Propeness of $\mathcal{X}^k_{n,N,v}$

① Throughout $\mathcal{X} := \mathcal{X}^{Fano}_{n,N,v}$
$\mathcal{X}^k := \mathcal{X}^k_{n,N,v}$

② Goal: $X^K_{n,N,v}$ is proper.

③ Recall the last Thm from Shing's talk
So our first goal is to construct such a numerical invariant μ on $\mathcal{X}$ satisfying all the assumptions in the Th

④ Let $f \in \underline{Map}(\mathbb{D}, \mathcal{X})$

Then $ev_1(f)$ is a pt in $\mathcal{X}$
i.e. $[(X, \Delta)]$

$ev_0(f)$ is the degeneration of (X, Δ)
i.e. central fibre $(\mathcal{X}_0, \Delta_0)$ of a t.c

(In particular, if f is an HN-filtra
then $ev_0(f) = \lim_{t\to 0} \lambda(t)\cdot X$

where λ optimal destabilizing 1-PS

⑤ S. f corresponds to a s.t.c. $(\mathcal{X}, \Delta)$ of (X, Δ)

- ① RMk of t.c.
- ② RMk of s.t.c. | ✱ X normal
(X_0, Δ_0) klt log Fa
$\mathcal{L} \sim_{\mathbb{Q}} -K_{\mathcal{X}/C}$

* (5) Let $(\mathcal{X}, \Delta)$ be a s.t.c. of (X, Δ)

The L^2-norm $\|\mathcal{X}, \Delta\|_2 := \|(\mathcal{X}_0, \Delta_0, \lambda)\|_2$
The m-norm $\|\mathcal{X}, \Delta\|_m := \|(\mathcal{X}_0, \Delta_0, \lambda)\|_m$
where λ is the $\mathbb{G}_m$-action on $\mathcal{X}_0$ induced by t

$$\lambda : \mathbb{G}_m \longrightarrow Aut(\mathcal{X}_0)$$

(6) Define $\mu(X) := (\mu_1(X), \mu_2(X))$

$$= \left(\frac{DF(X)}{\|X\|_m}, \frac{DF(X)}{\|X\|_2}\right)$$

$\mu(X) \in \Gamma' = \mathbb{R}^2$ with lexicographic order
(i.e. $(a,b) < (c,d)$ if $a < c$
or $a = c, b < d$.)

In particular, $\mu(X_{triv}) = (0, 0)$.

(7) Lemma: Given a family of log Fano pairs
$f : (\mathcal{X}, \Delta) \to S$
admitting fibrewise $\mathbb{G}_m$-action,

If S is connected,
then $\begin{cases} DF(\mathcal{X}_t, \Delta_t) \\ \|\mathcal{X}_t, \Delta_t\|_2 \\ \|\mathcal{X}_t, \Delta_t\|_m \end{cases}$ are independent of $t \in S$

(8) How does this $\Rightarrow \mu$ is a numerical invariant?

- Consider a family of such map parameterised by a connected scheme S

$$F : S \longrightarrow \underline{Mor}(\mathbb{CP}, \mathcal{X})$$

- By def
$\underline{Map}(\mathcal{H}, \mathcal{X})(S) = \underline{Map}(\mathcal{H} \times S, \mathcal{X})$

So each $F : S \to \underline{Map}(\mathcal{H}, \mathcal{X})$
corresponds to
$\mathcal{H} \times S \to \mathcal{X}$

$f : \mathcal{H} \to \mathcal{X}$ corresponds to a s.t.c.
So $\mathcal{H} \times S \to \mathcal{X}$ is a family of s.t.c. parameterised by S.

- Since S is connected,
it doesn't matter which fibre, $F(s)$, $s \in S$, we take.
$\Rightarrow \mu$ is locally constant on $\underline{Map}(\mathcal{H}, \mathcal{X})$

(2) Now we want to check it satisfies the destabilization property.

Analogue of Kempf's Thm
Thm: $\forall (X, \Delta)$ log Fano pair.

- $\exists$ a s.t.c. (X, Δ) s.t.
$\mu(X) = \inf_{(X', \Delta') \text{ s.t.c. of } (X, \Delta)} \mu(X')$

- Moreover, if (X, Δ) is K-unst
then $\mu(X) = \mu(X') \Leftrightarrow$ They induces the same divisor over,
i.e. μ is a numerical invariant on $\underline{Map}(\mathcal{H}, \mathcal{X})$.

⑨ • Define $M^{\mu}(X,\Delta) := \inf_{X \text{ s.t.c. of } (X,\Delta)} \mu(X)$

• Let X be a k-unstable s.t.c. of (X,Δ)

If $\mu(X) = M^{\mu}(X,\Delta)$ the X is called the μ-optimal destabilization.

• $\Gamma := \{ M^{\mu}(X,\Delta) \mid [X,\Delta] \in \mathcal{X}^{\text{Fano}} \}$

↑ The image of M^{μ}.

• If (X,Δ) is k-unstable, then $M^{\mu}(X,\Delta) = (S(X,\Delta)-1, M^{\mu_2}(X,\Delta))$

where $M^{\mu_2}(X,\Delta) = \inf\{\mu_2(X) \mid X \text{ s.t. } \mu_1(X) = S(X,\Delta)\}$

(Recall that $\mu(X) = (\mu_1(X), \mu_2(X))$

So "M^{μ} is the infimum of values of a invariant μ"

⑩ Thm: The fcn M^{μ} on $\mathcal{X}$ is constructible

① The values that M^{μ} takes is finite.
So Γ is a finite set.

② For each $m \in \Gamma$, define the open substack $\mathcal{X}_{\geq m} := \{x \in \mathcal{X} \mid M^{\mu}(x) \geq m\}$
It is constructible.

(3) ~~For~~ For each $\chi \geq m$
define the (H)-stratum
$$S_m := \{x \in \mathcal{X} \mid M^{\mathrm{u}}(x) = m\}$$

Then $\mathcal{X} = \coprod_{m \in \Gamma} S_m$.

(11) Lemma: R. K. k.
$(\mathcal{X}, \Delta) \to \mathrm{Spec}(R)$ family of log Fano pair over $\mathrm{Spec}(R)$

Then $M^{\mathrm{u}}(\mathcal{X}_K, \Delta_K) \geq M^{\mathrm{u}}(\mathcal{X}_k, \Delta_k)$

" Smaller M^{u} means more unstable.
So the s fibre is at least as unstable as the generic fibre "

(12) Check assumption 1

- For each $m \in \Gamma$,
define the subfunctor $\mathcal{X}^{\geq m}$ of $\mathcal{X}$
s.t. $\mathcal{X}^{\geq m}(T) = \{[(\mathcal{X}, \Delta)] \to T \mid M(\mathcal{X}_t, \Delta_t) \geq m \ \forall t \in T\}$
(family para. by T)
- Clearly, $\mathcal{X}^{\geq 0} = \mathcal{X}^{k}$

Prop: For each $m \in \Gamma$, $\mathcal{X}^{\geq m}$ is represented by an open substack
$$[\, M^{\geq m}/G \,]$$
of $\mathcal{X}$ of finite type.

where $M^{\geq m} := \{[(\mathcal{X}, \Delta)] \in M, M \text{ gauss.-proj} \text{ s.t. } M(\mathcal{X}, \Delta) \geq m\}$

(13) Check assumption 2.

Prop: Let (X, Δ) be a log Fano pair. Let $\mathcal{X}$ be μ-optimal destabilizing s.t.c. (i.e. $\mu(\mathcal{X}) = M^{\mu}(X, \Delta)$)

Since $\mathcal{X}$ s.t.c., its central fibre (X_0, Δ_0) is also log Fano pair. Denote it by (Y, Δ_Y)

Then $M^{\mu}(X, \Delta) = M^{\mu}(Y, \Delta_Y)$.

So passing to the central fibre does not change the "optimal instability level"

(14) Assumption 3 guaranteed for non-trivial s.t.c. of the generic fibre induced by a valuation v_k s.t.

$$\frac{A_{X_k, \Delta_k}(v_k)}{S(v_k)} \leq \min\{\delta(X_k, \Delta_k), 1\}.$$

(15) Thm: The numerical invariant M induces a well-ordered $\mathbb{Q}$-stratification on $\mathcal{X}$

(16) Cor: $X^k_{n,\Lambda,v}$ is proper.

Proof: · Given $\mathrm{Spec}\, k \to X^k_{n,\Lambda,v} =: \mathcal{X}^k$
Want $\mathrm{Spec}\, R \to \mathcal{X}^k$

· $\exists$ a finite extension of DVR's s.t. the generic fibre of the family is obtained by base change

$$(X'_{K'}, \Delta'_{K'}) \cong (X_K, \Delta_K) \times_K K'$$

and

an R'-pt of the moduli stack $\mathcal{X}$

$$[(X', \Delta') \to \mathrm{Spec}(R')]$$

- $\mathcal{X}$ admits a well-ordered $\mathbb{N}$-stratification
  ("eventually stabilizing of the isomorphic class")

- Semi-stable reduction Thm.
  $\exists$ a extension R'' of R' s.t.
  the generic fibre is obtained by the base change

$$(X''_{K''}, \Delta''_{K''}) \cong (X'_{K'}, \Delta'_{K'}) \times_{K'} K''$$

$$\cong \quad \cdots$$

- s.t. the ~~spe~~ generic fibre is K-ss.
  so as the special fibre. $\square$