Derivation of the Ampere's Law from the Continuity Equation

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1 The Continuity Equation

The continuity equation in electromagnetism can be derived considering the definition of the current density, in the scalar form,

$$J \equiv \frac{\mathrm{d}I}{\mathrm{d}a_{\perp}} \tag{1.1}$$

where I is the current flowing through the tube and $\mathrm{d}a_{\perp}$ is the infinitesimal cross-sectional area. Therefore, one can form the integral from Eq. (1.1)

$$\oint_{\mathcal{S}} J \,\mathrm{d}a_{\perp} = \oint_{\mathcal{S}} \mathbf{J} \cdot \mathrm{d}\mathbf{a} = \int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{J} \,\mathrm{d}\tau \tag{1.2}$$

for some surface $\mathcal{S}$ enclosing the volume $\mathcal{V}$, Eq. (1.2) can be obtained using the divergence theorem. From the conservation of charge, *whatever flows out through the surface must come at the expense of that remaining inside*[1]:

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{J} \,\mathrm{d}\tau = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} \rho \,\mathrm{d}\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} \right) \mathrm{d}\tau \tag{1.3}$$

this minus sign here represents that any outward flow decreases the charge left in the volume $\mathcal{V}$. Therefore, Eq. (1.3) yields that

$$\boldsymbol{\nabla} \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{1.4}$$

which is the exact mathematical statement of local charge conservation, i.e. the **continuity equation.**

[1] Griffiths, D. J. (2008) *Introduction to Electrodynamics.* San Francisco: Pearson/Benjamin Cummings.

2 A Simple, yet Useful Theorem

Theorem. For all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in the same vector space,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \boldsymbol{\nabla} \cdot \mathbf{B} \tag{2.1}$$

implies that

$$\mathbf{A} = \mathbf{B} + \boldsymbol{\nabla} \times \mathbf{C}. \tag{2.2}$$

Proof. Using Eq. (2.1), one may write

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{A} - \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \boldsymbol{\nabla} \cdot (\mathbf{A} - \mathbf{B}) &= 0 \end{aligned}$$

therefore, the difference $\mathbf{A} - \mathbf{B}$ is of the form $\boldsymbol{\nabla} \times \mathbf{C}$ for an arbitrary vector field in the same vector space $\mathbf{C}$. Thus,

$$\begin{aligned} \mathbf{A} - \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{C} \\ \mathbf{A} &= \mathbf{B} + \boldsymbol{\nabla} \times \mathbf{C}. \quad \text{QED} \end{aligned}$$

3 Derivation of the Ampere's Law

From the **Gauss' Law**, one has that

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \Rightarrow \rho &= \epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} \end{aligned} \tag{3.1}$$

substituting in Eq. (1.4), yields

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{J} &= -\frac{\partial}{\partial t} \left(\epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} \right) \\ &= -\boldsymbol{\nabla} \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \end{aligned} \tag{3.2}$$

here, Eq. (3.2) is of the form Eq. (2.1), then the theorem proved in Section 2 implies that

$$\mathbf{J} = \left(-\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) + \boldsymbol{\nabla} \times \mathbf{B}' \tag{3.3}$$

from Eq. (2.2). Here, $\mathbf{B}' = \frac{1}{\mu_0}\mathbf{B}$ where $\mathbf{B}$ is the magnetic field. Therefore, solving Eq. (3.3) for $\boldsymbol{\nabla} \times \mathbf{B}$, yields

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{3.4}$$

which is the **Ampere's Law** in the Maxwell's equations.