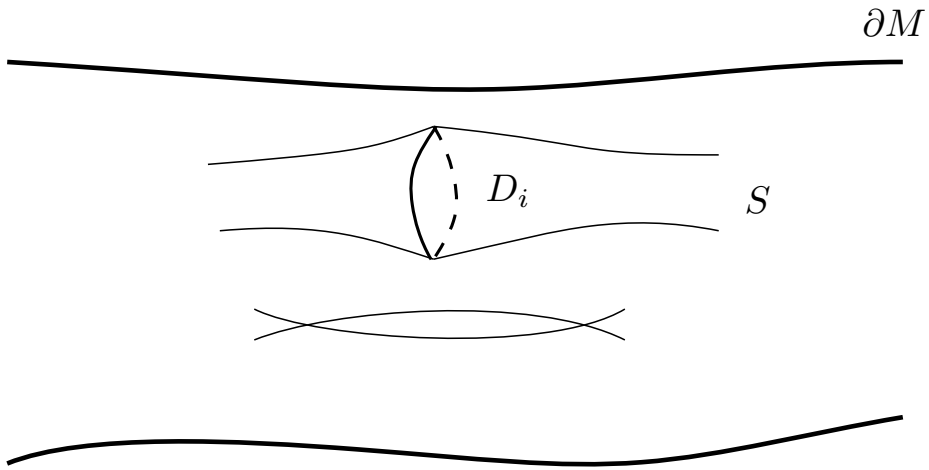
- Recall that H_g is irreducible.
- First case (S is incompressible):



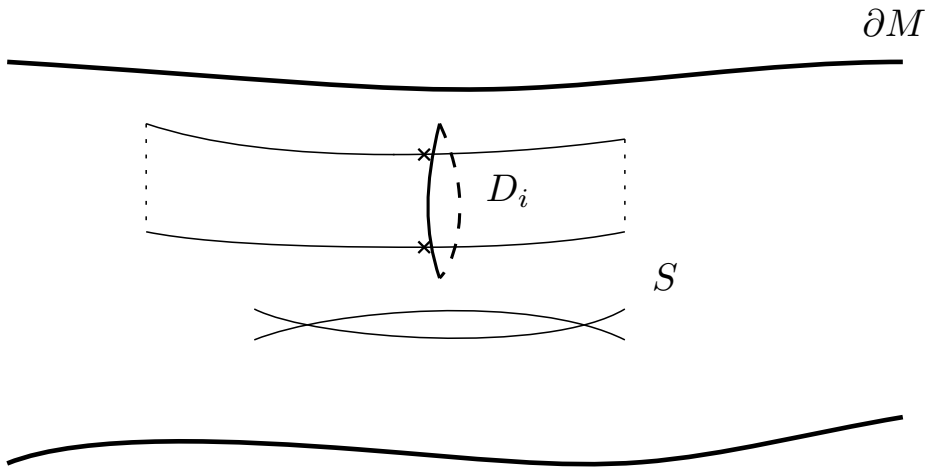
Second Case

- Second case (S is incompressible):



Third Case

- Third case (S is ∂-incompressible):



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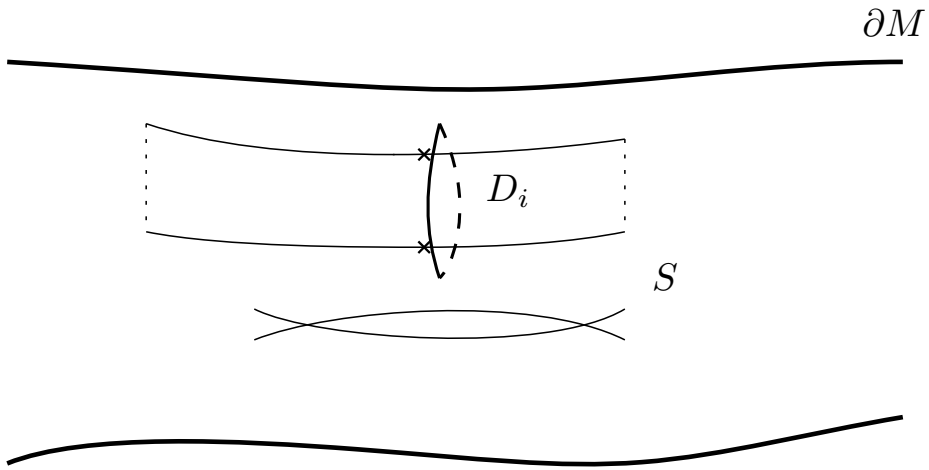

- After cut H_g along to $\sqcup D_i$, $S \subset B$, a ball.

- Contradiction.

OBRIGADO!

References

- Bruno Martelli. *An Introduction to Geometric Topology* (2016).
- Marc Lackenby. *Incompressible Surfaces*. Lecture notes available at this link.
- Allen Hatcher. *Algebraic Topology*.