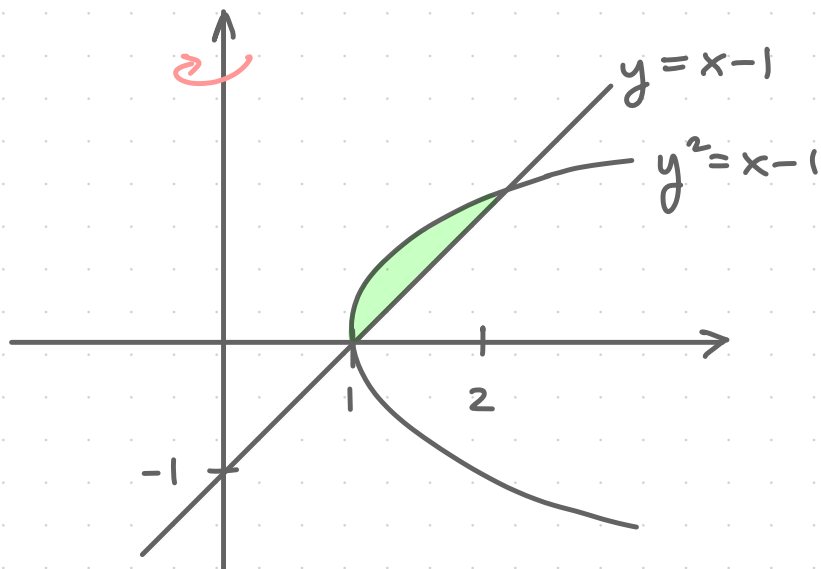


1. Find the volume of the solid that results from rotating the area between the curves $y^2 = x - 1$ and $y = x - 1$ around the y -axis.



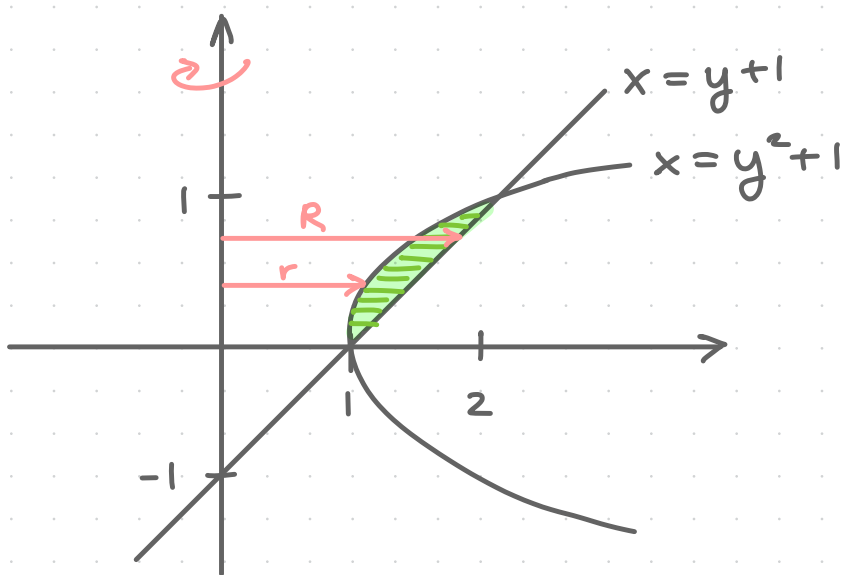
Can use Method of
Disks/Washers
OR Method of Shells

Using Disks/Washers:

Slice the region horizontally
 $\Rightarrow$ integrate in y

So we want our functions in
terms of y :

$$\begin{aligned}y^2 = x - 1 &\Rightarrow x = y^2 + 1 \\y = x - 1 &\Rightarrow x = y + 1\end{aligned}$$



inner radius: $r = y^2 + 1$
outer radius: $R = y + 1$

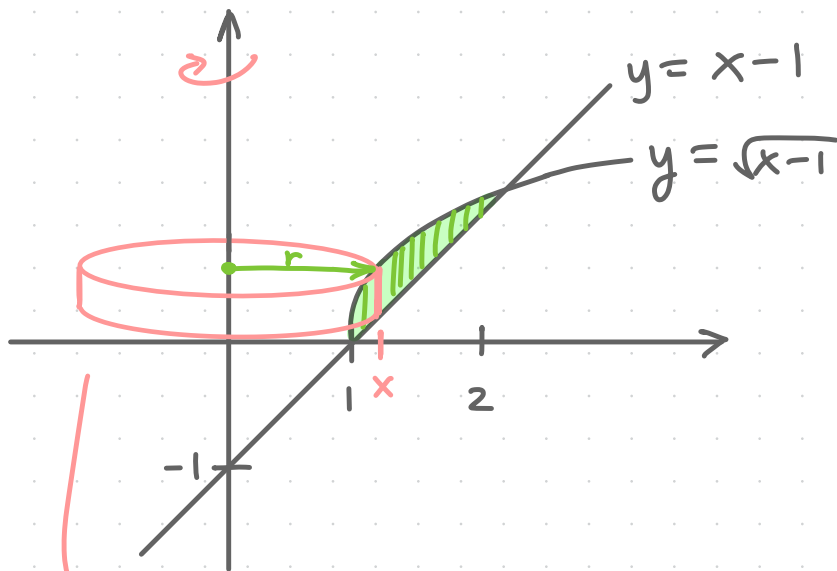
$$\begin{aligned}A(y) &= \pi(R^2 - r^2) \\&= \pi((y + 1)^2 - (y^2 + 1)^2)\end{aligned}$$

Bounds: y is between 0 & 1

$$\begin{aligned}
 V &= \int_0^1 A(y) dy = \int_0^1 \pi ((y+1)^2 - (y^2+1)^2) dy \\
 &= \pi \int_0^1 (y^2 + 2y + 1) - (y^4 + 2y^2 + 1) dy \\
 &= \pi \int_0^1 -y^4 - y^2 + 2y dy \\
 &= \pi \left(-\frac{1}{5}y^5 - \frac{1}{3}y^3 + y^2 \right) \Big|_0^1 \\
 &= \pi \left(-\frac{1}{5} - \frac{1}{3} + 1 \right) = \frac{7\pi}{15}
 \end{aligned}$$

OR

Using Shells: Slicing vertically $\Rightarrow$ integrate in x



roll it out $\rightarrow$

$$\underbrace{\hspace{10em}}_{2\pi r = 2\pi x} \sqrt{x-1} - (x-1)$$

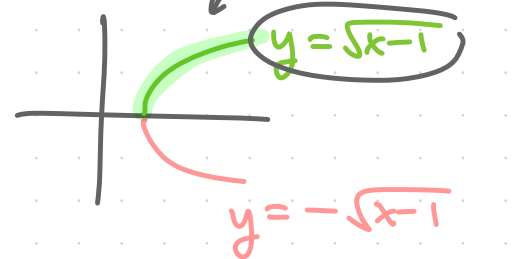
$$dV = 2\pi x (\sqrt{x-1} - (x-1)) dx$$

Bounds: x is between 1 and 2

Want our functions in terms of x :

$$y^2 = 1 - x \Rightarrow y = \pm \sqrt{x-1}$$

we only need the top piece



$$V = \int_1^2 2\pi x (\sqrt{x-1} - x + 1) dx$$

$$= 2\pi \int_1^2 x \sqrt{x-1} - x^2 + x dx$$

$$= 2\pi \left(\int_1^2 x \sqrt{x-1} dx + \int_1^2 -x^2 + x dx \right)$$

$$\boxed{u = x-1 \Rightarrow x = u+1}$$

$$du = dx$$

$$x=1 \Rightarrow u=0$$

$$x=2 \Rightarrow u=1$$

$$= 2\pi \left(\int_0^1 (u+1) \sqrt{u} du + \int_1^2 -x^2 + x dx \right)$$

$$= 2\pi \left(\int_0^1 u^{3/2} + u^{1/2} du + \left(-\frac{1}{3}x^3 + \frac{1}{2}x^2 \right) \Big|_1^2 \right)$$

$$= 2\pi \left(\left(\frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right) \Big|_0^1 + \left(\left(-\frac{8}{3} + \frac{4}{2} \right) - \left(-\frac{1}{3} + \frac{1}{2} \right) \right) \right)$$

$$= 2\pi \left(\frac{2}{5} + \frac{2}{3} - \frac{7}{3} + 2 - \frac{1}{2} \right)$$

$$= 2\pi \left(\frac{7}{30} \right) = \frac{7\pi}{15}$$

2. Find the average value of $f(x) = \sin^3 x$ over the interval $[0, \pi]$.

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{\pi - 0} \int_0^{\pi} \sin^3 x \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} \sin x \cdot (1 - \cos^2 x) \, dx \end{aligned}$$

$$\left. \begin{aligned} u &= \cos x \\ du &= -\sin x \, dx \end{aligned} \right\}$$

$$\begin{aligned} x=0 &\Rightarrow u=1 \\ x=\pi &\Rightarrow u=-1 \end{aligned}$$

$$= \frac{1}{\pi} \int_1^{-1} (1 - u^2)(-1) \, du$$

$$= \frac{1}{\pi} \int_1^{-1} (u^2 - 1) \, du$$

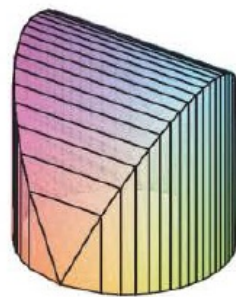
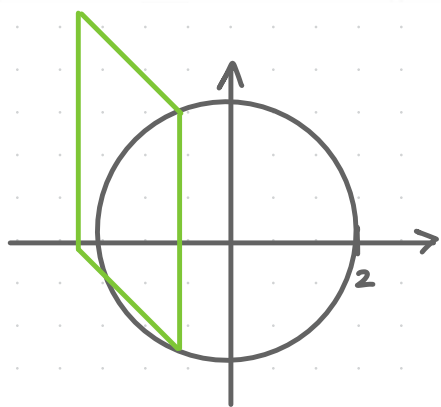
$$= \frac{1}{\pi} \left(\frac{1}{3} u^3 - u \right) \Big|_1^{-1}$$

$$= \frac{1}{\pi} \left(\left(-\frac{1}{3} + 1 \right) - \left(\frac{1}{3} - 1 \right) \right)$$

$$= \frac{1}{\pi} \left(2 - \frac{2}{3} \right)$$

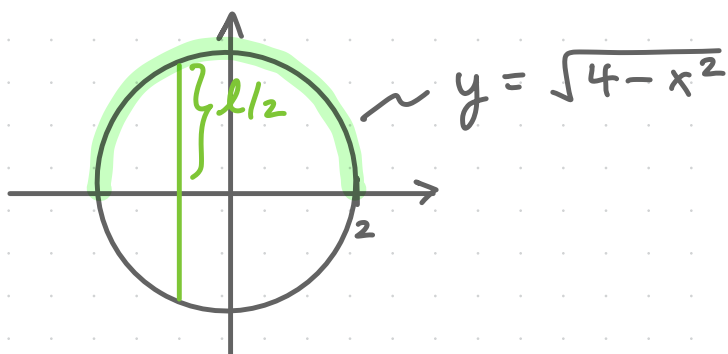
$$= \frac{4}{3\pi}$$

3. Find the volume of the solid whose base is bounded by the circle $x^2 + y^2 = 4$ and whose cross sections perpendicular to the x -axis are squares.



← looks like this

squares $\Rightarrow A = l^2$ ← need l in terms of x



$$\frac{l}{2} = \sqrt{4 - x^2}$$

$$l = 2\sqrt{4 - x^2}$$

$$\text{So } A(x) = (2\sqrt{4 - x^2})^2 = 4(4 - x^2)$$

$$\begin{aligned} V &= \int_{-2}^2 4(4 - x^2) dx \quad \leftarrow \text{even function} = 2 \int_0^2 4(4 - x^2) dx \\ &= 2 \cdot 4 \left(4x - \frac{1}{3}x^3 \right) \Big|_0^2 \\ &= 8 \left(8 - \frac{8}{3} \right) \\ &= \frac{128}{3} \end{aligned}$$

4. It requires a force of 12 N to hold a spring 0.1 m longer than its natural length. How much work is required to stretch it from its natural length to 0.1 m beyond its natural length?

Hooke's Law: $F(x) = kx$ (equilibrium at 0)

$$12 = k(0.1)$$

$$k = 120$$

$$W = \int_0^{0.1} 120x \, dx = 60x^2 \Big|_0^{0.1}$$

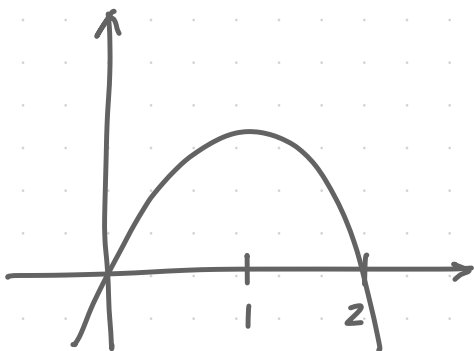
$$= 60(0.01)$$

$$= 0.6 \text{ J}$$

5. Find the center of mass of the lamina of constant density bounded by $y = 2x - x^2$ and the x -axis.

$$y = x(x-2)$$

roots: 0, 2



Center of Mass Formulas:

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx} = \frac{1}{A} \int_a^b x f(x) dx$$

(x-coordinate)

and

$$\bar{y} = \frac{M_x}{m} = \frac{\int_a^b \frac{1}{2} f(x)^2 dx}{\int_a^b f(x) dx} = \frac{1}{A} \int_a^b \frac{1}{2} f(x)^2 dx$$

(y-coordinate)

$$\begin{aligned} \text{First, } A = \text{area under } f &= \int_0^2 2x - x^2 dx \\ &= \left(x^2 - \frac{1}{3} x^3 \right) \Big|_0^2 \\ &= 4 - \frac{8}{3} - 0 = \frac{4}{3} \end{aligned}$$

Notice, $\bar{x} = 1$ by symmetry. Or compute:

$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_0^2 x (2x - x^2) dx = \frac{1}{(4/3)} \int_0^2 2x^2 - x^3 dx \\ &= \frac{3}{4} \left(\frac{2}{3} x^3 - \frac{1}{4} x^4 \right) \Big|_0^2 \\ &= \frac{3}{4} \left(\frac{16}{3} - 4 \right) \\ &= 1 \end{aligned}$$

$$\bar{y} = \frac{1}{(4/3)} \int_0^2 \frac{1}{2} (2x - x^2)^2 dx$$

$$= \frac{3}{4} \cdot \frac{1}{2} \int_0^2 4x^2 - 4x^3 + x^4 dx$$

$$= \frac{3}{8} \left(\frac{4}{3} x^3 - x^4 + \frac{1}{5} x^5 \right) \Big|_0^2$$

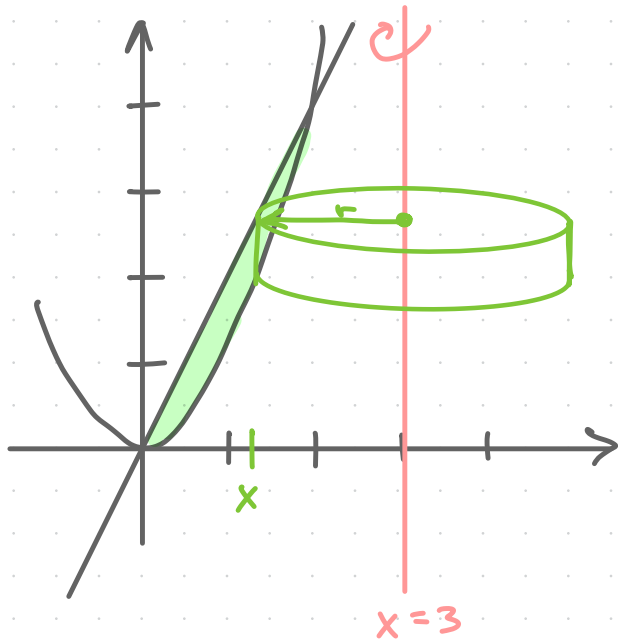
$$= \frac{3}{8} \left(\frac{32}{3} - 16 + \frac{32}{5} \right)$$

$$= \frac{3}{8} \left(\frac{16}{15} \right) = \frac{2}{5}$$

so $\bar{y} = \frac{2}{5}$

The center of mass is $(\bar{x}, \bar{y}) = (1, \frac{2}{5})$

6. Use the method of cylindrical shells to find the volume of the solid of revolution obtained by revolving the region bounded by the graphs of $y = x^2$ and $y = 2x$ about the line $x = 3$.



intersection points:

$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$x = 0, 2$$

$$2\pi r = 2\pi(3 - x)$$

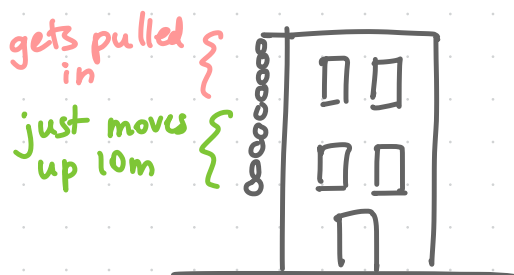
$$dV = 2\pi(3 - x)(2x - x^2)dx$$

$$V = \int_0^2 2\pi(3 - x)(2x - x^2)dx = \dots = \frac{16\pi}{3}$$

7. A 20 meter chain with a mass of 100 kilograms is hanging off the side of a 30 meter tall building. How much work is required to pull 10 meters of the chain to the top of the building?

$$W = F \cdot d = m \cdot g \cdot d$$

$\uparrow$
mass
 $\uparrow$
distance



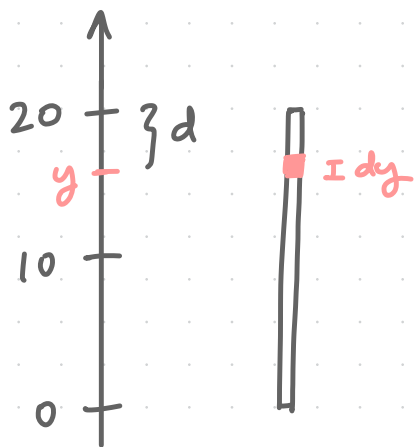
$$\text{total work} = \underbrace{W_b}_{\text{work for bottom half}} + \underbrace{W_t}_{\text{work for top half}}$$

For the bottom half:

$$\left. \begin{array}{l} m = 100/2 = 50 \text{ kg} \\ d = 10 \text{ m} \end{array} \right\} W_b = 50 \cdot g \cdot 10 = 500(9.8) = 4900 \text{ J}$$

For the top half: Need an integral.

* remember to establish your coordinate system!! *



$$dW = \text{work to raise a tiny chunk} = (\text{mass of chunk}) \cdot g \cdot (\text{distance})$$

$$\text{linear mass density: } \frac{100 \text{ kg}}{20 \text{ m}} = 5 \text{ kg/m}$$

$$\text{so } \text{mass} = 5 \cdot dy$$

$\underbrace{\hspace{2cm}}$
length of chunk

$$d = 20 - y$$

(in the coordinate system that I chose, yours might be different)

$$\text{So } dW = (5 dy) \cdot g \cdot (20 - y) = 5(9.8)(20 - y) dy$$

Bounds are 10 to 20 (just the top half)

$$W_t = \int_{10}^{20} 5(9.8)(20 - y) dy$$

$$= 49 \int_{10}^{20} 20 - y dy$$

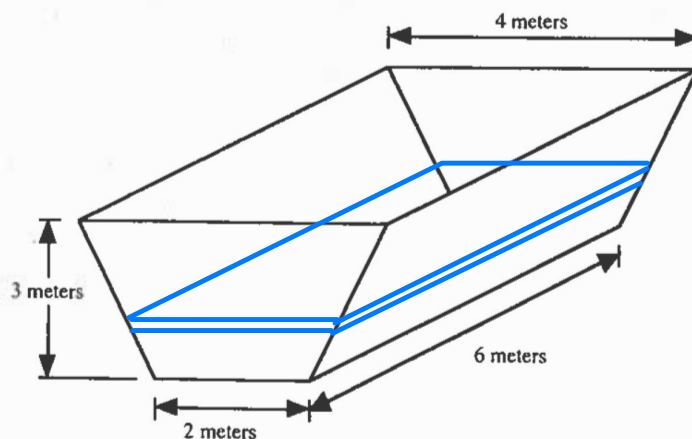
$$= 49 \left(20y - \frac{1}{2}y^2 \right) \Big|_{10}^{20}$$

$$= 49 ((400 - 100) - (200 - 50))$$

$$= 7350 \text{ J}$$

total work: $W = W_t + W_b = 12250 \text{ J}$

8. Suppose that the trough, pictured below, is completely filled with pure water (density = ρ kg/m³). Calculate the exact total amount of work that must be done to pump all of the water out of trough. (Use g m/s²)



← cross sections are rectangles

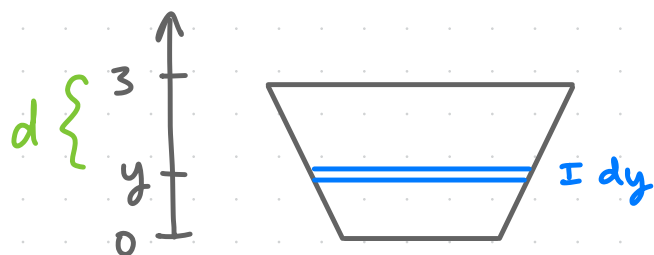
* remember, always use horizontal cross sections for water pumping problems

$$W = F \cdot d = m \cdot g \cdot d = \rho \cdot V \cdot g \cdot d$$

↑
volume of a slice of water

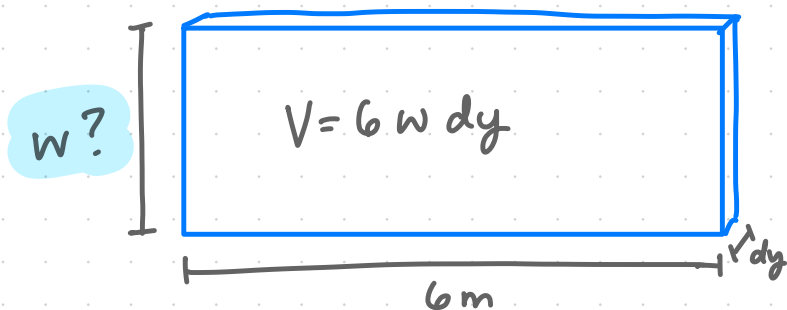
↑
distance to top of tank

Side View:

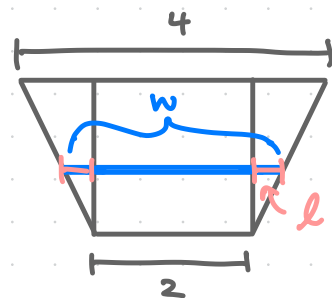


$$d = 3 - y$$

Now for V:

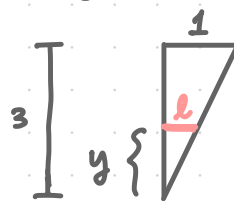


Need w in terms of y



$$w = 2 + 2l$$

and we can find l using similar triangles:



$$\frac{l}{y} = \frac{1}{3}$$

$$l = \frac{1}{3}y$$

$$\text{so } w = 2 + \frac{2}{3}y$$

$$\Rightarrow V = 6 \left(2 + \frac{2}{3}y \right) dy$$

Now we have

$$dW = \rho \cdot \left(6\left(2 + \frac{2}{3}y\right)dy\right) \cdot g \cdot (3-y)$$

$$W = \int_0^3 6\rho g \left(2 + \frac{2}{3}y\right)(3-y) dy$$

↑ if ρ and g are given, you should plug them in, but in this problem it tells you to use ρ for density and g for acceleration due to gravity

$$= 6\rho g \int_0^3 6 + \cancel{2y} - \cancel{2y} - \frac{2}{3}y^2 dy$$

$$= 6\rho g \left(6y - \frac{2}{9}y^3\right) \Big|_0^3$$

$$= 6\rho g \left(18 - \frac{2}{9} \cdot 27\right)$$

$$= 72\rho g \text{ J}$$