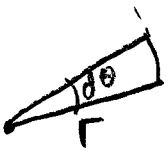
Purpose: To practice finding the area between polar curves and finding correct bounds for a given region.

Warm-up Question: Write down the formula for the area enclosed by a polar curve $r = f(\theta)$ for $a \leq \theta \leq b$ and explain how it comes from the formula for the area of a sector of a circle.

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta$$



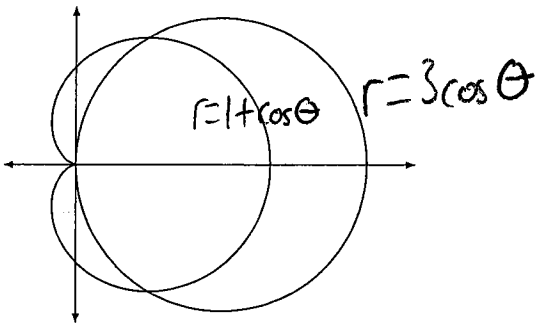
Area is proportional to total circle area.

$$A_s = \pi r^2 \cdot \frac{d\theta}{2\pi}$$

$$= \frac{1}{2} r^2 \, d\theta$$

The rest of the questions in this handout will have to do with the polar curves $r = 3\cos\theta$ and $r = 1 + \cos\theta$, graphed below.

1. Which curve is $r = 3\cos\theta$ and which is $r = 1 + \cos\theta$? Label them on the graph.



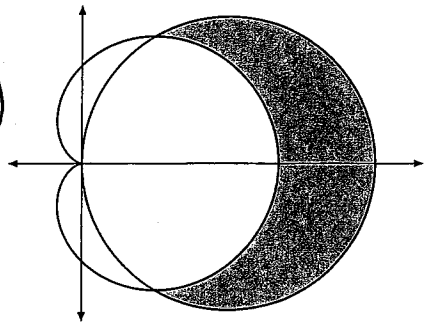
2. Find the intersection points of the two curves for θ in $[0, 2\pi]$. Can you explain what's going on at the origin?

$1 + \cos\theta = 3\cos\theta \Rightarrow 1 = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

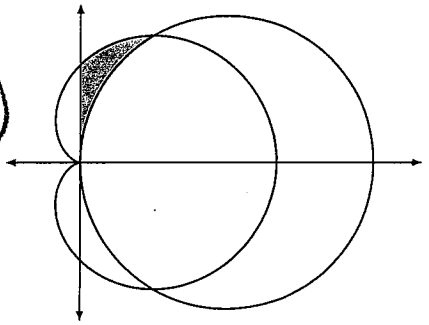
They both touch the origin, but don't have the same angle when they do, so we don't find it algebraically. Graphically, it's not at the same 'time' in the sweep through θ of the curve.

3. Match the shaded region with an integral that represents the area.

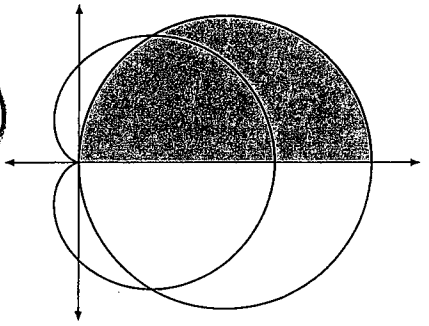
I. (e)



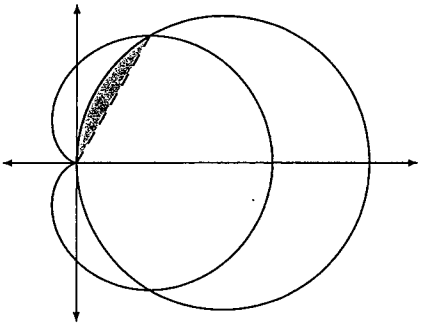
II. (c)



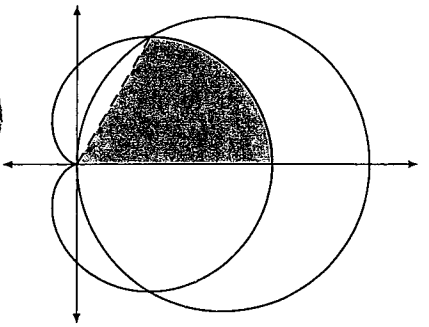
III. (h)



IV. (j)



V. (k)



(a) $\int_0^{\pi/2} \frac{1}{2}\Big((1+\cos\theta)^2 - (3\cos\theta)^2\Big)\,d\theta$

(b) $\int_{-\pi/3}^{\pi/3} \frac{1}{2}\Big((1+\cos\theta)^2 - (3\cos\theta)^2\Big)\,d\theta$

(c) $\int_{\pi/3}^{\pi/2} \frac{1}{2}\Big((1+\cos\theta)^2 - (3\cos\theta)^2\Big)\,d\theta$

(d) $\int_{\pi/3}^{5\pi/3} \frac{1}{2}\Big((3\cos\theta)^2 - (1+\cos\theta)^2\Big)\,d\theta$ ← Doesn't work for I as it sweeps out the wrong side.

(e) $\int_{-\pi/3}^{\pi/3} \frac{1}{2}\Big((3\cos\theta)^2 - (1+\cos\theta)^2\Big)\,d\theta$

(f) $\int_{\pi/3}^{\pi/2} \frac{1}{2}\Big((3\cos\theta)^2 - (1+\cos\theta)^2\Big)\,d\theta$

(g) $\int_0^{\pi} \frac{1}{2}(3\cos\theta)^2\,d\theta$

(h) $\int_0^{\pi/2} \frac{1}{2}(3\cos\theta)^2\,d\theta$

(i) $\int_0^{\pi/3} \frac{1}{2}(3\cos\theta)^2\,d\theta$

(j) $\int_{\pi/3}^{\pi/2} \frac{1}{2}(3\cos\theta)^2\,d\theta$

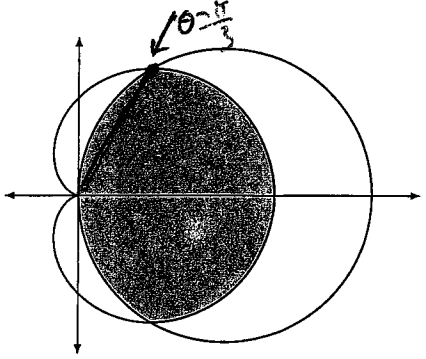
(k) $\int_0^{\pi/3} \frac{1}{2}(1+\cos\theta)^2\,d\theta$

(l) $\int_{\pi/3}^{\pi/2} \frac{1}{2}(1+\cos\theta)^2\,d\theta$

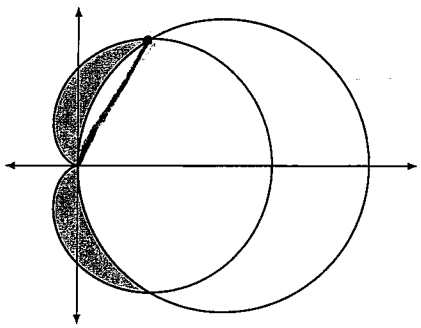
(m) $\int_0^{\pi} \frac{1}{2}(1+\cos\theta)^2\,d\theta$

4. The area of the shaded regions below cannot be represented with a single integral. (Can you see why?) Write down a **sum** of integrals that represents the shaded area for each figure.

(a)



(b)



(a) $A = \left[\int_0^{\pi/3} \frac{1}{2}[1+\cos\theta]^2\,d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}[3\cos\theta]^2\,d\theta\right] \cdot 2$

The top and botton halves are equal so we can do them by symmetry.
(Malle sure to write something when using this.)

(b) $A = 2\left[\int_{\pi/3}^{\pi/2} \left[\frac{1}{2}(1+\cos\theta)^2 - \frac{1}{2}(3\cos\theta)^2\right]d\theta + \int_{\pi/2}^{\pi} \frac{1}{2}(1+\cos\theta)^2\,d\theta\right]$

From $\frac{\pi}{3}$ to $\frac{\pi}{2}$, we have the circle as the inner curve, but from $\frac{\pi}{2}$ to π the circle is not involved.
The other half is again by symmetry.

Note: from $\frac{\pi}{2}$ to π, the circle $r = 3\cos\theta$ is present; since $r<0$ it appears in quadrant IV of a rectangular system there.