

# Invariant theory of Hamiltonian mechanics and related numerical analysis

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# Geometric mechanics



Isaac  
**New-**  
**ton,**  
1643-  
1727



Émilie  
**du**  
**Châtelet,**  
1706-  
1749



Joseph-  
Louis **de**  
**La-**  
**grange,**  
1736-  
1813



Siméon  
Denis  
**Poisson,**  
1781-  
1840



Marius  
Sophus  
**Lie,**  
1842-  
1899



Vladimir  
I.  
**Arnol'd,**  
1937-  
2010



André  
**Lich-**  
**nerow-**  
**icz,**  
1915-  
1998

and Carl Gustav Jacob **Jacobi** (1804 - 1851), Sir William Rowan **Hamilton** (1805 -1865), Sofia Vassilievna **Kovalevskaja** (1850-1891), Henri **Poincaré** (1854-1912), Jean-Marie **Souriau** (1922 - 2012), Alan David **Weinstein** (1943- )...

## Hamiltonian dynamics on Poisson manifolds

- Generalities on Poisson structures

- Hamiltonian systems

- A first approach to the numerical analysis on Poisson manifolds

## Symplectic groupoids

- Basic notions on groupoids

- Basics on symplectic groupoids

- Lagrangian bisections

- Birealisations

- Hamilton-Jacobi equation

## Hamiltonian Poisson integrators

- Butcher series

- Explicit construction through birealisations

- Numerical tests around a singularity: a Lotka-Volterra system

- Numerical tests of symmetry preservation: the rigid body

## Perspectives

- Poisson integrators in solid mechanics (with L. Le Marrec and V. Carlier)

- Deformation theory of symplectic groupoids and plasma physics

## Definition

$M$  be a smooth manifold.

Recall that a Lie bracket on  $\mathcal{C}^\infty(M)$  is a bilinear anti-symmetric map from  $\mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$  such that

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \quad (\text{Jacobi relation}).$$

## Definition (Flato, Lichnerowicz, Sternheimer, 1975)

A Poisson bracket on  $M$  is a Lie bracket on  $\mathcal{C}^\infty(M)$  such that for all  $f, g, h \in \mathcal{C}^\infty(M)$ ,

$$\{fg, h\} = f\{g, h\} + \{f, g\}h \quad (\text{Leibniz rule}).$$

## Remark

Anti-symmetry of  $\{.,.\} \Rightarrow$  for all  $H \in \mathcal{C}^\infty(M)$ ,  $\{H, H\} = 0$ .

## Consequence of the Leibniz rule: Poisson tensor of $\{\cdot, \cdot\}$

### Lemma

*An  $\mathbb{R}$ -bilinear map  $B: C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$  arises from a smooth bivector field  $\pi \in \Gamma(\bigwedge^2 TM)$  by the formula*

$$B(f, g) = \langle \pi, df \wedge dg \rangle$$

*if and only if  $B$  is skew-symmetric and verifies the Leibniz rule*

$$B(fg, h) = fB(g, h) + B(f, g)h$$

### Corollary

*Any Poisson bracket  $\{\cdot, \cdot\}$  gives rise to a smooth bivector field  $\pi \in \Gamma(\bigwedge^2 TM)$ . If  $(x_i)_i$  are coordinates on  $M$ ,  $\pi_{i,j} = \{x_i, x_j\}$ .*

### Remark

*This holds because we assumed  $M$  to be finite dimensional.*

## Two consequences of the Jacobi relation

Proposition (Poisson brackets are particular cases of bivector fields)

*There is a 1-to-1 correspondence between Poisson brackets on  $M$  and smooth bivector fields  $\pi$  such that the map*

$$(f, g) \in \mathcal{C}^\infty(M)^2 \mapsto \langle \pi, df \wedge dg \rangle$$

*satisfies the Jacobi relation.*

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### Definition

Let  $H \in \mathcal{C}^\infty(M)$ .  $f \in \mathcal{C}^\infty(M)$  is a first integral of  $H$  if  $\{H, f\} = 0$ .

### Proposition (Poisson theorem)

$$\{H, f\} = \{H, g\} = 0 \Rightarrow \{H, \{f, g\}\} = 0.$$

Jacobi cyclic relation implies a Lie algebra structure on the first integrals of  $H$ .

## A first example: symplectic manifolds

Let  $M$  be a symplectic manifold, meaning a manifold equipped with a closed 2-form  $\omega$  such that for all  $x \in M$ , the map:

$$\omega^\flat: X \in T_x M \mapsto \omega_x(X, \cdot) \in T_x^* M$$

is a bijection, with inverse denoted by  $\omega^\sharp$ . The Poisson bracket of a symplectic manifold is

$$\{f, g\} = \langle df, \omega^\sharp(dg) \rangle.$$

### Remark

*Jacobi relation for this Poisson bracket is equivalent to the closedness of  $\omega$ .*

In Darboux coordinates  $(q, p)$ , the Poisson tensor is given by the matrix  $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$  and the corresponding Poisson bracket

$$\{f, g\}(q, p) = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial g}{\partial q} \frac{\partial f}{\partial p}.$$

## Two more examples

2nd example: Lie-Poisson structure on  $\mathfrak{g}^*$

Let  $\mathfrak{g}$  be a Lie algebra.  $M = \mathfrak{g}^*$  comes canonically equipped with the Poisson bracket

$$\forall f, g \in \mathfrak{g}^*, \quad \forall x \in \mathfrak{g}, \quad \{f, g\}(x) = \langle x, [d_x f, d_x g] \rangle$$

where  $d_x f \in \mathfrak{g}^{**} \simeq \mathfrak{g}$ .

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3rd example: Cluster Poisson brackets

Let  $A = (a_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}_{n \times n}(\mathbb{R})$  anti-symmetric. Set  $M = \mathbb{R}^n$  and

$$\forall x \in \mathbb{R}^n, \quad \forall 1 \leq i, j \leq n, \quad (\pi(x))_{ij} = a_{ij} x_i x_j.$$

# Poisson morphisms

## Definition

Let  $(M_1, \{\cdot, \cdot\}_1)$  and  $(M_2, \{\cdot, \cdot\}_2)$  two Poisson manifolds. A smooth map  $\alpha: M_1 \rightarrow M_2$  is said to be a Poisson map, resp. an anti-Poisson map, if

$$\forall f, g \in C^\infty(M_2), \quad \{\alpha^*f, \alpha^*g\}_1 = \alpha^*\{f, g\}_2$$

$$\text{resp. } \{\alpha^*f, \alpha^*g\}_1 = -\alpha^*\{f, g\}_2.$$

## Example (Symplectic case)

Poisson automorphisms of a symplectic Poisson structure are called symplectomorphisms.

## Example (Linear case)

Linear Poisson automorphisms of the dual of a Lie algebra are transpose maps of automorphisms of the Lie algebra.

# Hamiltonian vector fields and flows

## Definition (Hamiltonian vector field)

Identifying  $\mathfrak{X}(M) \simeq \text{Der}(M)$ , the Hamiltonian vector field of  $H$  is set to be

$$X_H: \begin{array}{ccc} \mathcal{C}^\infty(M) & \rightarrow & \mathcal{C}^\infty(M) \\ f & \mapsto & \pi(dH \wedge df) \end{array} .$$

## Remark (Hamiltonian systems are local)

$X_H$  at  $x \in M$  only depend on the differential  $d_x H$  of  $H$  at  $x$ .

## Theorem

The Hamiltonian flow  $\phi_t^H$  of  $H$  is a Poisson automorphism:

$$\forall f, g \in \mathcal{C}^\infty(M), \quad \forall t, \quad (\phi_t^H)^* \{f, g\} = \{(\phi_t^H)^* f, (\phi_t^H)^* g\}$$

## Terminology

Hamiltonian flows are said to be inner automorphisms of  $\pi$ .

# 1st example of a Hamiltonian system: classical mechanics on $M = T^*X$

|             |                             |                           |
|-------------|-----------------------------|---------------------------|
| $X$         | a smooth manifold           | (the configuration space) |
| $\ \cdot\ $ | a Riemannian metric on $X$  | (the kinetic energy)      |
| $V$         | $\in \mathcal{C}^\infty(X)$ | (the potential energy).   |

$$H: \begin{array}{ccc} T^*X & \rightarrow & \mathbb{R} \\ \zeta & \mapsto & \frac{1}{2m} \|\zeta\|^2 + V \circ \mathfrak{p}(\zeta) \end{array}$$

where:  $\mathfrak{p}: T^*X \rightarrow V$  the cotangent projection  
 $m > 0$  the mass.

## Remark

*The first integrals of  $H$  depend on the physics. For instance: angular momentum.*

In cotangent coordinates  $(q, p)$ , the ODE of  $X_H$  is:

$$\begin{cases} \dot{q} = -\frac{p}{m} \\ \dot{p} = -\nabla V(q) \end{cases} \quad \text{i.e. 2nd law of Newton (1687) } m\ddot{q} = -\nabla V(q)$$

## 2nd example of a Hamiltonian system: mechanics on $\mathfrak{g}^*$ and the angular velocity of a rigid body

For the **canonical Poisson structure on  $\mathfrak{g}^*$** , the Hamiltonian system associated to  $H \in \mathcal{C}^\infty(\mathfrak{g}^*)$  is

$$\dot{x} = \text{ad}_{dH}^* x$$

In the case of  $\mathfrak{so}(3) \simeq \mathbb{R}^3$ , the Poisson bracket is

$$\{f, g\}(x) = (\nabla_x f)^T \cdot \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \cdot \nabla_x g$$

The Hamiltonian of the angular velocity of the rigid body is

$$H(x) = \frac{1}{2} \left( i_1(x_2^2 + x_3^2) + i_2(x_1^2 + x_3^2) + i_3(x_1^2 + x_2^2) \right)$$

$$\Rightarrow \dot{x} = -x \wedge Ix, \text{ where } I = \begin{pmatrix} i_1 & 0 & 0 \\ 0 & i_2 & 0 \\ 0 & 0 & i_3 \end{pmatrix} \text{ is the inertia tensor}$$

### 3rd example of a Hamiltonian system: Lotka-Volterra system on a cluster Poisson structure

**the cluster Poisson bracket**

$$\{f, g\}(x) = (\nabla_x f)^T \cdot \begin{pmatrix} 0 & x_1 x_2 & x_1 x_3 \\ -x_1 x_2 & 0 & x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & 0 \end{pmatrix} \cdot \nabla_x g$$

For the Hamiltonian

$$H(x) = \sum_{i=1}^3 x_i$$

being the sum of the species populations, the corresponding Hamiltonian system is

$$\Rightarrow \begin{cases} \dot{x}_1 = x_1(x_2 + x_3) \\ \dot{x}_2 = x_2(-x_1 + x_3) \\ \dot{x}_3 = -x_3(x_1 + x_2) \end{cases} \quad (1)$$

**Remark** (Vanhaecke et al., 2016)

*The equation (1) has been proven to be part of a family of completely integrable systems.*

# First integrals

## Theorem

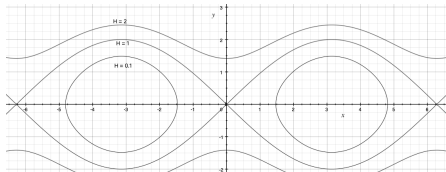
Let  $F \in \mathcal{C}^\infty(M)$  a first integral of  $H$ . The flow of  $H$  commutes with the one of  $F$ :

$$\forall s, t, \quad \phi_s^F \phi_t^H = \phi_t^H \phi_s^F$$

## Corollary

The Hamiltonian is always a preserved quantity along a trajectory of its Hamiltonian flow.

In  $\mathbb{R}^2$  equipped with  $\pi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , the data of the Hamiltonian levels of  $H$  is enough to compute any integral curve of the Hamiltonian vector field of  $H$ .



Hyperbolic pendulum  
 $H(q, p) = \frac{p^2}{2} + \cos q$

# Casimir functions

## Definition

$C \in \mathcal{C}^\infty(M)$  is said to be a Casimir if it is a first integral for all smooth functions on  $M$ . Equivalently, its Hamiltonian vector field vanishes.

## Example (The non-degenerate case)

A symplectic Poisson structure has no non-trivial Casimir.

## Example (The linear case)

Any element of the center of a Lie algebra  $\mathfrak{g}$  provides a Casimir on  $\mathfrak{g}^*$ .

## Example (A quadratic case)

Let  $A$  be a  $n \times n$  real anti-symmetric matrix and  $(a_i)_{1 \leq i \leq n} \in \ker A$ . Then, the function  $x \in \mathbb{R}^n \mapsto \prod_{i=1}^n x_i^{a_i}$  is a Casimir of the cluster Poisson bracket associated to the matrix  $A$ .

# Symplectic foliation

On a Poisson manifold  $M$ , not all the points can be connected by Hamiltonian flows. This foliates  $M$ .

## Theorem

*Any Poisson manifold  $(M, \pi)$  is foliated into symplectic leaves. This foliation is singular.*

A foliation is in particular a partition of the manifold into immersed submanifolds.

Let  $x \in M$  and  $\mathcal{F}_x$  the leaf of  $x$ .

- ▶  $\mathcal{F}_x$  is a symplectic manifold.
- ▶ The immersion  $\mathcal{F}_x \hookrightarrow M$  is a Poisson morphism.
- ▶ Let  $y \in M$ . The topology of  $\mathcal{F}_y$  might differ a lot from the one of  $\mathcal{F}_x$ , even if  $x$  and  $y$  are close.

## Remark

*Any Casimir must be constant on a symplectic leaf.*

# Symplectic foliation of the examples

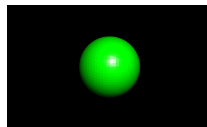
The symplectic Poisson structure

The symplectic foliation is made of one symplectic leaf: the whole space.

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Lie-Poisson structure on  $\mathfrak{g}^*$

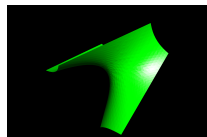
It is given by coadjoint orbits.



A symplectic leaf of  $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$

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Cluster Poisson brackets



Exercise !

A symplectic leaf of  $\{x_i, x_j\} = x_i x_j$ ,

# Properties of Hamiltonian flows

Let  $H \in \mathcal{C}^\infty(M)$ . Its Hamiltonian flow  $\phi_t^H$

- ▶ **preserves  $H$ :**  $H \circ \phi_t^H = H$ ,  
Hamiltonian flows are *conservative*.
- ▶ **preserves the Poisson tensor:**  $\phi_t^H \star \pi = \pi$ ,  
Hamiltonian flows are *inner automorphisms of the Poisson structure*.
- ▶ **stay on a leaf:**  $\forall x \in M, \quad \phi_t^H(x) \in \mathcal{F}_x$ ,  
this provides a *construction* of any symplectic leaf, cf control theory.

# Symplectic Runge-Kutta

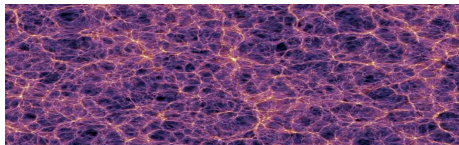
Define an integrator as a family:  $\varphi_\epsilon: x_n \mapsto x_{n+1}$  depending on the time-step  $\epsilon$ .

## Definition

In the space  $\{(q, p)\}$  of position-momenta, an integrator  $\varphi_\epsilon$  is said to be **symplectic** if it is a Poisson morphism for the Poisson

structure  $\pi(q, p) = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} := J$ .

Symplectic  
integrators are used  
in astrophysics,  
molecular dynamics,  
optics...



Virgo project, 2004

# How to cook up a symplectic integrator ?

Here:  $\pi = J$ .

A Runge-Kutta integrator for  $\dot{x} = J \cdot \nabla_x H$  is of the form

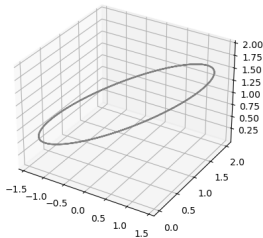
$$\begin{aligned}x_{n+1} &= x_n + \epsilon \sum_{i=1}^s b_i k_i \\ k_i &= J \cdot \nabla H(x_n + \epsilon \sum_{j=1}^s a_{ij} k_j)\end{aligned}$$

where  $x := (q, p)$ .

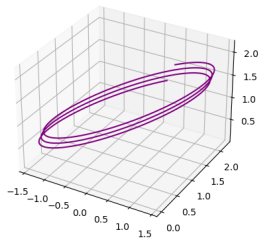
## Theorem

*If :  $\forall (i, j), b_i a_{ij} + b_j a_{ji} = b_i b_j$ ,  
then the integrator is symplectic.*

# Runge-Kutta Poisson ?



A Hamiltonian trajectory for a Poisson structure in  $\mathbb{R}^3$



An integrator that preserves this Poisson structure

In the general case, it is not enough to preserve the Poisson structure. Such an integrator, even of high order, may provide bad numerical simulations: *it destroys the foliation.*

## Exercises for tomorrow

1. Prove that the cluster Poisson brackets are indeed Poisson brackets, namely, that the brackets verify the Jacobi relation.
2. Compute the symplectic foliation of  $\mathbb{R}^3$  endowed with  $\{x_1, x_2\} = x_1 x_2$ ,  $\{x_1, x_3\} = 2x_1 x_3$ ,  $\{x_2, x_3\} = x_2 x_3$ . Namely: describe the partition of  $\mathbb{R}^3$  induced by these Poisson brackets.

# Bibliography for the first lecture

- ▶ *Symplectic Geometry and Analytical Mechanics*, Paulette Libermann and Charles-Michel Marke, Kluwer, 1987  
→ Poisson geometry and mechanics
- ▶ Chapter 1 of *Poisson Structures and their Normal Forms*, Jean-Paul Dufour and Nguyen Tien Zung, Birkhäuser, 2005  
→ Basics on Poisson geometry
- ▶ *Lectures on Poisson Geometry*, Marius Crainic, Rui Loja Fernandes, Ioan Mărcuț, American Mathematical Society, 2021  
→ More involved Poisson geometry

## Hamiltonian dynamics on Poisson manifolds

- Generalities on Poisson structures

- Hamiltonian systems

- A first approach to the numerical analysis on Poisson manifolds

## Symplectic groupoids

- Basic notions on groupoids

- Basics on symplectic groupoids

- Lagrangian bisections

- Birealisations

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- Explicit construction through birealisations

- Numerical tests around a singularity: a Lotka-Volterra system

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## Perspectives

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- Deformation theory of symplectic groupoids and plasma physics

# Groupoid

## Definition

A groupoid over a set  $M$  is a set  $\mathcal{G}$  with

1. Two maps  $\alpha$  and  $\beta: \mathcal{G} \rightarrow M$  (*source* and *target*). We write  $\mathcal{G}^{(k)} = \{(g_i)_{1 \leq i \leq k} \in \mathcal{G}^k, \quad \forall 1 \leq i \leq k-1, \alpha(g_{i+1}) = \beta(g_i)\}$
2. A *product*  $\cdot: \mathcal{G}^{(2)} \rightarrow \mathcal{G}$ , such that
  - 2.1  $\alpha(g \cdot h) = \alpha(h)$  and  $\beta(g \cdot h) = \beta(h)$  (compatibility with source and target)
  - 2.2  $\forall (g, h, k) \in \mathcal{G}^{(3)}, \quad (g \cdot h) \cdot k = g \cdot (h \cdot k)$  (associativity)
3. An *identity section*  $i: M \rightarrow \mathcal{G}$  such that  $g \cdot i(\alpha(g)) = i(\beta(g)) \cdot g = g$ .
4. An *inverse*  $(.)^{-1}: \mathcal{G} \rightarrow \mathcal{G}$  such that  $\alpha(g^{-1}) = \beta(g)$ ,  $\beta(g^{-1}) = \alpha(g)$  and

$$g^{-1} \cdot g = i(\alpha(g)), \quad g \cdot g^{-1} = i(\beta(g)).$$

$M$  is said to be the *objects* and  $\mathcal{G}$  is said to be the *arrows*.

# Lie groupoid

## Definition

A Lie groupoid is a groupoid  $\mathcal{G} \rightrightarrows M$  such that

- ▶  $\mathcal{G}$  and  $M$  are smooth manifolds
- ▶ the product  $\cdot: \mathcal{G}^{(2)} \rightarrow \mathcal{G}$  and the identity section  $i: M \rightarrow \mathcal{G}$  are smooth maps
- ▶  $\alpha, \beta: \mathcal{G} \rightarrow M$  are smooth submersions.

## Remark

*In the literature, it is common not to assume the differentiable topology of  $\mathcal{G}$  to be Hausdorff separable. In this lecture, all the manifolds are assumed to be separable except if they are explicitly introduced as Lie groupoids.*

More generally, the topology of groupoids is an intricate topic<sup>1</sup>.

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<sup>1</sup>Lie Groupoids and Lie Algebroids in Differential Geometry, Mackenzie, 1987

## Examples of Lie groupoids

The pair groupoid of a manifold  $M$

$$\mathcal{G} = M \times M$$

$$\alpha(x, y) = x, \beta(x, y) = y$$

$$(x, y) \cdot (y, z) = (x, z)$$

$$i(x) = (x, x)$$

---

The action groupoid  $\mathcal{G} = G \times M$  of an action of  $G$  on  $M$

$$\alpha(g, x) = x, \beta(g, x) = g.x$$

$$(g, x) \cdot (h, g.x) = (gh, x)$$

$$i(x) = (1, x)$$

---

$$G \times \mathfrak{g}^*$$

$$\alpha(g, x) = x, \beta(g, x) = \text{Ad}_g^* x$$

$$(g, x) \cdot (h, \text{Ad}_g^* x) = (gh, x)$$

$$i(x) = (1, x)$$

# Induced singular foliation

Any Lie groupoid  $\mathcal{G} \rightrightarrows M$  induces a partition of the base  $M$ :

$$x \sim y \iff \exists p \in \mathcal{G}, \quad \alpha(p) = x \quad \& \quad \beta(p) = y$$

## Theorem

*This partition is a singular foliation. In particular, each leaf is an immersed submanifold.*

## Example

- ▶ The foliation of a pair groupoid has only one leaf.
- ▶ The foliation of an action groupoid is made of the orbits of the action.
- ▶ The foliation of  $G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$  is made of the coadjoint orbits.

# Bisections

## Definition (Bisection)

A bisection is a submanifold  $L$  of  $\mathcal{G}$  such that  $\alpha|_L: L \rightarrow M$  and  $\beta|_L: L \rightarrow M$  are diffeomorphisms.

## Proposition

*The set of bisections is a group: for any bisections  $L_1$  and  $L_2$ ,*

$$L_1 \cdot L_2 = \{l_1 \cdot l_2 \in \mathcal{G}, l_1 \in L_1, l_2 \in L_2, \beta(l_1) = \alpha(l_2)\}.$$

## Proposition

*Any bisection  $L \subset \mathcal{G}$  induces a diffeomorphism*

$$\Phi^L: \begin{array}{ccc} M & \rightarrow & M \\ x & \mapsto & \beta \circ (\alpha|_L)^{-1}(x) \end{array}$$

*such that  $x \in M$  and  $\Phi^L(x) \in M$  remain on the same leaf.*

# Symplectic groupoids

## Definition (Symplectic groupoid)

A symplectic groupoid is a Lie groupoid  $\mathcal{G} \rightrightarrows M$  such that  $\mathcal{G}$  is equipped with a symplectic form  $\omega$  such that

$$m^*\omega = \text{pr}_1^*\omega + \text{pr}_2^*\omega$$

where  $m: \mathcal{G}^{(2)} \rightarrow \mathcal{G}$  is the product and  $\text{pr}_i: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$  is the projection on the  $i$ -th factor.

## Terminology

$\omega$  is said to be *multiplicative*.

Roughly speaking, “ $\omega(g \cdot h) = \omega(g) + \omega(h)$ ”.

## Remark

The graph of the product is isotropic –even Lagrangian– in  $\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$ , where  $\bar{\mathcal{G}}$  is equipped with  $-\omega$ .

# Some properties of symplectic groupoids

## Proposition

1. *We identify  $M$  with  $i(M) \subset \mathcal{G}$ . Then,  $M$  is Lagrangian in  $\mathcal{G}$ .*
2. *The inverse map is an anti-symplectomorphism:*

$$i^*\omega = -\omega$$

3. *The fibers of the source are symplectically orthogonal to the fibers of the target:*

$$\forall f, g \in C^\infty(M), \{\alpha^*f, \beta^*g\} = 0.$$

4. *There is a unique Poisson structure  $\pi$  on  $M$  such that the source map  $\alpha$  is a Poisson map and the target map  $\beta$  is an anti-Poisson map.  $\mathcal{G} \rightrightarrows M$  is said to be integrating  $(M, \pi)$ .*

## Remark (Informal)

*A symplectic groupoid desingularizes the symplectic foliation on the base.*

## Examples

The cotangent bundle  $T^*M$  of any manifold  $M$  – equipped with the canonical symplectic form on  $T^*M$  and source and target equal the cotangent projection – integrates the zero Poisson structure on  $M$ .

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$(M, \Omega)$  is integrated by the pair groupoid  $M \times \overline{M} \rightrightarrows M$ .

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The action groupoid  $T^*G \simeq G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$  – equipped with the canonical symplectic form on  $T^*G$  – integrates  $\mathfrak{g}^*$ .

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A symplectic groupoid for any cluster Poisson structure has been computed in *Symplectic groupoids for cluster manifolds*, Li Rupel, 2020.

# Lagrangian Bisections

Let  $(\mathcal{G}, \omega) \rightrightarrows M$  be a symplectic groupoid.

## Definition (Lagrangian bisection)

A Lagrangian bisection is a bisection which is Lagrangian as a submanifold of  $(\mathcal{G}, \omega)$ .

Lagrangian bisections are a subgroup of the group of bisections.

## Lemma

*The group of bisections acts on  $\mathcal{G}$ .*

## Lemma

*The group of Lagrangian bisections acts on  $\mathcal{G}$  by symplectomorphisms.*

# Induced diffeomorphisms

Let us assume that the fibers of  $\alpha$  are source-connected.

## Theorem

*Let  $L \subset \mathcal{G}$  be a Lagrangian bisection of  $(\mathcal{G}, \omega)$ . The induced diffeomorphism*

$$\Phi^L = \beta \circ (\alpha|_L)^{-1}: M \rightarrow M$$

- 1. is a Poisson diffeomorphism on the base*
- 2. and stays on a leaf.*

## Question

*Let  $H \in \mathcal{C}^\infty(M)$  a Hamiltonian and  $t \in \mathbb{R}$  some time. Which Lagrangian bisection  $L$  recovers the Hamiltonian flow  $\phi_t^H$  ?  
Namely, how to choose  $L$  such that*

$$\Phi^L = \phi_t^H?$$

# Weinstein tubular neighborhood theorem

Let  $(\mathcal{G}, \omega) \rightrightarrows M$  be a symplectic groupoid.

## Theorem (Weinstein tubular neighborhood theorem)

*There exists a symplectomorphism from an open neighborhood of  $M \subset \mathcal{G}$  onto an open neighborhood of the zero section of  $T^*M$ .*

One can transport any local symplectic groupoid towards a neighborhood of the identity section inside the cotangent bundle of the base. There, the symplectic form is known and provides many Lagrangian bisections:

### Remark

1. Any closed 1-form on  $M$  provides a Lagrangian submanifold of  $T^*M$ .
2. Locally, any image of a sufficiently small closed 1-form is a Lagrangian bisection.

# Birealisation

The notion of birealisation axiomatises this embedding.

## Definition (Birealisation)

A birealisation of a Poisson manifold  $(M, \pi)$  is a triple  $(\mathbb{G}, \alpha, \beta)$  such that

- ▶  $\mathbb{G} \subset T^*M$  is a neighborhood of the zero section
- ▶  $\alpha: \mathbb{G} \rightarrow M$  – *the source* – and  $\beta: \mathbb{G} \rightarrow M$  – *the target* – are smooth submersions such that
  1.  $\alpha \circ 0 = \beta \circ 0 = \text{Id}_M$
  2.  $\alpha$  is a Poisson and  $\beta$  an anti-Poisson morphism for the canonical Poisson bracket on  $\mathbb{G} \subset (T^*M, \omega)$
  3.  $\forall f, g \in C^\infty(M), \{\alpha^*f, \beta^*g\}_\omega = 0$ .

## Remark

*A birealisation is a local symplectic groupoid transformed by Weinstein tubular neighborhood theorem.*

## Examples of birealisations

for the canonical Poisson structure  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  of  $T^*\mathbb{R} \simeq \mathbb{R}^2$

$$\begin{cases} \alpha: T^*T^*\mathbb{R} \rightarrow T^*\mathbb{R}: (q, p, \xi_q, \xi_p) \mapsto (q - \frac{1}{2}\xi_p, p + \frac{1}{2}\xi_q) \\ \beta: T^*T^*\mathbb{R} \rightarrow T^*\mathbb{R}: (q, p, \xi_q, \xi_p) \mapsto (q + \frac{1}{2}\xi_p, p - \frac{1}{2}\xi_q) \end{cases} .$$

---

for the canonical Poisson structure on  $so(n)^* \simeq so(n)$

$$\begin{cases} \alpha: so(n) \times so(n) \rightarrow so(n): (A, x) \mapsto (1 + \frac{A}{2}).x.(1 - \frac{A}{2}) \\ \beta: so(n) \times so(n) \rightarrow so(n): (A, x) \mapsto (1 - \frac{A}{2}).x.(1 + \frac{A}{2}) \end{cases} .$$

---

for the cluster Poisson brackets

$$\begin{cases} \alpha: T^*\mathbb{R}^n \rightarrow \mathbb{R}^n: (x, p) \mapsto \left( e^{-\frac{1}{2} \sum_i a_{ij} x_i p_i} . x_j \right)_{j=1, \dots, n} \\ \beta: T^*\mathbb{R}^n \rightarrow \mathbb{R}^n: (x, p) \mapsto \left( e^{\frac{1}{2} \sum_i a_{ij} x_i p_i} . x_j \right)_{j=1, \dots, n} \end{cases} .$$

### Remark

*The computation of a birealisation for a generic Poisson structure is a hard task ! We will talk about it later.*

# Birealisations ease symplectic groupoid computations

- ▶ Cotangent lifts of Poisson automorphisms are morphisms of local symplectic groupoids
- ▶ Lagrangian bisections are images of closed 1-forms
- ▶ Smooth families of Lagrangian bisections are smooth families of closed 1-forms

# Hamilton-Jacobi equation

Let  $H \in \mathcal{C}^\infty(M)$ . Which Lagrangian bisection induces the flow of  $H$  as diffeomorphism on the base ?

Theorem (C., 2022)

1. *Hamilton-Jacobi equation*

$$\begin{cases} \partial_t S_t(x) &= H(\alpha(d_x S_t)) \\ S_0 &= 0 \end{cases} \quad (2)$$

*admits a unique solution  $(S_t)_t$  for  $t$  in a neighborhood of 0.  
 $(S_t)_t$  is the Hamilton-Jacobi transform of  $H$ .*

2. *Such a solution allows to compute the flow of  $H$  by:*

$$\begin{cases} \phi_t^H(x) = \beta(d_{\bar{x}} S_t) \\ \text{where } \bar{x} \text{ is the unique solution of } \alpha(d_{\bar{x}} S_t) = x \end{cases}$$

# Hamilton-Jacobi equation at finite order

## Theorem (C., 2022)

Let  $S_t^i = \sum_{j=1}^i \frac{t^j}{j!} S_j$  be a solution of Hamilton-Jacobi equation (2) at order  $i$ . Then,

$$\begin{cases} \alpha(d_{\bar{x}_n} S_{\Delta t}^i) &= x_n \\ x_{n+1} &= \beta(d_{\bar{x}_n} S_{\Delta t}^i) \end{cases} \quad (3)$$

provides a Hamiltonian Poisson integrator at order  $i$  and time-step  $\Delta t$  for the Hamiltonian flow of  $H$ .

## Remark

The resulting numerical methods are implicit: the non-linear equation (3) is to be solved at each time-step.

## Remark

The truncated Lagrangian bisection induces a time-dependent Hamiltonian flow. A **Hamiltonian Poisson integrator** is an integrator that follows the flow of a time-dependent Hamiltonian.

## Exercises for tomorrow

We use the Frobenius metric  $\langle A, B \rangle = \text{Tr}(AB)$  to identify  $so(d)^* \simeq so(d)$ .  $so(d)$  becomes then equipped with the Lie-Poisson structure coming from  $so(d)^*$ .

1. Use the content of the previous chapter to prove that the isospectral method

$$\begin{aligned} W_n &= (I + \frac{\Delta t}{2} \overline{W}_n) \overline{W}_n (I - \frac{\Delta t}{2} \overline{W}_n) \\ W_{n+1} &= (I - \frac{\Delta t}{2} \overline{W}_n) \overline{W}_n (I + \frac{\Delta t}{2} \overline{W}_n) \end{aligned}$$

preserves the Poisson structure of  $so(d)$ .

2. Find the Hamiltonian for which the map  $W_n \mapsto W_{n+1}$  is a Hamiltonian Poisson integrator at order 1.

## Bibliography for the second lecture

- ▶ Chapter 7 of *Poisson Structures and their Normal Forms*, Jean-Paul Dufour and Nguyen Tien Zung, Birkhäuser, 2005  
→ symplectic groupoids
- ▶ *Geometric Models for Noncommutative Algebras*, Ana Cannas da Silva, Alan Weinstein, American Mathematical Society, 1999  
→ groupoids
- ▶ *Symplectic groupoids for Poisson integrators*, O. C., Journal of Geometry and Physics, 2023

## Hamiltonian dynamics on Poisson manifolds

- Generalities on Poisson structures

- Hamiltonian systems

- A first approach to the numerical analysis on Poisson manifolds

## Symplectic groupoids

- Basic notions on groupoids

- Basics on symplectic groupoids

- Lagrangian bisections

- Birealisations

- Hamilton-Jacobi equation

## Hamiltonian Poisson integrators

- Butcher series

- Explicit construction through birealisations

- Numerical tests around a singularity: a Lotka-Volterra system

- Numerical tests of symmetry preservation: the rigid body

## Perspectives

- Poisson integrators in solid mechanics (with L. Le Marrec and V. Carlier)

- Deformation theory of symplectic groupoids and plasma physics

# Rooted trees

## Definition

Let  $T$  be the set of non-planar trees defined as follows. A non-planar Butcher tree is a rooted tree defined recursively by

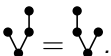
$$\bullet \in T, \quad (\tau_n, \dots, \tau_1)_\bullet \in T, \quad \tau_1, \dots, \tau_n \in T,$$

where the root is graphically represented at the bottom.

$(\tau_n, \dots, \tau_1)_\bullet$  denotes the tree with the root  $\bullet$  and the  $n$  trees  $\tau_n, \dots, \tau_1$  plugged to the root.

## Remark

*By non-planar, we mean that the order of the branches does not*

*matter: for instance,*   $\text{V} = \text{V}$ .

# Grafting

The grafting of trees  $\curvearrowright$  is defined as a product on  $T$  returning the sum of all possibilities (counted with multiplicity) of grafting the root of one tree on the nodes of another tree.

## Example

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \curvearrowright \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ / \backslash \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \quad \bullet \curvearrowright \begin{array}{c} \bullet \\ / \backslash \\ \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \\ / \backslash \\ \bullet \quad \bullet \end{array} + 2 \begin{array}{c} \bullet \\ | \\ \bullet \\ / \backslash \\ \bullet \quad \bullet \end{array}, \quad \begin{array}{c} \bullet \\ / \backslash \\ \bullet \quad \bullet \end{array} \curvearrowright \bullet = \begin{array}{c} \bullet \\ / \backslash \\ \bullet \quad \bullet \end{array}.$$

By extending  $\curvearrowright$  linearly on  $\mathcal{T} = \text{Span}_{\mathbb{R}}(T)$ , this defines the pre-Lie algebra  $(\mathcal{T}, \curvearrowright)$  of Butcher trees.

## Remark

A natural grading on  $\mathcal{T}$  is given by the number of nodes:  $|\begin{array}{c} \bullet \\ / \backslash \\ \bullet \quad \bullet \end{array}| = 3$ .

$$\forall \tau_1, \tau_2 \in T, \quad |\tau_1 \curvearrowright \tau_2| = |\tau_1| + |\tau_2|.$$

## The elementary differential map

$\mathfrak{p} : \mathbb{G} \rightarrow M$  the cotangent projection  
 $0 : M \rightarrow \mathbb{G}$  the zero section of  $\mathbb{G}$  . Let  $H \in \mathcal{C}^\infty(M)$ .

### Definition (Elementary differential map)

The elementary differential map  $\mathbb{F}^H : \mathcal{T} \rightarrow \mathcal{C}^\infty(M)$  associated to  $H$  is defined by

$$\mathbb{F}^H(\bullet) = H \quad \text{and}$$

$$\mathbb{F}^H\left((\tau_n, \dots, \tau_1)\bullet\right) = 0^*\{\{\dots\{\alpha^*H, \mathfrak{p}^*\mathbb{F}^H(\tau_1)\}_\omega, \dots\}_\omega, \mathfrak{p}^*\mathbb{F}^H(\tau_n)\}_\omega.$$

### Example

$$\mathbb{F}^H\left(\begin{smallmatrix} \bullet \\ \bullet \end{smallmatrix}\right) = 0^*\{\alpha^*H, \mathfrak{p}^*H\}_\omega.$$

### Idea

*Compute the terms of the Taylor series of the Hamilton-Jacobi equation by grafting trees and apply  $\mathbb{F}^H$ .*

# Butcher series

## Definition (Butcher series)

The B-series associated to  $H \in \mathcal{C}^\infty(M)$  is the following formal series indexed by a coefficient map  $a \in \mathcal{T}^*$ :

$$B^H: \begin{array}{ccc} \mathcal{T}^* & \rightarrow & \mathcal{C}^\infty(M)[[t]] \\ a & \mapsto & \sum_{\tau \in \mathcal{T}} \frac{a(\tau)}{\sigma(\tau)} \mathbb{F}^{tH}(\tau) \end{array}$$

where  $\sigma(\tau)$  is the number of graph automorphisms of  $\tau$ , also called the symmetry coefficient.<sup>2</sup>

## Remark

For any  $\tau \in \mathcal{T}$ ,  $\mathbb{F}^{tH}(\tau) \in \mathcal{C}^\infty(M)[t]$ . Actually,  $\deg(\mathbb{F}^{tH}(\tau)) = |\tau|$ .

---

<sup>2</sup>See Geometric Numerical Integration, Hairer et al., 2006, sec. III.1, for an explicit formula of the symmetry coefficient.

# Formal solution of HJ equation

## Lemma

For any  $(S_t)_{t \in I} \in \mathcal{C}^\infty(M \times I)$  and any  $(f_t)_{t \in I} \in \mathcal{C}^\infty(T^*M \times I)$ ,

$$\frac{\partial}{\partial t} ((dS_t)^* f_t) = (dS_t)^* \left( \{f_t, \mathfrak{p}^* \frac{\partial S_t}{\partial t}\}_\omega + \frac{\partial f_t}{\partial t} \right). \quad (4)$$

## Theorem (Busnot Laurent, O.C., 2025)

Let  $(S_t)_{t \in I} \in \mathcal{C}^\infty(M \times I)$  be the solution of the Hamilton-Jacobi equation (2), then its Taylor expansion is given by

$$B^H(e) \in \mathcal{C}^\infty(M)[[t]], \quad (5)$$

where  $e \in \mathcal{T}^*$  is given by

$$e(\bullet) = 1, \quad e(\tau) = \frac{1}{|\tau|} e(\tau_1) \dots e(\tau_n), \quad \tau = (\tau_n, \dots, \tau_1)_\bullet.$$

## Explicit construction through birealisations

Let  $\pi$  a Poisson structure on  $\mathbb{R}^d$ . Assume that a birealisation  $(\alpha, \beta)$  is constructed for  $\pi$ . Let  $H \in \mathcal{C}^\infty(\mathbb{R}^n)$ ,  $k \in \mathbb{N}$  and  $\Delta t > 0$  a time-step.

1. Solve the Hamilton-Jacobi equation up to order  $k$  using Butcher series of Equation (5). Obtain  $S_{\Delta t}^k \in \mathcal{C}^\infty(M)$ .
2. Let  $x_n \in \mathbb{R}^d$ . Solve the nonlinear equation

$$\alpha(\bar{x}_n, \nabla_{\bar{x}_n} S_{\Delta t}^k) = x_n$$

by a fixed point method, e.g., Newton descent.

3. One step of the obtained Hamiltonian Poisson integrator of order  $k$  and time-step  $\Delta t$  is then given by

$$x_{n+1} = \beta(\bar{x}_n, \nabla_{\bar{x}_n} S_{\Delta t}^k).$$

## Expected numerical behaviour

- ▶ **stay on a leaf** because the numerical methods are induced by bisections
- ▶ **stability near a singularity** because these bisections are Lagrangian
- ▶ **oscillate around first integrals** as symplectic numerical methods do

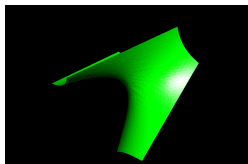
# Poisson structure and foliation of a cluster Poisson structure on $\mathbb{R}^3$

- ▶ the Poisson bracket

$$\{f, g\}(x) = (\nabla_x f)^T \cdot \begin{pmatrix} 0 & x_1 x_2 & x_1 x_3 \\ -x_1 x_2 & 0 & x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & 0 \end{pmatrix} \cdot \nabla_x g$$

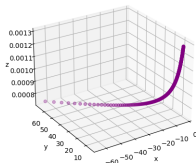
- ▶ a bi-realisation  $\left\{ \begin{array}{l} \alpha: (x, p) \mapsto \begin{pmatrix} e^{\frac{1}{2}(x_2 p_2 + x_3 p_3)} x_1 \\ e^{\frac{1}{2}(-x_1 p_1 + x_3 p_3)} x_2 \\ e^{-\frac{1}{2}(x_1 p_1 + x_2 p_2)} x_3 \end{pmatrix} \\ \beta: (x, p) \mapsto \alpha(x, -p) \end{array} \right. ,$

- ▶ a symplectic leaf



# A Hamiltonian vector field with an exploding trajectory

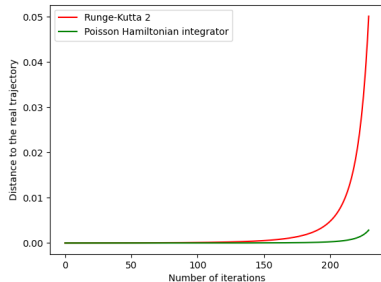
$$H(x) = \sum_{i=1}^3 x_i \Rightarrow \begin{cases} \dot{x}_1 = x_1(x_2 + x_3) \\ \dot{x}_2 = x_2(-x_1 + x_3) \\ \dot{x}_3 = -x_3(x_1 + x_2) \end{cases}$$



Exploding for  $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 10^{-3} \end{pmatrix}$   
(Vanhaecke et al. 2016)

$$\begin{cases} \lim_{t \rightarrow \infty} x_1(t) = -\infty \\ \lim_{t \rightarrow \infty} x_2(t) = \infty \\ \lim_{t \rightarrow \infty} x_3(t) = 0 \end{cases} .$$

# Numerical simulation: comparison of numerical errors



Error with respect to the analytical solution

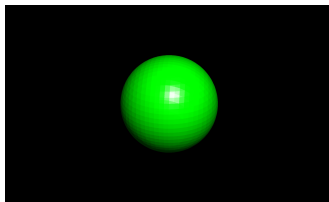
with initial point  $\begin{pmatrix} -3 \\ 5 \\ 10^{-3} \end{pmatrix}$  and  $\Delta t = 10^{-3}$ .

# Poisson structure and foliation of $\mathfrak{so}(3)^*$

- ▶ the Poisson bracket

$$\{f, g\}(x) = (\nabla_x f)^T \cdot \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \cdot \nabla_x g$$

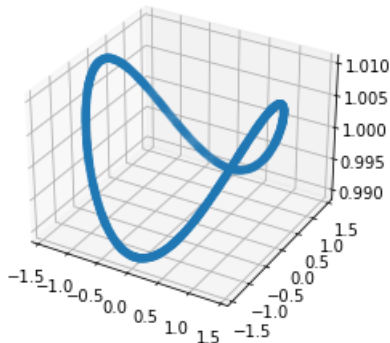
- ▶ a bi-realisation  $\mathbb{R}^3 \simeq \mathfrak{so}(3) \begin{cases} \alpha: (x, p) \mapsto (I + \frac{p}{2}) \cdot x \cdot (I - \frac{p}{2}) \\ \beta: (x, p) \mapsto \alpha(x, -p) \end{cases}$
- ▶ a symplectic leaf



# The Hamiltonian vector field of angular velocity of a rigid body in $\mathbb{R}^3$

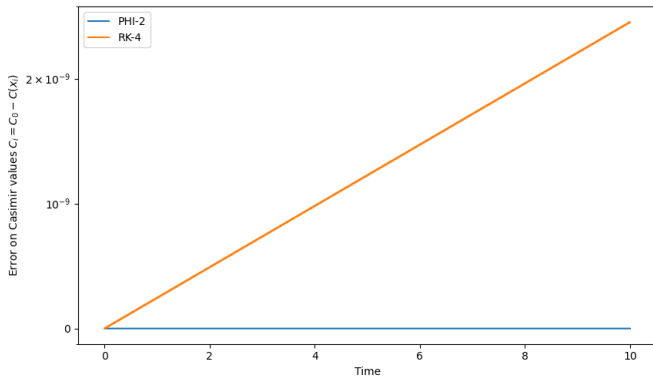
$$H(x) = \frac{1}{2} \left( i_1(x_2^2 + x_3^2) \right. \\ \left. + i_2(x_1^2 + x_3^2) \right. \\ \left. + i_3(x_1^2 + x_2^2) \right)$$

$$\Rightarrow \dot{x} = -x \wedge Ix$$



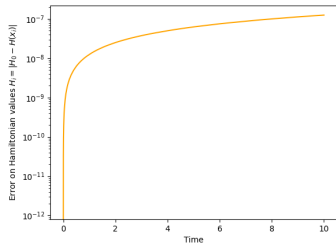
$$x(0) = (1, 1, 0)$$

# Preservation of Casimir levels

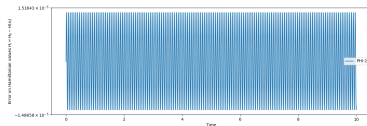


Error on Casimir values for PHI-2 and RK-4

# Controlled oscillations around Hamiltonian levels



Errors on Hamiltonian values for  
RK-4



Errors on Hamiltonian values for  
HPI-2

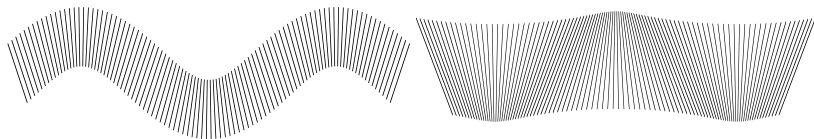
## Exercises

1. Use the lemma of Equation (4) to compute the terms of order 1 and 2 of the solution of the Hamilton-Jacobi equation (2).
2. Verify that the Butcher series of the equation (5) solves the Hamilton-Jacobi equation at order 2.

## Bibliography for the third lecture

- ▶ *Numerical Methods in Poisson Geometry and Applications to Mechanics*, O. C., C. Laurent-Gengoux, V. Salnikov, Mathematics and Mechanics of Solids, 2024
- ▶ *Butcher series for Hamiltonian Poisson integrators through symplectic groupoids*, A. Busnot Laurent, O. C., arXiv, 2025

# Poisson integrators for the Timoshenko model (w. L. Le Marrec, V. Carlier)



Two beam configurations

fig: Le Marrec

## Hamiltonian structure of Timoshenko model<sup>3</sup>

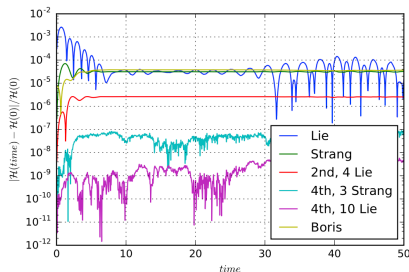
- ▶ compute a birealisation
- ▶ construct Hamiltonian Poisson integrators
- ▶ expect these numerical methods to track periodic orbits efficiently, detect instabilities...

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<sup>3</sup>*Timoshenko beam under finite and dynamic transformations: Lagrangian coordinates and Hamiltonian structures*, O.C. & Le Marrec, 2025

# Construction of birealisations by perturbative techniques

- ▶ Construct birealisations by studying deformation theory of symplectic groupoids  
→ several ongoing discussions with D. Calaque, A. Busnot Laurent, V. Dotsenko
- ▶ Poisson integrators in plasma physics with É. Sonnendrucker



Landau damping: energy error<sup>4</sup>

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<sup>4</sup>GEMPIC, Sonnendrucker et al., 2017