

HW 9

8. Use a power series to approximate the definite integral $\int_0^{0.3} \frac{x^2}{1+x^4} dx$ to six decimal places.

$$\frac{x^2}{1+x^4} = x^2 \left(\frac{1}{1+x^4} \right) = x^2 \left(\frac{1}{1-(-x^4)} \right)$$

$$= x^2 \sum_{n=0}^{\infty} (-x^4)^n$$

$$= \sum_{n=0}^{\infty} x^2 \cdot (-1)^n x^{4n}$$

combine

$$= \sum_{n=0}^{\infty} (-1)^n x^{4n+2}$$

$$\int_0^{0.3} \left(\sum_{n=0}^{\infty} (-1)^n x^{4n+2} \right) dx = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{4n+3} \right) \Big|_{x=0}^{x=0.3}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(0.3)^{4n+3}}{4n+3} - 0$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(0.3)^{4n+3}}{4n+3}$$

a number

Use AST Remainder Estimate to find N needed to get value within 6 decimal places:

$$|R_N| \leq 0.0000005 = \frac{5}{10^7}$$

↑
error

$$|R_N| \leq \underbrace{b_{N+1}}_{\text{finish...}} \leq \frac{\text{desired error}}{10^7} = \frac{5}{10^7}$$

Taylor and Maclaurin Series

If f is a function with a power series representation

$$\sum_{n=0}^{\infty} C_n (x-a)^n$$

then the coefficients C_n are the Taylor coefficients: $C_n = \frac{f^{(n)}(a)}{n!}$

(Minor issue: sometimes f has a P.S. representation but f isn't equal to its power series, even on its IOC)

↑ we'll address next week

The Taylor series for $f(x)$ centered at $x=a$ is

$$\underline{3} \cdot \underline{2} \cdot \underline{1} = 3!$$

$$0! = 1$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$= \frac{f^{(0)}(a)}{0!} (x-a)^0 + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots$$

The Maclaurin series is Taylor series where $a=0$.

ex: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

← we saw this yesterday
(using geometric series formula)

↑
 $c_n = 1, a = 0$

Start similar to finding the Taylor poly:

$f(x) = \frac{1}{1-x} = (1-x)^{-1}$ chain rule

$f'(x) = - (1-x)^{-2} (-1) = (1-x)^{-2}$

$f''(x) = -2 (1-x)^{-3} (-1) = 2 (1-x)^{-3} = \frac{2!}{(1-x)^3}$

$f^{(3)}(x) = -6 (1-x)^{-4} (-1) = 6 (1-x)^{-4} = \frac{3!}{(1-x)^4}$

$f^{(4)}(x) = -24 (1-x)^{-5} (-1) = 24 (1-x)^{-5}$
 $= \frac{4!}{(1-x)^5}$

⋮

$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$ ← $f^{(n)}(0) = \frac{n!}{(1-0)^{n+1}} = n!$

Taylor coefficients: $C_n = \frac{f^{(n)}(a)}{n!}$ basept.

$$= \frac{f^{(n)}(0)}{n!}$$

$$= \frac{n!}{n!} = 1$$

ex: Maclaurin series for $f(x) = e^x$ and its radius of convergence
base point is $a=0$

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f''(x) = e^x$$

$\vdots$

$$f^{(n)}(x) = e^x \quad \text{and} \quad f^{(n)}(0) = e^0 = 1$$

$$\text{so } C_n = \frac{1}{n!}$$

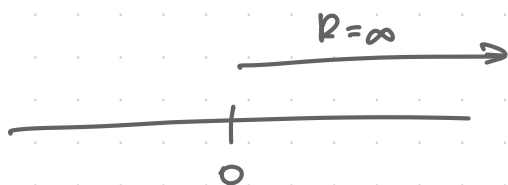
$$e^x = \sum_{n=0}^{\infty} C_n (x-0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

radius of convergence:

Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$

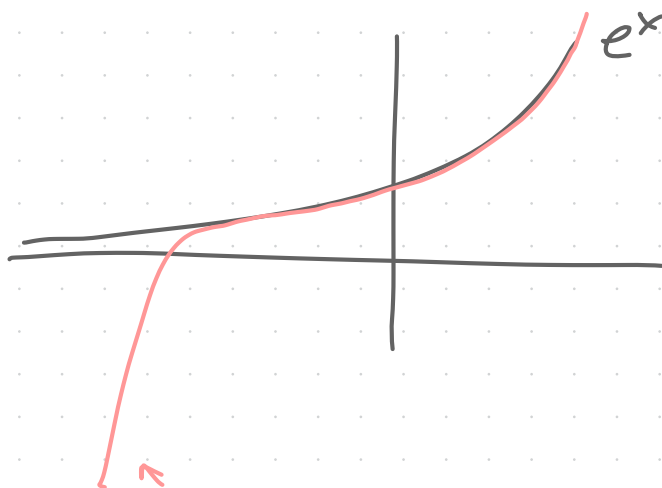
$\swarrow a_{n+1}$ $\swarrow \frac{1}{a_n}$

$$= \lim_{n \rightarrow \infty} \left| \frac{\cancel{x^n} \cdot x \cdot \cancel{n!}}{(n+1) \cdot \cancel{n!} \cdot \cancel{x^n}} \right|$$
$$= \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| \leftarrow \text{blows up}$$
$$= 0 < 1 \text{ always so cogs for all } x$$



$\Rightarrow$ radius of cogs is $\boxed{R = \infty}$

$$\left[\begin{array}{l} \text{Ratio Test Limit: } 0 \Rightarrow R = \frac{1}{0} = \infty \\ \qquad \qquad \qquad \infty \Rightarrow R = \frac{1}{\infty} = 0 \\ \qquad \qquad \qquad \frac{1}{5} |x-a| \Rightarrow R = \frac{1}{(1/5)} = 5 \end{array} \right]$$



e^x equals its
Maclaurin series
at every point
 x in $\mathbb{R}$

(we'll prove)

ex: Taylor series for $f(x) = e^{-2x}$ about
 $a = 4$

Need a formula for $f^{(n)}(x)$

$$f(x) = e^{-2x}$$

$$f'(x) = (-2)e^{-2x}$$

$$f''(x) = (-2)(-2)e^{-2x} = (-2)^2 e^{-2x}$$

$$f'''(x) = (-2)^2 (-2)e^{-2x} = (-2)^3 e^{-2x}$$

$\vdots$

$$f^{(n)}(x) = (-2)^n e^{-2x}$$

$$f^{(n)}(4) = (-2)^n e^{-8} = \left(\frac{(-2)^n}{e^8} \right) \quad *$$

$$C_n = \frac{f^{(n)}(4)}{n!} = \frac{\left(\frac{(-2)^n}{e^8} \right)}{\left(\frac{n!}{1} \right)} = \frac{(-2)^n}{e^8} \cdot \frac{1}{n!}$$

Plug in to Taylor series:

$$\begin{aligned} \sum_{n=0}^{\infty} C_n (x-a)^n &= \sum_{n=0}^{\infty} C_n (x-4)^n \\ &= \sum_{n=0}^{\infty} \frac{(-2)^n}{e^8 \cdot n!} \cdot (x-4)^n \end{aligned}$$

Taylor series at $a=4$

Recap of previous ex:

$$f(x) = e^{-2x}, \text{ base point } a=4$$

Want Taylor coefficients

$$C_n = \frac{f^{(n)}(4)}{n!} \quad \swarrow \text{Need this}$$

$$\sum_{n=0}^{\infty} C_n (x-4)^n$$

$\uparrow$
Taylor coeff.

$$\text{We found } C_n = \frac{\left(\frac{(-2)^n}{e^8}\right)}{n!} \quad \}$$

$$\text{So } \sum_{n=0}^{\infty} \frac{(-2)^n}{e^8 \cdot n!} \cdot (x-4)^n$$

(radius of convergence is $R=\infty$, (can check))

ex: Maclaurin series

$$f(x) = e^x$$

$$f(2) = e^2$$

$$\int e^{-x^2} dx$$



no closed form for integral



ex: Maclaurin series for $f(x) = \sin x$
and its radius of convergence

↖ $a=0$

Want to find $f^{(n)}(0)$ ↘

$$\sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(x) = \sin x \quad \rightarrow \quad f(0) = 0 \quad \leftarrow$$

$$f'(x) = \cos x \quad f'(0) = 1$$

$$f''(x) = -\sin x \quad f''(0) = 0 \quad \leftarrow$$

$$f'''(x) = -\cos x \quad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0 \quad \leftarrow$$

$$f^{(5)}(x) = \cos x \quad f^{(5)}(0) = 1$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \cancel{f(0)} + \frac{f'(0)}{1!} x + \cancel{\frac{f''(0)}{2!} x^2} + \frac{f'''(0)}{3!} x^3 + \cancel{\frac{f^{(4)}(0)}{4!} x^4} + \frac{f^{(5)}(0)}{5!} x^5 + \dots$$

$$= \frac{f'(0)}{1!} x + \frac{f'''(0)}{3!} x^3 + \frac{f^{(5)}(0)}{5!} x^5 + \dots$$

$$= \frac{1}{1!} x + \frac{(-1)}{3!} x^3 + \frac{1}{5!} x^5 + \frac{(-1)}{7!} x^7 + \dots$$

$$n=0$$

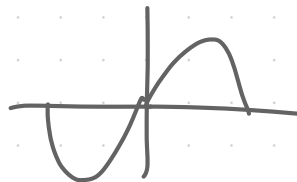
$$n=1$$

$$n=2$$

$$n=3$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}$$

Maclaurin series for $\sin x$



can check that rad. of conv. is $R = \infty$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

} can derive similarly

$R = \infty$ too

← $a=0$

Standard Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Radius of Convergence

$$R = 1 \quad \leftarrow \text{IOC is } (-1, 1)$$

$$\underline{R = \infty}$$

$$\underline{R = \infty}$$

$$\underline{R = \infty}$$

$$R = 1$$

$$R = 1$$