

6.6 Applications to Physics and Engineering - Work

In Physics, a **force** is a push or pull on an object.

Newton's 2nd Law of Motion: If an object moves along a straight line and its position is given by a function $s(t)$, then the force F on the object is given by

$$F = ma$$

"Force equals mass times acceleration."

When a force *moves* an object, we say the force does **work** on the object. Work can be thought of as the amount of energy it takes to move an object.

The work W done by exerting a constant force F over a distance d is given by

$$W = Fd.$$

"Work equals force times distance."

Relevant Units

In the SI metric system:

- Force is measured in Newtons ($N = \text{kg} \cdot \text{m}/\text{s}^2$)
- Mass is measured in kilograms (kg)
- Displacement is measured in meters (m)
- Time is measured in seconds (s)
- Work is measured in Newton-meters ($\text{N}\cdot\text{m}$) a.k.a. joules (J)*

In the US Customary system:

- Force is measured in pounds (lbs)
- Work is measured in foot-pounds ($\text{ft}\cdot\text{lb}$)

dist meas. in ft

Note:

- In the US system, the fundamental unit is the unit of force.
- The **weight** of an object is the force on the object due to gravity, NOT its mass.
- Acceleration due to gravity is approximately $9.8 \text{ m}/\text{s}^2$ in SI units and approximately $32 \text{ ft}/\text{s}^2$ in US units.
- The density of water is approximately $1000 \text{ kg}/\text{m}^3$.

*One joule is the amount of work done by a force of 1 N over a distance of 1 meter.

weight = mass $\cdot$ g where g is accel. due to gravity
ex: a 100 kg object weight $100 \cdot (9.8) = 980 \text{ N}$
vs a 100 lb object weighs 100 lbs.

simple ex: How much work is done to lift a 200 kg rock to a height of 3m?

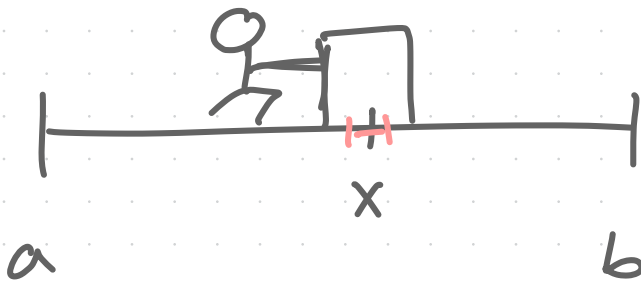
$$m = 200 \text{ kg}, d = 3 \text{ m}$$

Using $F = m \cdot g$ and $W = F \cdot d$

$$\begin{aligned} \text{weight} &= 200 \cdot g = 200(9.8) \\ &= 1960 \text{ N} \end{aligned}$$

$$W = F \cdot d = (1960) \cdot 3 = \boxed{5880 \text{ J}}$$

What if the force being applied is not constant?



(F moves the obj. from a to b)

Force is a function $F(x)$.

Let $dW =$ work done by $F(x)$ over a small interval of length dx

$$= \underbrace{F(x)}_{\text{force}} \cdot \underbrace{dx}_{\text{dist.}}$$

$$\boxed{dW = F(x)dx} \Rightarrow W = \int_a^b dW$$

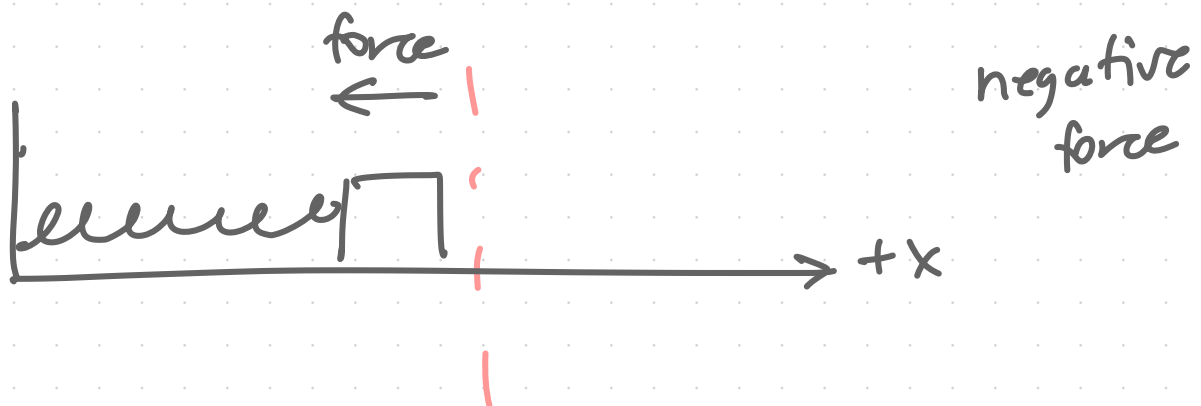
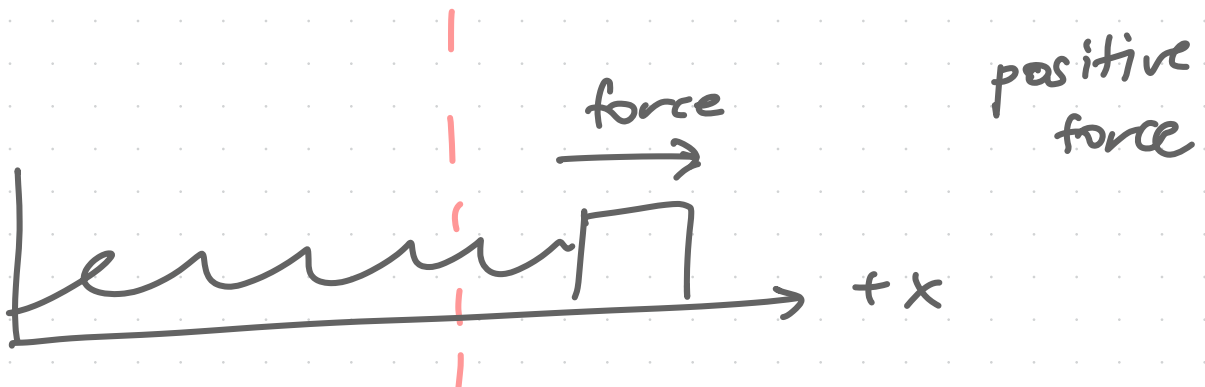
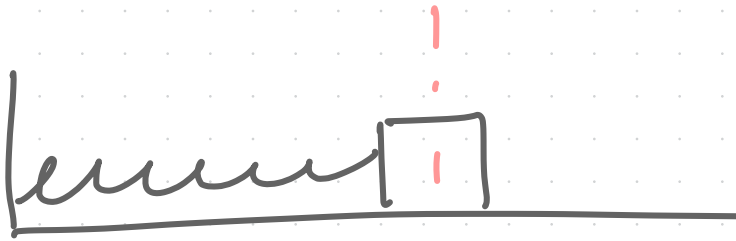
$$\boxed{W = \int_a^b F(x)dx}$$

Work is the integral of force.

Work done to stretch/compress a spring

Consider a block sitting on a frictionless surface, attached to a spring:

$x = x_0$ (equilibrium)



Hooke's Law: $F(x) = k(x - x_0)$

↓
 $y = kx$

↑
force needed
to stretch/compress
from x_0 to x

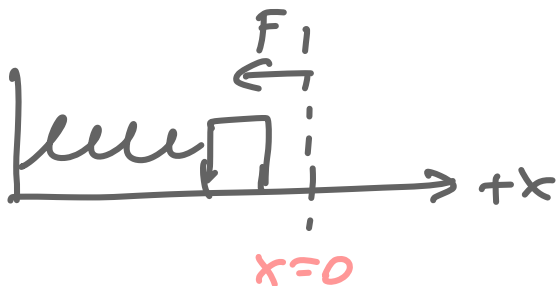
↑
spring constant

(usually can
choose $x_0 = 0$)

6.6 Work Examples

$$F = -10 \quad x = -0.2$$

Example 1: It takes a force of 10 N to compress a spring 0.2 m from equilibrium. How much work is done to stretch the spring 0.5 m from equilibrium?

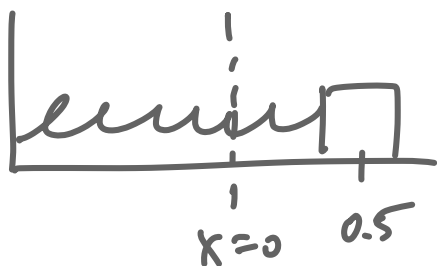


Hooke's Law: $F = -kx$
 $-10 = k(-0.2)$
 $k = \frac{10}{0.2} = 50$

So $F(x) = 50x$

$$W = \int_0^{0.5} F(x) dx = \int_0^{0.5} 50x dx = 50 \left(\frac{1}{2} x^2 \right) \Big|_0^{0.5} = 6.25 \text{ J}$$

Example 2: If 10 J of work is needed to stretch a spring 0.5 m beyond its natural length, how much work is needed to stretch it from 1 to 2 m beyond its natural length?



$$W = 10 \text{ J}$$

$$10 = \int_0^{0.5} F(x) dx$$

$$10 = \int_0^{0.5} kx dx$$

$$= k \left(\frac{1}{2} x^2 \right) \Big|_0^{0.5}$$

$$10 = k \cdot \frac{1}{8} \Rightarrow k = 80$$

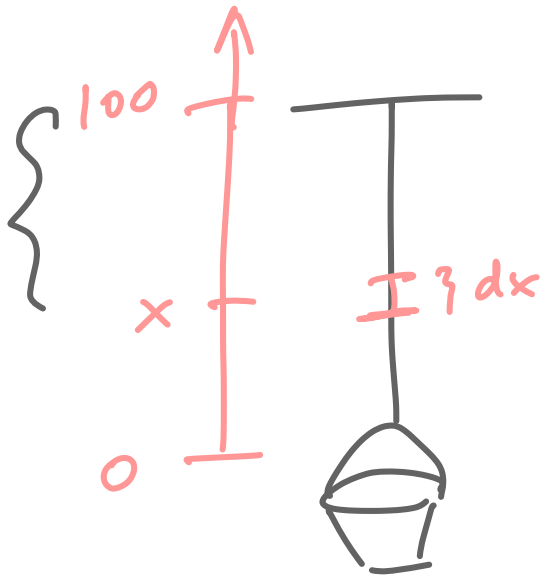
$$W = \int_1^2 80x dx$$

$$= 80 \left(\frac{1}{2} x^2 \right) \Big|_1^2$$

$$= 40(4 - 1) = 120 \text{ J}$$

Example 3: A cable that weighs 2 lbs/ft is attached to a bucket filled with water that weighs 40 lbs. The bucket is initially at the bottom of a 100 ft well. Determine the amount of work necessary to lift the bucket to the top of the well.

** establish a coordinate system*



We'll find dW and integrate to get W .

needs an integral.
↓

$$\text{total work} = W_{\text{bucket}} + W_{\text{cable}}$$

$$W_{\text{bucket}} = F \cdot d = 40 \text{ lbs} \cdot 100 \text{ ft} = 4000 \text{ ft-lbs}$$

For the cable, let dW be the work needed to raise a tiny chunk of cable to the top of the well

$$\text{weight of chunk} = (2 \text{ lbs/ft}) (dx \text{ ft}) = 2 dx \text{ lbs}$$

← a number

$$\begin{aligned} \text{So } dW &= F \cdot d = (\text{weight}) \cdot (\text{dist travelled}) \\ &= (2 dx) \cdot (100 - x) = 2(100 - x) dx \end{aligned}$$

$$W_{\text{cable}} = \int_0^{100} dW = \int_0^{100} 2(100 - x) dx = 10,000 \text{ ft-lbs}$$

$$\text{total work: } W = 4000 + 10000 \\ = 14000 \text{ ft-lbs}$$

An aside:

2 lbs/ft is a linear weight density
↖ along a line

If instead we were told the cable
200 lbs total and is 100 ft long, then
we would need to compute

$$\frac{\text{total weight}}{\text{total length}} = \frac{200 \text{ lbs}}{100 \text{ ft}} = 2 \text{ lbs/ft}$$

If given total mass, compute the
linear mass density:

ex: a 75 kg cable, 50 m long

$$\text{linear density} = \frac{75}{50} \text{ kg/m} = \frac{3}{2} \text{ kg/m}$$



a chunk of length 2m would
weighs

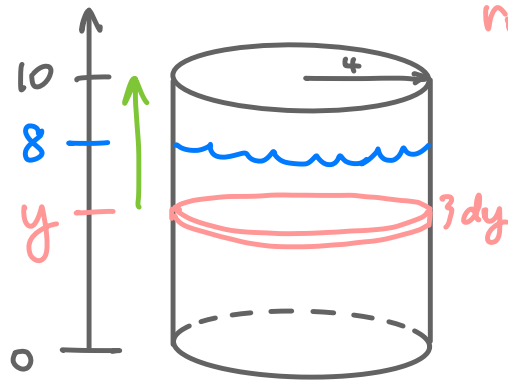
$$\frac{3}{2} \cdot 2 = 3 \text{ kg}$$

density length

SI units

Example 4: A cylindrical tank with radius 4 m and height 10 m is filled with water to a level of 8 m. Find the work required to pump the water out of the top of the cylindrical tank.

Need to lift all the water to the top of the tank. Water at the bottom travels further (so requires more work) $\Rightarrow$ need an integral!



Strategy: think of thin (horizontal) slices of water & compute the work needed to raise a slice to top

(density of water is $\rho = 1000 \text{ kg/m}^3$) *

Let dW = work needed to raise a slice at height y

$$= \underbrace{(\text{weight of slice})}_{?} \cdot \underbrace{(\text{distance to top})}_{10-y}$$

$$\begin{aligned} \text{weight} &= \text{mass} \cdot g \\ &= \rho \cdot \text{vol} \cdot g \\ &= \rho \cdot g \cdot (16\pi dy) \end{aligned}$$

$$\text{vol} = \pi (4)^2 dy$$

$$\begin{aligned} dW &= (\rho \cdot g \cdot 16\pi dy) (10-y) \\ &= \rho \cdot g \cdot 16\pi \cdot (10-y) dy \end{aligned}$$

← where the water ends, not the tank

$$W = \int_0^8 dW$$

$$= \int_0^8 \rho \cdot g \cdot 16\pi (10-y) dy$$

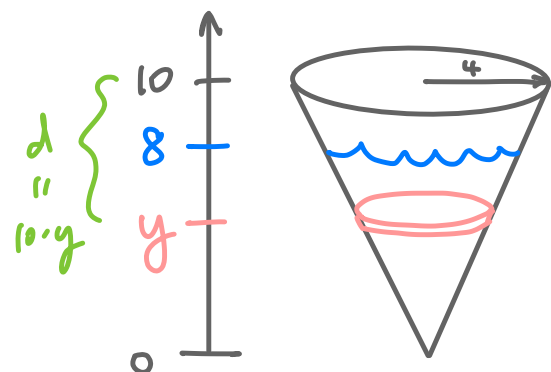
$$= \int_0^8 \underbrace{(1000)(9.8)16\pi}_{\text{plug-in } \rho=1000 \text{ and } g=9.8} (10-y) dy$$

$$= 9800 \cdot 16\pi \int_0^8 10-y dy$$

$$= \quad \quad \cdot \left(10y - \frac{1}{2}y^2\right) \Big|_0^8$$

$$= 9800 \cdot 16\pi (80 - 32 - 0)$$

Example 5: A conical tank with base radius 4 m and height 10 m is filled with water to a level of 8 m. Find the work required to pump the water out of the top of the ~~cylindrical~~ tank.



$$W = F \cdot d = (m \cdot g) d = \rho \cdot \text{vol} \cdot g \cdot d$$

these change from problem to problem

$$d = 10 - y$$

vol = volume of a slice
(not the tank)

d = distance to top of tank
(or spout)

Not likely to come up, BUT...

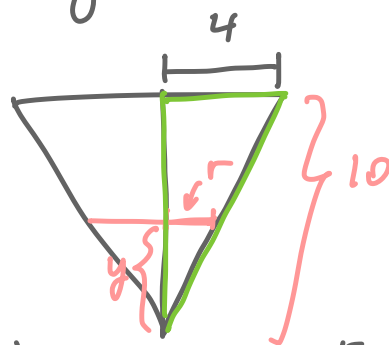
If in US units, drop g and use $\rho = 62.4 \text{ lbs/ft}^3$ (for water)
 $\Rightarrow dW = \rho \cdot \text{vol} \cdot d$

volume of a slice?



$$\text{vol} = \pi r^2 \cdot dy$$

need r as a function of y :



by similar triangles, $\frac{r}{y} = \frac{4}{10}$
 $r = \frac{2}{5}y$

$$\text{So } \text{vol} = \pi \left(\frac{2}{5}y \right)^2 dy$$

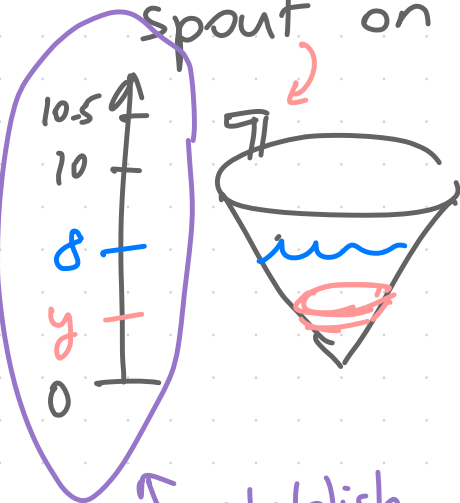
$$dW = \rho \cdot \left(\pi \left(\frac{2}{5}y \right)^2 dy \right) \cdot g \cdot (10-y)$$

$$W = \int_0^8 \rho \cdot g \pi \left(\frac{2}{5}y \right)^2 (10-y) dy$$

$$= \int_0^8 (1000)(9.8) \pi \left(\frac{2}{5}y \right)^2 (10-y) dy$$

example 5.5

same tank and water level as example 5, BUT we want to pump the water out of a 0.5m spout on top.



same tank $\Rightarrow$ same slices so
Same volume

same water level $\Rightarrow$ same bounds

d is only change

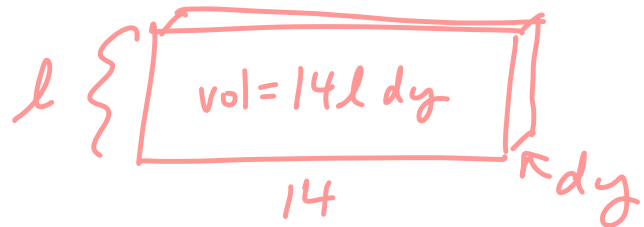
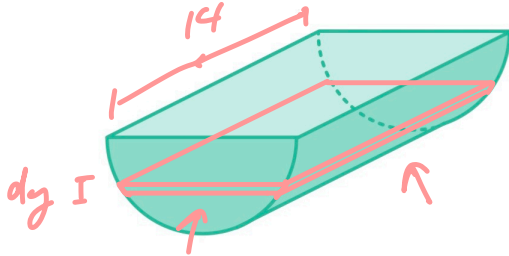
$$d = (10.5 - y)$$

$\uparrow$ establish
your coordinate
system

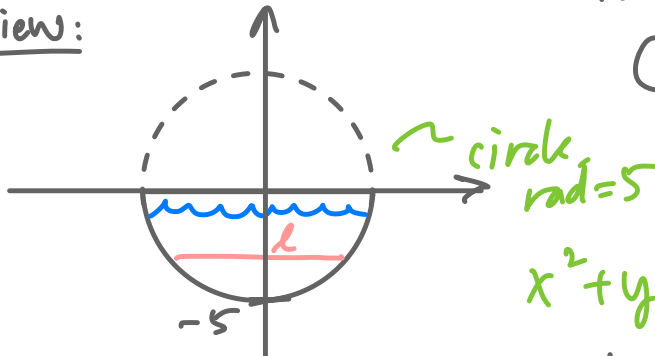
$$dW = \rho \cdot \pi \left(\frac{2}{5}y\right)^2 dy \cdot g \cdot (10.5 - y)$$

Example 6: A tank is formed by taking half of a cylinder of radius 5 m and height 14 m and laying it on its side. It is filled with water to a level of 4 m. Find the work required to pump the water out of the top of the cylindrical tank.

slices are rectangles



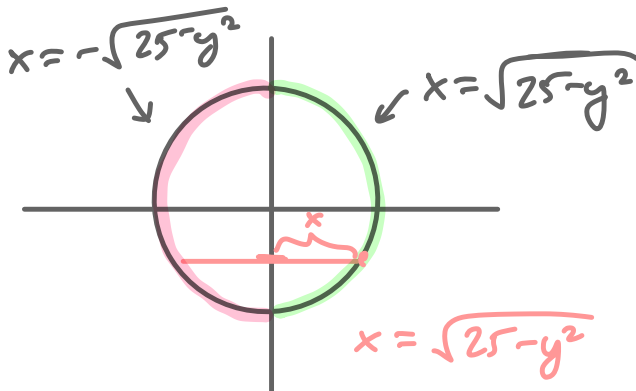
Side View:



Need l in terms of y ✓
(for vol) and d

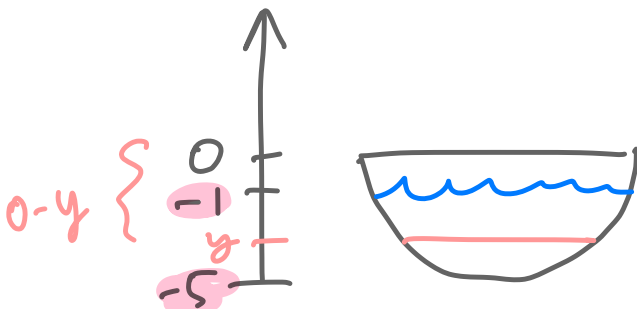
$$x^2 + y^2 = 25$$

$$\hookrightarrow x = \pm \sqrt{25 - y^2}$$



$$x = \sqrt{25 - y^2} \quad \text{so} \quad l = 2x = 2\sqrt{25 - y^2}$$

$$vol = 14l dy = 14 \cdot 2 \sqrt{25 - y^2} dy$$



$$d = -y$$

$$dW = F \cdot d = m \cdot g \cdot d = \rho \cdot \text{vol} \cdot g \cdot d$$

$$= \rho (28 \sqrt{25 - y^2} dy) \cdot g (-y)$$

$$W = \int_{-5}^1 \rho \cdot g \cdot 28 \sqrt{25 - y^2} (-y) dy$$