

Master's Thesis

Dirac Structures and Examples of Variational Principles

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Abstract

This thesis introduces the framework of Lie algebroids, foliations, horizontal cohomology, and Dirac geometry, and their applications to classical mechanics. Dirac structures are particular cases of Lie algebroids. Poisson, and in particular symplectic structures are specific examples of Dirac structures. Every Lie algebroid also induces a foliation of its base manifold, which in turn gives rise to the notion of horizontal Lie algebroid cohomology. This cohomological framework provides a natural setting for a generalized variational principle that extends the classical Principle of Least Action in Hamiltonian Mechanics. Using horizontal cohomology, we investigate this generalized variational principle on symplectic and Poisson structures.

Plagiarism Declaration

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- I have provided the sources for all figures, diagrams, tables, data, etc., that are not my own work.
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Chapter 1

Introduction

The *Principle of Least Action* is one of the fundamental principles of classical mechanics. In Hamiltonian mechanics, the dynamics of a system with configuration manifold Q and Hamiltonian function $H: T^*Q \rightarrow \mathbb{R}$ are determined by Hamilton's equations. A classical result states that these equations admit the following variational characterization: a solution

$$\gamma(t) = (q(t), p(t)): [t_0, t_1] \longrightarrow T^*Q$$

of Hamilton's equations is a critical point of the action functional

$$\mathcal{A}(\gamma) = \int_{t_0}^{t_1} p \, dq - H \, dt,$$

with respect to variations fixing the configuration endpoints. This formulation, which can be found for example in [1], provides a geometric interpretation of Hamiltonian dynamics.

The aim of this thesis is to place this classical variational principle into a more general geometric framework, which was done in [6]. The central objects are *Lie algebroids*, which generalize both Lie algebras and tangent bundles. Every Dirac structure naturally carries the structure of a Lie algebroid. At the same time, the symplectic and Poisson formalisms of classical mechanics give rise to canonical examples of Dirac structures. Dirac geometry therefore provides a common framework encompassing both symplectic and Poisson geometry, in which one may try to extend the former variational principle.

A Lie algebroid naturally induces a foliation of its base manifold. This additional structure leads to the notion of horizontal Lie algebroid cohomology, which plays a central role in the variational theory developed in [6]. Under suitable cohomological assumptions, we establish a generalized variational principle for Hamiltonian curves on Dirac structures. As applications, we recover the classical Principle of Least Action in Hamiltonian mechanics. Further, we construct a second family of examples arising from Lie-Poisson structures, a family of examples mentioned in [6] but made concrete in this text.

The thesis is organized as follows. We begin by introducing the necessary background on Lie algebroids, and Dirac structures. After developing the required cohomological tools, we prove the general variational principle. The final chapters illustrate this theory through the two principal examples mentioned above.

Conventions. Throughout this thesis, all manifolds are assumed to be smooth, real, finite-dimensional, Hausdorff, and second countable. In particular, they are paracompact. Unless

stated otherwise, the word 'smooth' will be omitted. Vector bundles and maps between manifolds are always assumed to be smooth. All vector spaces and Lie algebras are real and finite-dimensional.

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Chapter 2

Lie Algebroids

For a concise presentation of Lie algebroids we refer to [9] and [17]. Further we adopted also examples from [6].

Definition 2.1. [9] A *Lie algebroid* $(A, \rho, [\cdot, \cdot])$ is a vector bundle A over M , together with a bundle homomorphism $\rho: A \rightarrow TM$ and a bilinear, skew-symmetric map $[\cdot, \cdot]: \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$ such that $[\cdot, \cdot]$ satisfies the Jacobi identity and the Leibniz rule

$$[s, fs'] = f[s, s'] + \rho_*(s)(f) \cdot s', \text{ for all } f \in C^\infty(M), s, s' \in \Gamma(A).$$

We refer to $[\cdot, \cdot]$ as the (Lie) bracket of the Lie algebroid, and ρ is called the anchor map. Further, we call the anchor map *transitive* when ρ is surjective, i.e., when $\text{Im}(\rho) = TM$.

In particular, the anchor map ρ is a bundle homomorphism that extends to $\rho_*: \Gamma(A) \rightarrow \Gamma(TM)$, as stated in Remark A.9. The Leibniz rule and the Jacobi identity imply that ρ_* is a Lie algebra homomorphism. Indeed, let $\alpha, \beta, \gamma \in \Gamma(A)$ and $f \in C^\infty(M)$. We conclude ([9]):

$$\begin{aligned} 0 &= [[\alpha, \beta], f\gamma] + [[\beta, f\gamma], \alpha] + [[f\gamma, \alpha], \beta] \\ &= f[[\alpha, \beta], \gamma] + \rho_*([\alpha, \beta])(f)\gamma \\ &\quad + f[[\beta, \gamma], \alpha] - \rho_*(\alpha)(f)[\beta, \gamma] + \rho_*(\beta)(f)[\gamma, \alpha] - \rho_*(\alpha)(\rho_*(\beta)(f))\gamma \\ &\quad + f[[\gamma, \alpha], \beta] - \rho_*(\beta)(f)[\gamma, \alpha] + \rho_*(\alpha)(f)[\gamma, \beta] - \rho_*(\beta)(\rho_*(\alpha)(f))\gamma \\ &= \rho_*([\alpha, \beta])(f)\gamma - [\rho_*(\alpha), \rho_*(\beta)](f)\gamma. \end{aligned} \tag{2.1}$$

We used the Jacobi identity in the last equation.

Example 2.2. Consider the tangent bundle $\pi: TM \rightarrow M$ of M . By combining the bracket $[\cdot, \cdot]_{\mathfrak{X}(M)}$ on vector fields $\Gamma(TM) = \mathfrak{X}(M)$ and anchor map $\rho = \text{id}_{TM}$, one obtains the *tangent Lie algebroid* $(TM, [\cdot, \cdot]_{\mathfrak{X}(M)}, \text{id}_{TM})$.

Example 2.3 [9]. Given a Poisson manifold (M, π) . The cotangent bundle T^*M carries a natural Lie algebroid structure. We define the anchor map by $\rho := \pi^\sharp: T^*M \rightarrow TM$. We will revisit the map $\pi^\sharp$ later. Note that $\Gamma(T^*M) = \Omega^1(M)$. We define a bracket on $\Omega^1(M)$ as follows:

$$\begin{aligned} [\cdot, \cdot]_\pi: \Omega^1(M) \times \Omega^1(M) &\rightarrow \Omega^1(M) \\ (\alpha, \beta) &\mapsto \mathcal{L}_{\rho_*(\alpha)}\beta - \mathcal{L}_{\rho_*(\beta)}\alpha - d\pi(\alpha, \beta) \end{aligned}$$

The Lie algebroid $(T^*M, [\cdot, \cdot]_\pi, \pi^\sharp)$ is called the *cotangent Lie algebroid* of the Poisson manifold (M, π) .

Example 2.4. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra. Let $M = \{*\}$ be a one-point manifold. Hence, $TM \simeq \{*\} \times \{0\}$. Then, $\mathfrak{g}$ can be considered as a Lie algebroid over M as follows. The vector bundle structure $E = \{*\} \times \mathfrak{g} \simeq \mathfrak{g}$ is determined by the projection $\pi_{\mathfrak{g}} = \text{const}_*: \mathfrak{g} \rightarrow \{*\}$. The anchor map is given by $\rho = \text{const}_{(*,0)}: \mathfrak{g} \rightarrow TM$. Note that $\Gamma(\mathfrak{g}) \simeq \mathfrak{g}$. Therefore, we can define a Lie bracket on the vector bundle $\mathfrak{g}$ by using the Lie bracket of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$. $(\mathfrak{g}, [\cdot, \cdot], \rho)$ becomes a Lie algebroid.

Example 2.5 [9]. A *Lie algebra action* of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ on M is a Lie algebra homomorphism $\Phi: \mathfrak{g} \rightarrow \mathfrak{X}^1(M)$. Such an action induces the so called *transformation algebroid* $\mathfrak{g} \ltimes M$ as follows.

The vector bundle structure is given by the trivial vector bundle $A = \mathfrak{g} \times M$ over M . Further, by defining

$$\rho: A \rightarrow TM, (\xi, p) \mapsto \Phi(\xi)_p$$

and

$$\begin{aligned} [\cdot, \cdot]: C^\infty(M, \mathfrak{g}) \times C^\infty(M, \mathfrak{g}) &\rightarrow C^\infty(M, \mathfrak{g}) \\ [\alpha, \beta](p) &:= [\alpha(p), \beta(p)]_{\mathfrak{g}} + (\Phi(\alpha(p)))_p(\beta) - (\Phi(\beta(p)))_p(\alpha), \text{ for all } p \in M, \end{aligned}$$

we obtain a Lie algebroid $(A, \rho, [\cdot, \cdot])$. Note that we used the identification $\Gamma(\mathfrak{g} \times M) \simeq \{s: M \rightarrow \mathfrak{g} \mid s \text{ smooth}\} =: C^\infty(M, \mathfrak{g})$. For a one-point manifold $M = \{*\}$, the resulting Lie algebroid in this example is the same as in the previous example.

Remark 2.6. Suppose we have a left action $\delta: G \times M \rightarrow M, (g, p) \mapsto g.p$ of a Lie group G on M . Let $\xi \in \mathfrak{g}$ be an element of the associated Lie algebra of G . The fundamental vector field K^ξ of the action induced by ξ is given at a point $p \in M$ by

$$K_p^\xi = \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi).p,$$

where $\exp: \mathfrak{g} \rightarrow G$ is the exponential map. This construction gives us a map

$$\begin{aligned} \hat{\theta}: \mathfrak{g} &\rightarrow \mathfrak{X}^1(M) \\ \xi &\mapsto -K^\xi, \end{aligned}$$

which is a Lie algebra action on M ([16]). Therefore, we obtain a corresponding transformation algebroid.

Example 2.7 [6]. Let $F \subseteq TM$ be an involutive subbundle, i.e.,

$$[\Gamma(F), \Gamma(F)] \subseteq \Gamma(F).$$

In particular, $\Gamma(F)$ is a Lie subalgebra of $\mathfrak{X}(M)$. Consequently, the Lie algebroid structure on TM over M can be restricted to one on F over M .

2.1 Regular and Singular Distributions

Now, we consider a Lie algebroid $(A, \rho, [\cdot, \cdot])$ over M . Let $\mathcal{D}(A) := \text{Im}(\rho) \subseteq TM$. Thus,

$$\mathcal{D}(A)_x = \text{Im}(\rho_x: A_x \rightarrow T_x M) \subseteq T_x M, \text{ for all } x \in M.$$

As the rank of ρ may not be constant, the set $\mathcal{D}(A)$ is no subbundle of TM in general. Examples will follow. Therefore, the standard discussion of integrating an involutive

distribution using the Frobenius theorem cannot be applied here. However, there is an alternative approach. We will relax the definition and will distinguish between *regular distributions*, which are treated in standard differential geometry textbooks, and *singular distributions*. We will introduce this concept now, following [9]. In this section, it is useful to be more precise with the term 'smooth'. This is because, when talking about distributions, one might assume that one simply means the general assignment of subspaces to tangent spaces.

Definition 2.8 (distribution). A *smooth singular distribution* on M is an assignment

$$M \ni p \mapsto \mathcal{D}_p \subseteq T_p M$$

such that $\mathcal{D}_p$ is a vector subspace of $T_p M$ for each $p \in M$, and for all points $p \in M$ and vectors $v \in \mathcal{D}_p$, there is a neighborhood $U_p \subseteq M$ of p and a smooth vector field $X \in \Gamma(TM|_{U_p})$ with $X_p = v$ which is tangent to the distribution, i.e.,

$$X_q \in \mathcal{D}_q, \text{ for all } q \in U_p.$$

Note that $\dim(\mathcal{D}_p)$ depends on $p \in M$.

If it does not depend on $p \in M$, we call $\mathcal{D}$ a *regular distribution*. In particular, a regular distribution is a singular one.

Definition 2.9. Let M be n -dimensional and $\mathcal{F} = \{N_\nu\}$ a partition of M into a disjoint union of immersed submanifolds $N_\nu \subseteq M$. Let $p \in M$ and $N_p \in \mathcal{F}$ such that $p \in N_p$. Set $k := \dim(N_p)$. $\mathcal{F}$ fulfils the *local foliation property* at p if and only if there is a chart (U, ϕ) with coordinates $x^1, \dots, x^n$ such that

- $\phi(U)$ is a cube in $\mathbb{R}^n$.
- $\{(x^1, \dots, x^n) \mid x^{k+1} = \dots = x^n = 0\} = [p] \in \pi_0(N_p \cap U)$.
- For all $c \in \mathbb{R}^{n-k}$, there exists $N_{\hat{p}} \in \mathcal{F}$ for which

$$\{(x^1, \dots, x^n) \in \mathbb{R}^n \mid (x^{k+1}, \dots, x^n) = c\} \subseteq N_{\hat{p}}.$$

Definition 2.10 (singular foliation). A *smooth singular foliation* of M is a partition $\mathcal{F} = \{N_\nu\}$ of M into a disjoint union of immersed connected submanifolds N_ν , called *leaves*, which satisfies the local foliation property at each $p \in M$.

If $\dim(N_\nu)$ is the same for all $N_\nu \in \mathcal{F}$, we call $\mathcal{F}$ a *regular foliation*. Note that a regular foliation is, in particular, a singular one.

Let $\mathcal{F} = \{N_\nu\}$ be a smooth singular foliation of M . M has a natural smooth distribution $\mathcal{D}^\mathcal{F}$

$$M \supseteq N_\nu \ni p \mapsto T_p N_\nu,$$

which we call the natural associated *tangent distribution* of $\mathcal{F}$.

Note: From now on, we will omit the word 'smooth' again. Therefore, when we refer to singular foliations, we mean smooth ones, and so on.

Definition 2.11. Let $\mathcal{D}$ be a singular distribution on M . A connected immersed submanifold $N \subseteq M$ is called an *integral submanifold* of $\mathcal{D}$ if

$$T_y N \subseteq \mathcal{D}_y \quad \text{is a vector subspace}$$

for all $y \in N$. Furthermore, an integral submanifold N is said to be

- *maximal* if it is not contained within any other integral submanifold.
- of *maximal dimension* if $T_y N = \mathcal{D}_y$ for every $y \in N$.

Definition 2.12. Let $\mathcal{D}$ be a singular distribution on M and let $X \in \mathfrak{X}(M)$. Further, let $C = \{X_i : i \in I\} \subseteq \mathfrak{X}(M)$ be a family of vector fields.

- $\mathcal{D}$ is called *integrable* if each point $p \in M$ is contained in a maximal integral submanifold of $\mathcal{D}$ of maximum dimension.
- $\mathcal{D}$ is generated by C if $\mathcal{D}_p = \text{span}_{i \in I} \{X_i|_p\}$ for each $p \in M$.
- $\mathcal{D}$ is called invariant with respect to X if for the local flow φ_X of X there holds

$$d\varphi_X^t|_p(\mathcal{D}_p) = \mathcal{D}_{\varphi_X^t(p)}$$

for all $p \in M$ and $t \in \mathbb{R}$ such that $\varphi_X^t(p)$ is well-defined.

- $\mathcal{D}$ is called *invariant* with respect to C if $\mathcal{D}$ is invariant with respect to each $Y \in C$.

Theorem 2.13 (Stefan-Sussmann), [9]. *Let $\mathcal{D}$ be a singular distribution on a manifold M . Then the following three conditions are equivalent:*

- (i) $\mathcal{D}$ is integrable.
- (ii) $\mathcal{D}$ is generated by a family C of vector fields, and is invariant with respect to C .
- (iii) $\mathcal{D}$ is the tangent distribution $\mathcal{D}^{\mathcal{F}}$ of a singular foliation $\mathcal{F}$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $\mathcal{D}$ is integrable. Let C be the family of all smooth vector fields on M which are tangent to $\mathcal{D}$. This family exists by smoothness of $\mathcal{D}$ and using partition of unity. Even better: smoothness of $\mathcal{D}$ implies that $\mathcal{D}$ is generated by C . We will show invariancy with respect to C by showing that $\mathcal{D}$ is invariant with respect to an arbitrary smooth vector field X , which is tangent to $\mathcal{D}$. Let X be such a vector field. Let $p \in M$ be arbitrary, and denote by $\mathcal{F}_p \subseteq M$ the maximal integral submanifold of maximum dimension through p , which exists by integrability of $\mathcal{D}$. Maximal dimension implies that

$$T_y \mathcal{F}_p = \mathcal{D}_y, \text{ for all } y \in \mathcal{F}_p. \quad (2.2)$$

Hence, $X|_{\mathcal{F}_p}$ is tangent to $\mathcal{F}_p$. In particular, the local flow φ_X can be restricted to $\mathcal{F}_p$. Therefore, $\mathcal{F}_p$ is invariant for φ_X . Let $\tau \in \mathbb{R}_{>0}$ be such that $\varphi_X^\tau(p)$ is well-defined. Denote $M_\tau := \{p \in M : (\tau, p) \in D\}$ where $D \subseteq \mathbb{R} \times M$ is the flow domain of φ_X . As $\mathcal{F}_p$ is maximal, it follows that $\varphi_X^\tau(p) \in \mathcal{F}_p$. X is tangent to $\mathcal{F}_p$ and $\varphi_X^\tau|_{\mathcal{F}_p \cap M_\tau}$ is a diffeomorphism. Consequently, uniqueness of the flow implies that

$$d\varphi_X^\tau|_p(T_p \mathcal{F}_p) = T_{\varphi_X^\tau(p)} \mathcal{F}_p.$$

Using (2.2) for $y = \varphi_X^\tau(p)$ and noticing that $T_p \mathcal{F}_p = \mathcal{D}_p$ implies that $d\varphi_X^\tau|_p(\mathcal{D}_p) = \mathcal{D}_{\varphi_X^\tau(p)}$. Because p is arbitrary, we conclude that $\mathcal{D}$ is invariant with respect to C .

(ii) $\Rightarrow$ (iii). Suppose that $\mathcal{D}$ is generated by a family C of vector fields, and that $\mathcal{D}$ is invariant with respect to C . We prepare the situation: Let $p \in M$ and $d := \dim(\mathcal{D}_p)$. Let $X_1, \dots, X_d \in C$ with $\mathcal{D}_p = \text{span}_{i \in \{1, \dots, d\}} \{X_i|_p\}$. Denote the local flows of $X_1, \dots, X_d$ by

$\varphi_{X_1}, \dots, \varphi_{X_d}$, respectively.

We obtain a map

$$\begin{aligned} \Psi: \mathbb{R}^d \supseteq Q &\rightarrow N \\ (t_1, \dots, t_d) &\mapsto \varphi_{X_1}^{t_1} \circ \dots \circ \varphi_{X_d}^{t_d}(p), \end{aligned}$$

which is a local diffeomorphism between a d -dimensional cuboid Q onto a d -dimensional submanifold $N \subseteq M$ that contains p .

Let $e_k \in \mathbb{R}^d$ be the k -th standard basis vector and $x := (s_1, \dots, s_d) \in \mathbb{R}^d$. We denote $\psi_{s_l} := \psi(0, \dots, 0, s_l, \dots, s_d)$ and $\varphi_{t_l, \dots, t_d} := \varphi_{X_l}^{t_l} \circ \dots \circ \varphi_{X_d}^{t_d}$ for $l \in \{1, \dots, d\}$ and obtain for each $x \in \mathbb{R}^d$ by using $\mathbb{R}^d \simeq T_x \mathbb{R}^d$:

$$\begin{aligned} d\psi_x(e_k) &= \left. \frac{d}{dt} \right|_{t=0} \varphi_{X_1}^{s_1} \circ \dots \circ \varphi_{X_k}^{s_k+t} \circ \dots \circ \varphi_{X_d}^{s_d}(p) \\ &= d\varphi_{X_1}^{t_1}|_{\psi_{s_d}} \circ \dots \circ d\varphi_{X_{k-1}}^{t_{k-1}}|_{\psi_{s_{k-1}}} (X_k|_{\varphi_{t_{k+1}, \dots, t_d}(p)}). \end{aligned}$$

Using the fact that $\mathcal{D}$ is invariant with respect to $\{X_1, \dots, X_d\}$ and that $X_1, \dots, X_d$ span D at p , it follows that $N \subseteq M$ is an integral submanifold of maximum dimension. We then glue these local integral submanifolds together wherever they intersect, in order to obtain a partition of M into a disjoint union of connected, immersed integral submanifolds of maximum dimension. These submanifolds are called leaves. The maximum dimension implies that $\mathcal{D}$ is the tangent distribution to the singular foliation formed by the partition above, provided that the constructed partition fulfils the local foliation property. It therefore remains to show that the local foliation property holds.

We proceed by induction on the dimension of $\mathcal{D}_p$. If $\dim(\mathcal{D}_p) = 0$, there is nothing to show because the local foliation property at p is empty. For the case when $\dim(\mathcal{D}_p) > 0$, there is a vector field $X \in C$ such that $X_p \neq 0$, i.e., p is a regular point of X . The *Canonical Form Near a Regular Point* ([16]) tells us that there exist coordinates (x^i) in a neighborhood U of p such that X has the coordinate representation $\partial/\partial x^1$. The trajectories of X lie on the leaves. Taking the quotient $\pi: U \rightarrow U/\sim$ of U by the trajectories of X reduces the dimension of M and of the leaves by 1. Note that in these coordinates π reads just as the projection $(x^1, \dots, x^n) \mapsto (x^2, \dots, x^n)$. We define $\tilde{\mathcal{D}}_{[p]} := d\pi_p(\mathcal{D}_p)$ for each $p \in U$. The invariance of $\mathcal{D}$ with respect to X implies well-definedness of $\tilde{\mathcal{D}}$: Given $[p] = [q] \in U/\sim$, we obtain $p = \varphi_X^t(q)$ for some $t \in \mathbb{R}$. Therefore

$$\begin{aligned} \tilde{\mathcal{D}}_{[p]} &= d\pi_{\varphi_X^t(q)}(\mathcal{D}_{\varphi_X^t(q)}) \\ &= d(\pi \circ \varphi_X^t)_q(\mathcal{D}_q) \\ &= \tilde{\mathcal{D}}_{[q]}. \end{aligned}$$

For the coordinates (x^i) , take the slice through p (just assume that $x^1(p) = 0$), i.e., take $V := \{x^1 = 0\} \simeq U/\sim$. Let $Y \in C$ and define

$$\tilde{Y}_x := d\pi_x(Y_x), \quad x \in V.$$

We obtain a family $\tilde{C}$ of vector fields on $V \simeq U/\sim$. $\tilde{C}$ generates $\tilde{\mathcal{D}}$ because C generates $\mathcal{D}$. Moreover, $\tilde{\mathcal{D}}$ is invariant with respect to $\tilde{C}$. By the induction hypothesis, $\tilde{\mathcal{D}}$ has the local foliation property at $[p] \in U/\sim$. Using the representation of π in the coordinates (x^i) , we deduce that $\mathcal{D}$ has the local foliation property at $p \in M$.

(iii) $\Rightarrow$ (i). Remind that a foliation is in particular a partition $\mathcal{F} = \{N_\nu\}$ of M into a disjoint union of smooth immersed connected submanifolds $N_\nu \subseteq M$. If $\mathcal{D} = \mathcal{D}^{\mathcal{F}}$ is the

tangent distribution of a singular foliation $\mathcal{F}$, then $\mathcal{D}_p^{\mathcal{F}} = T_p N_\nu$ for $N_\nu \subseteq M$ with $p \in N_\nu$. Therefore the leaves of $\mathcal{F}$ are integral submanifolds of maximum dimension for $\mathcal{D}$. $\square$

Remark 2.14. Let $\mathcal{D}$ be a regular distribution. Suppose $\mathcal{D}$ is involutive, i.e.,

$$[\Gamma(\mathcal{D}), \Gamma(\mathcal{D})] \subseteq \Gamma(\mathcal{D}).$$

One can show that $\mathcal{D}$ is invariant to the family of vector fields which are tangent to $\mathcal{D}$. Then the *Frobenius Theorem* can be seen as a special case of the [Stefan-Sussmann Theorem](#).

2.2 The Characteristic Distribution of a Lie Algebroid

Given a Lie algebroid $(A, \rho, [\cdot, \cdot])$ over M . We define the *characteristic distribution* of A as

$$\mathcal{D}(A): p \mapsto \text{Im}(\rho_p: A_p \rightarrow T_p M), p \in M.$$

In general, the characteristic distribution is a singular distribution because the rank of ρ may vary.

Lemma 2.15. *Let $(A, \rho, [\cdot, \cdot])$ be a Lie algebroid over M . The characteristic distribution $\mathcal{D}(A)$ of A is smooth.*

Proof. We denote the rank of the vector bundle A by k . Let $p \in M$ and $v \in \rho_p(A_p)$. Let $w \in \rho_p^{-1}(v)$. The existence of local frames (see Lemma [A.2](#)) provides a neighborhood U of p and local sections $(\sigma_i: U \rightarrow A) \in \Gamma(A|_U)$, where $i \in \{1, \dots, k\}$, such that $w = \sum_{i=1}^k \lambda_i \sigma_i(p)$ for some $\lambda_i \in \mathbb{R}$. For all $q \in U$, let $f_i(q) := \lambda_i$ and denote $\xi_i := \rho_*(\sigma_i)$. The section

$$X := \sum_{i=1}^k f_i \xi_i \in \Gamma(TM|_U) = \mathfrak{X}(U)$$

satisfies $X_p = v$ and is tangent to $\mathcal{D}(A)$. Since p and v were arbitrary, this proves that $\mathcal{D}(A)$ is smooth. $\square$

The image of the induced map $\rho: \Gamma(A) \rightarrow \mathfrak{X}(M)$ provides a family of vector fields on M . We denote this family by

$$C(A) := \text{Im}(\rho_*) = \{\rho_*(\xi) \mid \xi \in \Gamma(A)\}.$$

Lemma 2.16. *The family $C(A) \subseteq \mathfrak{X}(M)$ generates $\mathcal{D}(A)$. Furthermore, $\mathcal{D}(A)$ is invariant with respect to $C(A)$.*

Proof. Let $p \in M$. We have to show that $\mathcal{D}(A)_p = \text{span}_{\xi \in \Gamma(A)} \{\rho_*(\xi)_p\}$. As we saw in the proof of Lemma [2.15](#), there is a neighborhood U of p and local sections $(\sigma_i: U \rightarrow A) \in \Gamma(A|_U)$, where $i \in \{1, \dots, k\}$, such that $A_p = \text{span}_{i \in \{1, \dots, k\}} \{\sigma_i(p)\}$. The *Existence of Smooth Bump Functions* ([\[16\]](#)) provides a function $\Psi \in C^\infty(M)$ such that $\Psi(p) = 1$ and $\text{supp}(\Psi) \subseteq U$. We define

$$\xi_i(q) := \begin{cases} \Psi(q) \sigma_i(q), & q \in U \\ 0, & \text{else} \end{cases}$$

for each $q \in M$, which defines sections $\xi_i \in \Gamma(A)$ with

$$\mathcal{D}(A)_p = \text{span}_{i \in \{1, \dots, k\}} \{\rho_*(\xi_i)_p\}.$$

As p was arbitrary, this shows that $\mathcal{D}(A)$ is generated by $C(A)$.

Now, we show that $\mathcal{D}(A)$ is invariant with respect to $C(A)$ by following an argument of [26]. Let $X = \rho_*(\hat{\xi}) \in C(A)$ and $p \in M$. We denote $\mathcal{J}_p := \{t_0 \in \mathbb{R} \mid (t_0, p) \in D\}$ where $D \subseteq \mathbb{R} \times M$ is the flow domain of the flow φ of X . Similarly to the construction above, let $Y_i := \rho_*(\xi_i) \in C(A)$, where $i \in \{1, \dots, k\}$, such that $\mathcal{D}(A)_p = \text{span}_{i \in \{1, \dots, k\}} \{Y_i|_p\}$. Remind that the anchor map is a Lie algebra homomorphism (see [2.1]). Further, remind that the diffeomorphism φ^t induces a pushforward $\varphi_*^t: \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ on vector fields. It follows that

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=t_0} \varphi_*^{-t}(Y_i)_p &= \varphi_*^{-t_0}(\mathcal{L}_X(Y_i))_p \\ &= \varphi_*^{-t_0}([\rho_*(\hat{\xi}), \rho_*(\xi_i)])_p \\ &= \varphi_*^{-t_0}(\rho_*[\hat{\xi}, \xi_i])_p \end{aligned} \quad (2.3)$$

for all $t_0 \in \mathcal{J}_p$ and each $i \in \{1, \dots, k\}$. In the first equation we used a fact from differential geometry (see Proposition [C.2]). But $\hat{X}_i := \rho_*([\hat{\xi}, \xi_i]) \in C(A)$ and there are functions $f_j^i \in C^\infty(M)$ such that

$$\hat{X}_i|_p = \sum_{j=1}^k f_j^i(p) Y_j|_p.$$

Hence, by denoting $W_i(t) := \varphi_*^{-t}(Y_i)_p$ for each $i \in \{1, \dots, k\}$, Equation (2.3) boils down to the differential equation

$$\left. \frac{d}{dt} \right|_{t=t_0} W_i(t) = \sum_{j=1}^k f_j^i(p) W_j(t_0), \quad t_0 \in \mathcal{J}_p \quad (2.4)$$

with initial condition $W_i(0) = Y_i|_p$. As it is done for linear systems of ODEs as in [5], we write the $W_i(t)$'s into a vector

$$W(t) := \begin{pmatrix} W_1(t) \\ \vdots \\ W_k(t) \end{pmatrix} \in \mathbb{R}^{n \cdot k}, \quad t \in \mathcal{J}_p,$$

where $n := \dim(M)$. Denoting $I_k \in \mathbb{R}^{k \times k}$ for the identity matrix we can rewrite (2.4) as

$$\left. \frac{d}{dt} \right|_{t=t_0} W(t) = (A \otimes I_k) W(t_0), \quad t_0 \in \mathcal{J}_p \quad (2.5)$$

for some $A \in \mathbb{R}^{n \times n}$. The $\otimes$ indicates the Kronecker product of matrices[†]. The theory for ODEs ([5]) gives a solution for (2.5) by use of the matrix exponential

$$\begin{aligned} W(t) &= \exp(t(A \otimes I_k)) W(0) \\ &= (\exp(tA) \otimes I_k) W(0), \quad t \in \mathcal{J}_p. \end{aligned} \quad (2.6)$$

[†]For matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{k \times l}$ we define the *Kronecker product* of A and B as

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \in \mathbb{R}^{mk \times nl}$$

Here, we used the fact that if we have a matrix $A \in \mathbb{R}^{m \times m}$ and the identity matrix $I_k \in \mathbb{R}^{k \times k}$, then we have $\exp(A \otimes I_k) = \exp(A) \otimes I_k$ (see, for example, [23]). Equation (2.6) implies that we can express $W_i(0)$ by the vectors $\{W_1(t), \dots, W_k(t)\}$ for each $i \in \{1, \dots, k\}$ and $t \in \mathcal{I}_p$. Because $\{W_i(0) \mid i \in \{1, \dots, k\}\} = \{Y_i|_p \mid i \in \{1, \dots, k\}\}$ spans $\mathcal{D}_p$ and because $\exp(t(A \otimes I_k))$ is invertible for all $t \in \mathcal{I}_p$, we conclude that

$$\{W_1(t), \dots, W_k(t)\} = \{\varphi_*^{-t}(Y_i)_p \mid i \in \{1, \dots, k\}\}$$

spans $\mathcal{D}_p$ for all $t \in \mathcal{I}_p$, too. As φ^{-t_0} is a diffeomorphism and $d\varphi^{t_0}|_p(\varphi_*^{-t_0}(Y_i)_p) = Y_i|_{\varphi^{t_0}(p)}$, we conclude that

$$d\varphi^{t_0}|_p(\mathcal{D}_p) = \mathcal{D}_{\varphi^{t_0}(p)}$$

for all $t_0 \in \mathcal{I}_p$. As p was arbitrary, we showed that $\mathcal{D}(A)$ is invariant with respect to $C(A)$. $\square$

Let $(A, \rho, [\cdot, \cdot])$ be a Lie algebroid. A consequence of the preceding Lemma is that the characteristic distribution $\mathcal{D}(A)$ is integrable. Even better, according to the [Stefan-Sussmann Theorem](#), $\mathcal{D}(A)$ is the tangent distribution of a singular foliation $\mathcal{F}$. Therefore we obtain:

Corollary 2.17. *Any Lie algebroid A has an associated singular foliation $\mathcal{F}(A)$ induced by its characteristic distribution $\mathcal{D}(A)$.*

Definition 2.18. The singular foliation $\mathcal{F}(A)$ is called the *characteristic foliation* of the Lie algebroid A .

2.3 Lie Algebroid Cohomology

Definition 2.19. Let $(A, \rho, [\cdot, \cdot])$ be a Lie algebroid. Let $d_A: \Gamma(\Lambda^\bullet A^*) \rightarrow \Gamma(\Lambda^{\bullet+1} A^*)$ be the degree 1 differential defined by

$$\begin{aligned} (d_A \eta)(\xi_1, \dots, \xi_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \rho_*(\xi_i)(\eta(\xi_1, \dots, \hat{\xi}_i, \dots, \xi_{n+1})) \\ &\quad + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \eta([\xi_i, \xi_j], \xi_1, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_{n+1}), \end{aligned} \tag{2.7}$$

where $\eta \in \Gamma(\Lambda^\bullet A^*)$ and $\xi_i \in \Gamma(A)$ for each $i \in \{1, \dots, n+1\}$. We call d_A the *exterior differential* on A . As usual, the hat stands for omitting the respective entry.

The differential d_A satisfies $d_A \circ d_A = 0$, which makes $(\Gamma(\Lambda^\bullet A^*), d_A)$ into a chain complex with associated cohomology, the *Lie algebroid cohomology*:

$$H^\bullet(A) := \frac{\ker(d_A: \Gamma(\Lambda^\bullet A^*) \rightarrow \Gamma(\Lambda^{\bullet+1} A^*))}{\text{Im}(d_A: \Gamma(\Lambda^{\bullet-1} A^*) \rightarrow \Gamma(\Lambda^\bullet A^*))}.$$

We are already familiar with the Lie algebroid cohomology of some of the examples we presented at the beginning of this chapter.

Example 2.20. When we apply the definition of Lie algebroid cohomology to the tangent Lie algebroid TM over M (see Example 2.2), this yields the de Rham cohomology of M , i.e., $H^\bullet(TM) = H_{dR}^\bullet(M)$.

Example 2.21. The Lie algebroid cohomology of the cotangent algebroid $(T^*M, [\cdot, \cdot]_\pi, \pi^\#)$ over M (see Example 2.3) is precisely the Poisson cohomology of M , i.e. $H^\bullet(T^*M) = H_\pi^\bullet(M)$, which we will revisit later.

Example 2.22. As discussed in Example 2.4, a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ induces a Lie algebroid structure on $\mathfrak{g}$. Noting that $\Gamma(\Lambda^\bullet \mathfrak{g}^*) = \Lambda^\bullet \mathfrak{g}^*$, the Lie algebroid cohomology of $\mathfrak{g}$ is the cohomology of the differential complex

$$d_{\mathfrak{g}}: \Lambda^\bullet \mathfrak{g}^* \rightarrow \Lambda^{\bullet+1} \mathfrak{g}^*$$

with

$$(d_{\mathfrak{g}}\omega)(v_1, \dots, v_{n+1}) = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \omega([\xi_i, \xi_j]_{\mathfrak{g}}, v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{n+1})$$

for all $v_i \in \mathfrak{g}$, $i \in \{1, \dots, n+1\}$. Note that, in this case, the first part of (2.7) vanishes because the base manifold of $\mathfrak{g}$ is $M = \{*\}$. The differential graded complex $(\Lambda^\bullet \mathfrak{g}^*, d_{\mathfrak{g}})$ is called the *Chevalley-Eilenberg complex*. The resulting Lie algebroid cohomology $H^\bullet(\mathfrak{g})$ is also known as the *Lie algebra cohomology* of the Lie algebra $\mathfrak{g}$.

Definition 2.23 [10, 24]. Let $(A_1, \rho_1, [\cdot, \cdot]_1)$ and $(A_2, \rho_2, [\cdot, \cdot]_2)$ be Lie algebroids over M_1 and M_2 , respectively. A *Lie algebroid morphism* between A_1 and A_2 covering $f: M_1 \rightarrow M_2$ is a bundle homomorphism $F: A_1 \rightarrow A_2$

$$\begin{array}{ccc} A_1 & \xrightarrow{F} & A_2 \\ \pi_1 \downarrow & \# & \downarrow \pi_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array} \quad (2.8)$$

compatible with the anchor maps

$$\begin{array}{ccc} A_1 & \xrightarrow{F} & A_2 \\ \rho_1 \downarrow & \# & \downarrow \rho_2 \\ TM_1 & \xrightarrow{df} & TM_2 \end{array} \quad (2.9)$$

and compatible with the brackets:

Suppose $\alpha, \beta \in \Gamma(A_1)$ can be pushed forward to A_2 , i.e., there exist $\alpha', \beta' \in \Gamma(A_2)$ such that $F_*(\alpha) = \alpha' \circ f$ and $F_*(\beta) = \beta' \circ f$. Hence, the way we would like to define compatibility with the brackets would be

$$F_*([\alpha, \beta]_1) = [\alpha', \beta']_2 \circ f. \quad (2.10)$$

Unfortunately, that is not always the case. However, the solution lies in using the pullback bundle $f^*A_2 = \{(p, a) \in M_1 \times A_2 \mid f(p) = \pi_2(a)\}$ of A_2 by f :

Let $\alpha, \beta \in \Gamma(A_1)$. By using Remark A.13, there is a representation

$$\begin{aligned} F_*(\alpha) &= \sum_{i=1}^n c_i f^*(\alpha_i) \\ \text{and } F_*(\beta) &= \sum_{j=1}^m c'_j f^*(\beta_j) \end{aligned} \quad (2.11)$$

where $c_i, c'_j \in C^\infty(M_1)$ and $\alpha_i, \beta_j \in \Gamma(A_2)$. Then, compatibility with the brackets means

$$\begin{aligned} F_*([\alpha, \beta]_1) &= \sum_{i,j} c_i c'_j f^*([\alpha_i, \beta_j]_2) \\ &+ \sum_j \rho_{1*}(\alpha)(c'_j) f^*(\beta_j) - \sum_i \rho_{1*}(\beta)(c_i) f^*(\alpha_i). \end{aligned} \quad (2.12)$$

Remark 2.24. When we come back to the case when we can push forward, i.e., when

$$\begin{aligned} F_*(\alpha) &= \alpha' \circ f = F_*(\alpha) = \mathbb{1}_{M_1} f^*(\alpha') \\ \text{and } F_*(\beta) &= \beta' \circ f = F_*(\beta) = \mathbb{1}_{M_2} f^*(\beta'), \end{aligned}$$

the general formula (2.12) reduces to (2.10) because derivations evaluate to zero when they are applied to constant functions.

Remark 2.25. The composition of Lie algebroid morphisms is again a Lie algebroid morphism. In particular, the class of Lie algebroids together with Lie algebroid morphisms forms a category.

Exercise 2.26 [17]. Given a Lie algebroid $(A, \rho, [\cdot, \cdot])$, check that the anchor map $\rho: A \rightarrow TM$ is a Lie algebroid morphism from A to the tangent Lie algebroid TM .

Exercise 2.27 [17]. Given a map $f: M \rightarrow N$ between the manifolds M and N . The tangent map $df: TM \rightarrow TN$ is a bundle homomorphism. When considering the tangent Lie algebroids $(TM, [\cdot, \cdot]_{\mathfrak{X}(M)}, \text{id}_{TM})$ and $(TN, [\cdot, \cdot]_{\mathfrak{X}(N)}, \text{id}_{TN})$, check that df can be seen as a Lie algebroid morphism.

2.4 Induced Maps in Lie Algebroid Cohomology

Let $(A_1, \rho_1, [\cdot, \cdot]_1)$ and $(A_2, \rho_2, [\cdot, \cdot]_2)$ be Lie algebroids over M_1 and M_2 , respectively. Let $F: A_1 \rightarrow A_2$ be a Lie algebroid morphism covering $f: M_1 \rightarrow M_2$. In particular, we obtain the commuting diagrams (2.8) and (2.9), i.e.,

$$\begin{aligned} \pi_2 \circ F &= f \circ \pi_1 \\ \rho_2 \circ F &= df \circ \rho_1. \end{aligned}$$

Let $k \in \mathbb{N}_0$. By extending Lemma A.12a, we obtain

$$\begin{aligned} f^*: \Gamma(\Lambda^k A_2^*) &\rightarrow \Gamma(f^* \Lambda^k A_2^*) \\ \sigma &\mapsto [M \ni p \mapsto (p, (\sigma \circ f)(p))]. \end{aligned}$$

Additionally, with Lemma A.12b we also obtain a map

$$\begin{aligned} F_*: \Gamma(A_1) &\rightarrow \Gamma(f^* A_2) \\ \sigma &\mapsto [M \ni p \mapsto (p, (F \circ \sigma)(p))]. \end{aligned}$$

Let $\omega \in \Gamma(\Lambda^k A_2^*)$. We can combine the two preceding maps in order to define an element $F^* \omega \in \Gamma(\Lambda^k A_1^*)$ by

$$F^* \omega(\xi_1, \dots, \xi_k) := (f^* \omega)(F_*(\xi_1), \dots, F_*(\xi_k)) \in C^\infty(M_1) \quad (2.13)$$

for all $\xi_i \in \Gamma(A_1)$, and $i \in \{1, \dots, k\}$.

More concretely, the application of F^* to ω leads to

$$F^*(\omega)_p = \omega_{f(p)}(F|_{A_{1p}}(\cdot), \dots, F|_{A_{1p}}(\cdot)) : A_{1p} \times \dots \times A_{1p} \rightarrow \mathbb{R} \quad (2.14)$$

for all $p \in M$ where we used the identification

$$\begin{aligned} (f^* A_2^*) &\cong (f^* A_2)^* \\ (p, \varphi) &\mapsto [(p, a) \mapsto \varphi(a)] \end{aligned}$$

Note that for the case when $k = 0$, we obtain $F^*(g) = g \circ f \in C^\infty(M_1)$, for all $g \in C^\infty(M_2)$.

Example 2.28. It is no coincidence that this looks familiar. Suppose we have the situation from Exercise 2.27. Let $p \in M$. The Lie algebroid morphism $df : TM \rightarrow TN$ restricted to $T_p M$ is precisely $df|_{T_p M} = d_p f$. Therefore, for $\alpha \in \Gamma(\Lambda^k T^* N) = \Omega^k(N)$, Equation 2.14 yields

$$(df)^*(\omega)_p = \omega_{f(p)}(df_p(\cdot), \dots, df_p(\cdot)) : T_p M \times \dots \times T_p M \rightarrow \mathbb{R}.$$

As for differential forms, the exterior differential of a Lie algebroid is compatible with Lie algebroid morphisms. Lie algebroid morphisms are often simply defined to be compatible with the exterior differential. However, this compatibility is a consequence of the above definition of a Lie algebroid morphism. It is more instructive to start with a structure-preserving map, which is automatically compatible. Concretely, we obtain

Theorem 2.29. Let $(A_1, \rho_1, [\cdot, \cdot]_1)$ and $(A_2, \rho_2, [\cdot, \cdot]_2)$ be Lie algebroids over M_1 and M_2 , respectively. Let $F : A_1 \rightarrow A_2$ be a Lie algebroid morphism covering $f : M_1 \rightarrow M_2$. The pullback map

$$\begin{aligned} F^* : \Gamma(\Lambda^k A_2^*) &\rightarrow \Gamma(\Lambda^k A_1^*) \\ \omega &\mapsto F^* \omega \end{aligned}$$

defined in Equation (2.13) fulfills

$$d_{A_1} F^* \omega = F^* d_{A_2} \omega.$$

In particular, F^* is a cochain map.

Proof. We start with the case when $k = 0$.

Applying Lemma A.12a to the commuting diagram in (2.9) yields the following commuting diagram:

$$\begin{array}{ccc} \Gamma(A_1) & \xrightarrow{F_*} & \Gamma(f^* A_2) \\ \rho_{1*} \downarrow & \# & \downarrow \hat{\rho}_{2*} \\ \Gamma(TM_1) & \xrightarrow{df_*} & \Gamma(f^* TM_2), \end{array}$$

where we set $\hat{\rho}_2 := \rho_2|_{f^* A_2}$. Remember that $df_p(v)(g) = v(g \circ f)$ for all $p \in M$, $v \in T_p M$ and $g \in C^\infty(M_2)$. Therefore,

$$X(g \circ f) = df_*(X)(g) \circ f$$

for all $X \in \mathfrak{X}(M_1)$ and $g \in C^\infty(M_2)$.

Now, let $g \in C^\infty(N)$ and $\xi \in \Gamma(A)$. It follows that

$$\begin{aligned}
 d_{A_1} F^* g(\xi) &= d_{A_1} (g \circ f)(\xi) \\
 &= \rho_{1*}(\xi)(g \circ f) \\
 &= df_*(\rho_{1*}(\xi))(g) \circ f \\
 &= \hat{\rho}_{2*}(F_*(\xi))(g) \circ f \\
 &= f^*(d_{A_2} g)(F_*(\xi)) = F^*(d_{A_2} g)(\xi).
 \end{aligned} \tag{2.15}$$

Therefore, $d_{A_1} F^* g = F^* d_{A_2} g$, for all $g \in C^\infty(M_2)$.

We continue with the case when $k = 1$.

Let $\omega \in \Gamma(A_2^*)$ and $\xi_1, \xi_2 \in \Gamma(A_1)$. Applying the definition of the exterior differential on A_1 yields (2.7):

$$d_{A_1} F^* \omega(\xi_1, \xi_2) = \rho_{1*}(\xi_1)(F^* \omega(\xi_2)) - \rho_{1*}(\xi_2)(F^* \omega(\xi_1)) \tag{2.16}$$

$$- F^* \omega([\xi_1, \xi_2]_1). \tag{2.17}$$

Again, because $\Gamma(f^* A_2)$ is generated by $\text{Im}(f^*: \Gamma(A_1) \rightarrow \Gamma(f^* A_2))$ as a $C^\infty(M_1)$ -module, this yields a representation

$$F_*(\xi_1) = \sum_{i=1}^n c_i f^*(\alpha_i) \quad \text{and} \quad F_*(\xi_2) = \sum_{j=1}^m c'_j f^*(\beta_j), \tag{2.18}$$

where $c_i, c'_j \in C^\infty(M_1)$ and $\alpha_i, \beta_j \in \Gamma(A_2)$. Therefore, by using $C^\infty(M_1)$ -linearity and the fact that the vector field $\rho_{1*}(\xi_2)$ can be regarded as a derivation, we conclude that

$$\begin{aligned}
 \rho_{1*}(\xi_2)(F^* \omega(\xi_1)) &= \rho_{1*}(\xi_2)(f^* \omega(F_*(\xi_1))) \\
 &= \sum_{i=1}^n \rho_{1*}(\xi_2)(c_i f^* \omega(f^* \alpha_i)) \\
 &= \sum_{i=1}^n \rho_{1*}(\xi_2)(c_i) f^*(\omega(\alpha_i)) + \sum_{i=1}^n c_i \rho_{1*}(\xi_2)(f^*(\omega(\alpha_i))).
 \end{aligned}$$

The procedure for $\rho_{1*}(\xi_1)(F^* \omega(\xi_2))$ is similar. By applying (2.12), we deduce using $C^\infty(M_1)$ -linearity:

$$\begin{aligned}
 F^* \omega([\xi_1, \xi_2]_1) &= f^* \omega(F_*([\xi_1, \xi_2]_1)) \\
 &= \sum_{i,j} c_i c'_j f^*(\omega([\alpha_i, \beta_j]_2)) \\
 &\quad + \sum_j \rho_{1*}(\xi_1)(c'_j) f^*(\omega(\beta_j)) - \sum_i \rho_{1*}(\xi_2)(c_i) f^*(\omega(\alpha_i)).
 \end{aligned}$$

Thus, by combining the preceding two equations, (2.16) becomes

$$\begin{aligned}
 d_{A_1} F^* \omega(\xi_1, \xi_2) &= - \sum_{i,j} c_i c'_j f^*(\omega([\alpha_i, \beta_j]_2)) \\
 &\quad - \sum_{i=1}^n c_i \rho_{1*}(\xi_2)(f^*(\omega(\alpha_i))) + \sum_j c'_j \rho_{1*}(\xi_1)(f^*(\omega(\beta_j))).
 \end{aligned}$$

By using (2.15), the preceding equation yields

$$d_{A_1} F^* \omega(\xi_1, \xi_2) = \sum_{i,j} c_i c'_j f^* \left(-\omega([\alpha_i, \beta_j]_2) - \rho_{2*}(\beta_j)(\omega(\alpha_i)) + \rho_{2*}(\alpha_i)(\omega(\beta_j)) \right).$$

On the other hand, by inserting (2.18), we obtain

$$\begin{aligned} F^*(d_{A_2} \omega)(\xi_1, \xi_2) &= f^*(d_{A_2} \omega)(F_*(\xi_1), F_*(\xi_2)) \\ &= \sum_{i,j} c_i c'_j f^*(d_{A_2} \omega)(\alpha_i, \beta_j) \\ &= \sum_{i,j} c_i c'_j f^* \left(\rho_{2*}(\alpha_i)(\omega(\beta_j)) \right. \\ &\quad \left. - \rho_{2*}(\beta_j)(\omega(\alpha_i)) - \omega([\alpha_i, \beta_j]_2) \right). \end{aligned}$$

Therefore, $d_{A_1} F^* \omega = F^* d_{A_2} \omega$ for all $\omega \in \Gamma(A_2)$.

The general formula,

$$d_{A_1} F^* \omega = F^* d_{A_2} \omega \text{ for all } \omega \in \Gamma(\Lambda^k A_2^*),$$

follows analogously. $\square$

Lemma 2.30. *Let $(A_i, \rho_i, [\cdot, \cdot]_i)$ be Lie algebroids over M_i where $i \in \{1, 2, 3\}$. Consider Lie algebroid morphisms $F: A_1 \rightarrow A_2$ and $G: A_2 \rightarrow A_3$ covering the maps $f: M_1 \rightarrow M_2$ and $g: M_2 \rightarrow M_3$, respectively. Then, the pullback is (contravariantly) functorial, i.e.,*

$$(\text{id}_{A_1})^* = \text{id}_{\Gamma(\Lambda^\bullet A_1^*)}$$

and

$$(G \circ F)^* = F^* \circ G^*: \Gamma(\Lambda^\bullet A_3^*) \rightarrow \Gamma(\Lambda^\bullet A_1^*)$$

Proof. The assertion about the identity follows from applying the definitions.

As the composition of Lie algebroid morphisms is again a Lie algebroid morphism, $(G \circ F)^*$ is well-defined.

Let $k \in \mathbb{N}$. For the case when $k = 0$, let $h \in C^\infty(M_3)$. By definition, it follows that $(G \circ F)^*(h) = h \circ (g \circ f) = F^*(G^*(h))$.

Now, consider the case when $k > 0$. Let $\alpha \in \Gamma(\Lambda^k A_3^*)$. Again, due to $C^\infty(M_3)$ -linearity it suffices to show the equality pointwise. Therefore, let $p \in M_1$ and $\xi_j \in A_1|_p$, where $j \in \{1, \dots, k\}$. We conclude that

$$\begin{aligned} (F^* \circ G^*)(\alpha)_p(\xi_1, \dots, \xi_k) &= G^*(\alpha)_{f(p)}(F|_p(\xi_1), \dots, F|_p(\xi_k)) \\ &= \alpha_{g(f(p))}(G|_{f(p)}(F|_p(\xi_1)), \dots, G|_{f(p)}(F|_p(\xi_k))) \\ &= (G \circ F)^*(\alpha)_p(\xi_1, \dots, \xi_k). \end{aligned}$$

$\square$

Corollary 2.31. *Let $F: A_1 \rightarrow A_2$ and $G: A_2 \rightarrow A_3$ be Lie algebroid morphisms as in Lemma 2.30. The map*

$$\begin{aligned} H^\bullet(F): H^\bullet(A_2) &\rightarrow H^\bullet(A_1) \\ [\alpha] &\mapsto [F^* \alpha] \end{aligned}$$

is well-defined. The brackets mark the respective equivalence classes.

Moreover, $H^\bullet(\cdot)$ is (contravariantly) functorial, meaning that

$$H^\bullet(\text{id}_{A_1}) = \text{id}_{H^\bullet(A)}$$

and

$$H^\bullet(G \circ F) = H^\bullet(F) \circ H^\bullet(G).$$

2.5 The Pullback Lie Algebroid

Let $f: M \rightarrow N$ be a map between the manifolds M and N . Let $(A, \rho, [\cdot, \cdot]_A)$ be a Lie algebroid over N . Suppose that the fiber product

$$f^!A := A \times_{TN} TM = \{(a, v) \in A \times TM \mid \rho(a) = df(v)\}$$

with projection

$$\begin{aligned} \pi_{f^!A}: \quad & f^!A \rightarrow M \\ & (a, v) \mapsto p, \text{ where } (a, v) \in A_{f(p)} \times T_p M \end{aligned} \quad (2.19)$$

is a vector subbundle of $\pi: A \times_N TM \rightarrow M$.

Define the anchor

$$\rho_{f^!A} := \text{pr}_2: f^!A \rightarrow TM.$$

Additionally, define the bracket

$$[\cdot, \cdot]_{f^!A}: \Gamma(f^!A) \times \Gamma(f^!A) \rightarrow \Gamma(f^!A)$$

by using the formula

$$[(f^*(\alpha), X), (f^*(\beta), Y)]_{f^!A} := (f^*([\alpha, \beta]_A), [X, Y]) \quad (2.20)$$

for all $\alpha, \beta \in \Gamma(A)$ and $X, Y \in \mathfrak{X}(M)$. The bracket on general sections of $f^!A$ is obtained by extending (2.20) using the Leibniz identity. We get ([24]):

$$\begin{aligned} & \left[\left(\sum_i c_i f^*(\alpha_i), X \right), \left(\sum_j c'_j f^*(\beta_j), Y \right) \right]_{f^!A} \\ &= \left(\sum_{i,j} c_i c'_j f^*([\alpha_i, \beta_j]_A) + \sum_j X(c'_j) f^*(\beta_j) - \sum_i Y(c_i) f^*(\alpha_i), [X, Y] \right) \end{aligned} \quad (2.21)$$

for all $\alpha_i, \beta_j \in \Gamma(A)$, $X, Y \in \mathfrak{X}(M)$ and $c_i, c'_j \in C^\infty(M)$. The sums in (2.21) are finite. This representation is justified by the fact that $\Gamma(f^*A)$ is generated by Remark A.13 and $\Gamma(f^!A) \simeq \Gamma(f^*A) \oplus \Gamma(TM)$.

Definition 2.32 (Pullback algebroid), [7]. The vector bundle $f^!A$ together with $\rho_{f^!A}$ and $[\cdot, \cdot]_{f^!A}$ becomes a Lie algebroid. We call $f^!A$ the *pullback Lie algebroid* of A induced by $f: M \rightarrow N$.

The following diagrams commute by the definition of:

$$\begin{array}{ccc} f^!A & \xrightarrow{\text{pr}_2} & A \\ \pi_{f^!A} \downarrow & \# & \downarrow \pi_A \\ M & \xrightarrow{f} & N \end{array} \qquad \begin{array}{ccc} f^!A & \xrightarrow{\text{pr}_2} & A \\ \rho_{f^!A} \downarrow & \# & \downarrow \rho \\ TM & \xrightarrow{df} & TN \end{array}$$

Furthermore, by comparing (2.21) and (2.12), compatibility with the brackets is fulfilled. Hence, we obtain

Corollary 2.33. *The projection $\text{pr}_2: f^!A \rightarrow A$ is a Lie algebroid morphism covering $f: M \rightarrow N$.*

Let $(A, \rho, [\cdot, \cdot])$ be a Lie algebroid over M with characteristic foliation $\mathcal{F}(A) = \{N_\nu\}$. First, we note that

$$TN_\nu = \rho(A)|_{N_\nu} = \{\rho_p(A_p) \mid p \in N_\nu\}. \quad (2.22)$$

Let N_ν be a leaf with injective immersion $i: N_\nu \hookrightarrow M$. By using (2.22), i induces

$$\begin{aligned} i^!A &= A \times_{TM} TN_\nu = \{(a, v) \in A \times TN_\nu \mid \rho(a) = di(v)\} \\ &\simeq \{a \in A_p \mid p \in N_\nu\} \\ &\simeq i^*A, \end{aligned} \quad (2.23)$$

which is a smooth vector bundle. Hence, we get a well-defined pullback Lie algebroid

$$A|_{N_\nu} := i^!A$$

with anchor

$$\begin{aligned} \rho|_{N_\nu} &:= \rho_{i^!A}: i^*A \rightarrow TN_\nu \\ (a, x) &\mapsto \rho_x(a) \end{aligned}$$

and bracket

$$[(\alpha|_{N_\nu}, X), (\beta|_{N_\nu}, Y)]_{i^!A} := (([\alpha, \beta]_A|_{N_\nu}), [X, Y]),$$

where $\alpha, \beta \in \Gamma(A)$ and $X, Y \in \mathfrak{X}(N_\nu)$. We set $\eta|_{N_\nu} := i^*\eta$ for all $\eta \in \Gamma(A)$.

Remark 2.34. When using Corollary 2.33 and the identification in (2.23), the map $\text{pr}_2: i^!A \rightarrow A$ becomes

$$\begin{aligned} i_{A|_{N_\nu}}: \{a \in A_p \mid p \in N_\nu\} &\hookrightarrow A \\ a &\mapsto a. \end{aligned}$$

The definition of the pullback on forms in (2.13) yields that we can interpret i as a Lie algebroid morphism, namely we obtain

$$i^*\eta = i_{A|_{N_\nu}}^* \eta \quad (2.24)$$

for all $\eta \in \Gamma(A)$.

2.6 The Horizontal Cohomology of a Lie Algebroid

Suppose we have a Lie algebroid $(A, \rho, [\cdot, \cdot])$. After we introduced the cochain complex $(\Gamma(\Lambda^\bullet A^*), d_A)$, we now take a look at an interesting subcomplex, the subcomplex of ρ -horizontal forms.

Definition 2.35 [6]. Let $(A, \rho, [\cdot, \cdot])$ be a Lie algebroid over M . We define

- the subspace of ρ -horizontal multilinear forms at $p \in M$ as

$$(\Lambda^\bullet A_p^*)^{hor} := \left\{ \alpha \in \Lambda^\bullet A_p^* \mid \iota_v \alpha = 0, \text{ for all } v \in \ker(\rho|_p : A_p \rightarrow T_p M) \right\}. \quad (2.25)$$

- the subspace of ρ -horizontal forms as

$$\Gamma(\Lambda^\bullet A^*)^{hor} := \{ \alpha \in \Gamma(\Lambda^\bullet A^*) \mid \alpha_p \text{ and } (d_A \alpha)_p \text{ are horizontal for all } p \in M \}.$$

We set $\Gamma(\Lambda^0 A^*)^{hor} := C^\infty(M)$. Further, $\Gamma(\Lambda^k A^*)^{hor} := 0$ for all $k \in \mathbb{Z}_{<0}$.

- the *horizontal cohomology* of A as the quotient

$$H_{hor}^\bullet(A) := \frac{\ker(d_A : \Gamma(\Lambda^\bullet A^*)^{hor} \rightarrow \Gamma(\Lambda^{\bullet+1} A^*)^{hor})}{\text{Im}(d_A : \Gamma(\Lambda^{\bullet-1} A^*)^{hor} \rightarrow \Gamma(\Lambda^\bullet A^*)^{hor})}.$$

By definition, the exterior differential d_A (see (2.7)) restricts to the ρ -horizontal forms $\Gamma(\Lambda^\bullet A^*)^{hor}$. In particular, $(\Gamma(\Lambda^\bullet A^*)^{hor}, d_A)$ is a subcomplex of $(\Gamma(\Lambda^\bullet A^*), d_A)$ such that $H_{hor}^\bullet(A)$ is well-defined.

Proposition 2.36. Let $(A_1, \rho_1, [\cdot, \cdot]_1)$ and $(A_2, \rho_2, [\cdot, \cdot]_2)$ be Lie algebroids over M_1 and M_2 , respectively. Let $F : A_1 \rightarrow A_2$ be a Lie algebroid morphism covering the map $f : M_1 \rightarrow M_2$.

Then, the cochain map $F^* : \Gamma(\Lambda^\bullet A_2^*) \rightarrow \Gamma(\Lambda^\bullet A_1^*)$ restricts to the ρ -horizontal forms. In particular, it induces a well-defined map

$$H_{hor}^\bullet(F) : H_{hor}^\bullet(A_2) \rightarrow H_{hor}^\bullet(A_1).$$

Proof. For the case when $k = 0$, we obtain

$$\Gamma(\Lambda^0 A_i^*)^{hor} = \Gamma(\Lambda^0 A_i^*) = C^\infty(M_i)$$

for each $i \in \{1, 2\}$. Therefore, we conclude that F^* restricts for the case when $k = 0$.

Now, let $k \in \mathbb{N}_{>0}$ and $\alpha \in \Gamma(\Lambda^k A_2^*)^{hor}$. Further, let $p \in M_1$ and $v \in \ker(\rho_1|_p) \subseteq A_1|_p$. Because the diagram (2.9) commutes, this implies that $F_p(v) \in \ker(\rho_2|_{f(p)})$ and therefore

$$\iota_v(F^* \alpha)_p = \iota_{F_p(v)} \alpha_{f(p)} = 0.$$

Additionally, by using Theorem 2.29 we obtain

$$\iota_v(d_{A_1} F^* \alpha)_p = \iota_v(F^* d_{A_2} \alpha)_p = \iota_{F_p(v)}(d_{A_2} \alpha)_{f(p)} = 0.$$

Note that, in both equations, we used the fact that α is ρ_2 -horizontal. $\square$

Example 2.37. Again, consider the tangent Lie algebroid $(TM, [\cdot, \cdot]_{\mathfrak{X}(M)}, \text{id}_{TM})$ (see Example 2.2). The horizontal and Lie algebroid cohomologies coincide since $\ker(\text{id}_{TM}|_p : T_p M \rightarrow T_p M) = 0$ for all $p \in M$ (see (2.25)). Therefore, using Example 2.20, we conclude that

$$H_{hor}^\bullet(TM) = H_{dR}^\bullet(M) = H^\bullet(TM).$$

Proposition 2.38. *Let $(A_1, \rho_1, [\cdot, \cdot]_1)$ and $(A_2, \rho_2, [\cdot, \cdot]_2)$ be Lie algebroids over M_1 and M_2 , respectively, and a Lie algebroid morphism $F: A_1 \rightarrow A_2$ covering the map $f: M_1 \rightarrow M_2$, as in Proposition 2.36.*

(i) *The anchor map induces a natural map*

$$H_{hor}^\bullet(\rho_1): H_{dR}^\bullet(M_1) \rightarrow H_{hor}^\bullet(A_1)$$

(ii) *We have a well-defined natural map*

$$\begin{aligned} \hat{\iota}_{A_1}: H_{hor}^\bullet(A_1) &\rightarrow H^\bullet(A_1) \\ [[\alpha]] &\mapsto [\alpha], \end{aligned}$$

where $[[\cdot]]$ refers to the equivalence class in the quotient $H_{hor}^\bullet(A_1)$ and $[\cdot]$ in $H^\bullet(A_1)$.

Naturality in (i) and (ii) refers respectively to the commutativity of the diagrams

$$\begin{array}{ccc} H_{dR}^\bullet(M_2) & \xrightarrow{H_{hor}^\bullet(\rho_2)} & H_{hor}^\bullet(A_2) \\ H_{dR}^\bullet(f) \downarrow & \# & \downarrow H_{hor}^\bullet(F) \\ H_{dR}^\bullet(M_1) & \xrightarrow{H_{hor}^\bullet(\rho_1)} & H_{hor}^\bullet(A_1) \end{array} \quad \text{and} \quad \begin{array}{ccc} H_{hor}^\bullet(A_2) & \xrightarrow{\hat{\iota}_{A_2}} & H^\bullet(A_2) \\ H_{hor}^\bullet(F) \downarrow & \# & \downarrow H^\bullet(F) \\ H_{hor}^\bullet(A_1) & \xrightarrow{\hat{\iota}_{A_1}} & H^\bullet(A_1). \end{array}$$

Proof. (i) The anchor map and the tangent map df are Lie algebroid morphisms by Exercises 2.26 and 2.27. Because F is a Lie algebroid morphism, we conclude $F \circ \rho_2 = \rho_1 \circ df$. Naturality then follows from Corollary 2.31.

(ii) By definition, $(\Gamma(\Lambda^\bullet A^*)^{hor}, d_A)$ is a subcomplex of $(\Gamma(\Lambda^\bullet A^*), d_A)$. Therefore, it follows that $\hat{\iota}_{A_1}$ is well-defined. Now, let $\alpha \in \ker(d_A: \Gamma(\Lambda^\bullet A^*)^{hor} \rightarrow \Gamma(\Lambda^{\bullet+1} A^*)^{hor})$. We conclude that

$$F^*(\hat{\iota}_{A_1}[[\alpha]]) = F^*[\alpha] = [F^*(\alpha)] = \hat{\iota}_{A_2}[[F^*(\alpha)]] = \hat{\iota}_{A_2}F^*[[\alpha]].$$

This shows naturality. □

Lemma 2.39. *Let $(A, \rho, [\cdot, \cdot])$ be a transitive Lie algebroid over M . The anchor map induces an isomorphism*

$$H_{dR}^k(M) \simeq H_{hor}^k(A)$$

for all $k \in \mathbb{N}_0$.

Proof. The case when $k = 0$ is clear. Indeed, it holds that

$$\Omega^0(M) = C^\infty(M) = \Gamma(\Lambda^0 A^*)^{hor}.$$

Let $k = 1$. We will show that the induced map $\rho^*: \Omega^1(M) \rightarrow \Gamma(A^*)$ restricts to an isomorphism

$$\rho^*: \Omega^1(M) \xrightarrow{\sim} \Gamma(A^*)^{hor}$$

on complexes. This will show that $H_{dR}^1(M) \simeq H_{hor}^1(A)$. At first, we show that ρ^* indeed maps into the ρ -horizontal forms. Let $\alpha \in \Omega^1(M)$. Since $\rho^*(\alpha) = \alpha(\rho(\cdot))$, the covector $\rho^*(\alpha)_p$ is ρ -horizontal for each $p \in M$. Furthermore, Theorem 2.29 implies that

$$d_A \rho^*(\alpha) = \rho^* d\alpha.$$

Therefore, $d_A \rho^*(\alpha)_p$ is also ρ -horizontal for each $p \in M$, and hence $\text{Im}(\hat{\rho}^*) \subseteq \Gamma(A^*)^{hor}$.

Now, let $\alpha, \beta \in \Omega^1(M)$ such that $\rho^*(\alpha) = \rho^*(\beta)$. Because differential forms are $C^\infty(M)$ -linear, they do only depend on their values at each point $p \in M$. For each $p \in M$, we conclude from

$$\alpha_p(\rho_p(\cdot)) = \beta_p(\rho_p(\cdot))$$

that α and β coincide because ρ_p is onto. It follows that ρ^* is injective.

Let $\rho': T^*M \rightarrow A^*$ be the dual bundle homomorphism of ρ . Because ρ is onto, ρ' has constant rank and is injective. We therefore obtain a vector bundle isomorphism

$$\rho': T^*M \xrightarrow{\sim} \text{Im}(\rho')$$

from which it follows that

$$\Gamma(\text{Im}(\rho')) = \text{Im}(\rho^*). \quad (2.26)$$

Because ρ is surjective, it has constant rank and hence $\ker(\rho) \subseteq A$ is a vector subbundle. Furthermore, vector bundle theory implies that

$$\ker \rho^\circ = \text{Im}(\rho'), \quad (2.27)$$

where $\ker \rho^\circ$ denotes the annihilator subbundle

$$\ker \rho^\circ = \coprod_{p \in M} \{a \in A_p^* \mid a(v) = 0 \text{ for all } v \in \ker \rho_p\} \subseteq A^*.$$

Let $\omega \in \Gamma(A^*)^{hor}$. It follows that ω is a section of $\ker \rho^\circ$. By combining (2.26) and (2.27), we conclude that $\omega \in \text{Im}(\rho^*)$. Hence, $\text{Im}(\rho^*) = \Gamma(A^*)^{hor}$.

The discussion for a general $k \in \mathbb{N}$ is analogous. The surjectivity may not be immediately clear. The idea is to first show $\Gamma(\Lambda^k A^*)^{hor} \subseteq \Gamma(\Lambda^k(\ker(\rho)^\circ))$. Then, using $\Lambda^k(\ker(\rho)^\circ) = \text{Im}(\Lambda^k \rho')$ and $\Gamma(\text{Im}(\Lambda^k \rho')) = \text{Im}(\rho^*)$, one shows that any element of $\Gamma(\Lambda^k A^*)^{hor}$ lies in the image of

$$\rho^*: \Omega^k(M) \rightarrow \Gamma(\Lambda^k A^*)^{hor}.$$

□

Lemma 2.40 [6]. *Let $(A, \rho, [\cdot, \cdot])$ be a Lie algebroid over the manifold M . Let $N \subseteq M$ be a leaf of A with immersion $i: N \hookrightarrow M$, and let $\eta \in \Gamma(\Lambda^k A^*)^{hor}$ a ρ -horizontal form. Then the following hold:*

- (i) $\eta|_N$ is a $\rho|_N$ -horizontal k -form on $A|_N$ and induces a unique k -form $\eta_N \in \Omega^k(N)$.
- (ii) η is completely determined by the collection $\{\eta_N \mid N \text{ leaf of } A\}$.
- (iii) The exterior differential respects the construction in (i), i.e., $(d_A \eta)_N = d(\eta_N)$.
- (iv) Let $[\eta] = 0 \in H_{hor}^k(A)$, then $[\eta_N] = 0 \in H_{dR}^k(N)$.

Proof. (i) Let $p \in N$ and $v \in \ker((\rho|_N)_p: A_p \rightarrow T_p N \subseteq T_p M)$. Then,

$$\iota_v(\eta|_N)_p = \iota_v(i^*\eta)_p = \iota_v\eta_p = 0.$$

When we consider (2.22), we deduce that $(A|_N, \rho_N, [\cdot, \cdot]_{i^!A})$ is transitive. Hence, Lemma 2.39 provides a unique k -form $\eta_N \in \Omega^k(M)$, such that $\rho_N^*(\eta_N) = \eta|_N$.

(ii) Let $p \in M$. Let $i: N \hookrightarrow M$ be the leaf through p . Let $v_j \in A_p$, where $j \in \{1, \dots, k\}$. Remark 2.34 yields

$$\begin{aligned} \eta_p(v_1, \dots, v_k) &= (i_{A|_N}^* \eta)_p(v_1, \dots, v_k) \\ &= (i^* \eta)_p(v_1, \dots, v_k) \\ &= (\eta|_N)_p(v_1, \dots, v_k) \\ &= \rho_N^*(\eta_N)_p(v_1, \dots, v_k) \\ &= (\eta_N)_p((\rho|_N)_p(v_1), \dots, (\rho|_N)_p(v_k)). \end{aligned}$$

Subsequently η is determined by the collection $\{\eta_N \mid N \text{ leaf of } A\}$.

(iii) First, let $k \in \mathbb{N}_0$ and $[\eta] = 0 \in H_{hor}^k(A)$. Because the immersion $i: N \hookrightarrow M$ can be seen as a Lie algebroid morphism (Remark 2.34), we conclude, by the naturality of d_A (see Theorem 2.29), that

$$\rho_N^*((d_A \eta)_N) = (d_A \eta)|_N = i^*(d_A \eta) = d_{A|_N}(i^* \eta) = d_{A|_N}(\eta|_N).$$

But $\rho|_N$ is also a Lie algebroid morphism (see Exercise 2.26). Therefore,

$$d_{A|_N}(\eta|_N) = d_{A|_N}(\rho_N^*(\eta_N)) = \rho_N^*(d(\eta_N)).$$

Since $A|_N$ is transitive, Lemma 2.39 provides that ρ_N^* is an isomorphism. Hence the preceding equations imply that $(d_A \eta)_N = d(\eta_N)$.

(iv) We start with the case when $k = 0$. Let

$$[f] = 0 \in H_{hor}^0(A) = \ker(d_A: C^\infty(M) \rightarrow \Gamma(A^*)^{hor}).$$

Therefore, $d_A f = 0$. Subsequently, with (iii), we get $0 = (d_A f)_N = d(f_N)$, where the first equality follows by uniqueness (see (i)). Hence, $f_N \in H_{dR}^0(N) = \ker(d: C^\infty(M) \rightarrow \Omega^1(M))$.

Now, let $k \in \mathbb{N}_{>0}$ and $[\eta] = 0 \in H_{hor}^k(A)$. It follows that there exists $\zeta \in H_{hor}^{k-1}(A)$ such that $d_A \zeta = \eta$. From (iii) it follows that

$$\rho_N^*(\eta_N) = \rho_N^*((d_A \zeta)_N) = \rho_N^*(d(\zeta_N)).$$

Since $A|_N$ is transitive, the same argument as above implies that $\eta_N = d(\zeta_N)$, and the claim follows. □

Chapter 3

Dirac Structures

We denote the Whitney sum of the tangent and the cotangent bundle of M with

$$\mathbb{T}M := TM \oplus T^*M$$

and refer to it as the *generalized tangent bundle* of M as it is called in [10] or the *Pontryagin bundle* as it is called in [29].

There are the canonical projections pr_1 and pr_2 on $\mathbb{T}M$:

$$\begin{array}{ccc} & \mathbb{T}M & \\ \text{pr}_1 \swarrow & & \searrow \text{pr}_2 \\ TM & & T^*M. \end{array} \quad (3.1)$$

Using the isomorphism $\Gamma(TM \oplus T^*M) \simeq \Gamma(TM) \oplus \Gamma(T^*M)$, we denote a section s of $\mathbb{T}M$ as $s = (X, \alpha)$, where $X \in \mathfrak{X}(M)$ and $\alpha \in \Omega^1(M)$.

The Pontryagin bundle $\mathbb{T}M$ has a natural pairing on the space of sections $\Gamma(\mathbb{T}M)$, namely

$$\begin{aligned} \langle \cdot, \cdot \rangle : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) &\rightarrow C^\infty(M) \\ ((X, \alpha), (Y, \beta)) &\mapsto \iota_Y \alpha + \iota_X \beta. \end{aligned} \quad (3.2)$$

Furthermore, there is a bracket on $\Gamma(\mathbb{T}M)$: the *Courant bracket*

$$\begin{aligned} [\![\cdot, \cdot]\!] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) &\rightarrow \Gamma(\mathbb{T}M) \\ ((X, \alpha), (Y, \beta)) &\mapsto \left([X, Y], \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + \frac{1}{2} d(\alpha(Y) - \beta(X)) \right). \end{aligned} \quad (3.3)$$

We follow [4].

Definition 3.1. Let $D \subseteq \mathbb{T}M$ be a vector subbundle.

D is said to be an *almost Dirac structure* if it is isotropic, i.e.,

$$D = D^\perp, \text{ where the } \perp \text{ is wrt. } \langle \cdot, \cdot \rangle. \quad (3.4)$$

A subbundle D which satisfies condition (3.4) is also called a Lagrangian subbundle.

Further, an isotropic subbundle $D \subseteq \mathbb{T}M$ is called a *Dirac structure* if it is involutive with respect to the Courant bracket, i.e.,

$$[\![\Gamma(D), \Gamma(D)]\!] \subseteq \Gamma(D). \quad (3.5)$$

Condition (3.5) is referred to as the *integrability condition*.

Let A be a vector bundle. Given a bracket $(\cdot, \cdot) : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$, we define the *Jacobiator* of $a, b, c \in \Gamma(A)$ with respect to $(\cdot, \cdot)$ by:

$$\text{Jac}_{(\cdot, \cdot)}(a, b, c) := ((a, b), c) + ((c, a), b) + ((b, c), a).$$

There is another bracket on the Pontryagin-bundle $\mathbb{T}M$.

Definition 3.2. The *Dorfman bracket* is the bilinear map

$$\begin{aligned} [\cdot, \cdot]_D : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) &\rightarrow \Gamma(\mathbb{T}M) \\ ((X, \alpha), (Y, \beta)) &\mapsto ([X, Y], \mathcal{L}_X \beta - \iota_Y d\alpha). \end{aligned} \quad (3.6)$$

Note that, using Cartan's formula, one verifies that the Courant bracket, defined in (3.3), is the skew-symmetrization of the Dorfman bracket. In what follows, it may be helpful to recall the identities of Cartan calculus (see Proposition C.1).

Lemma 3.3. *Let $D \subseteq \mathbb{T}M$ be a subbundle. Then the following holds:*

- (i) *Let D be an almost Dirac structure. The Dorfman and the Courant bracket coincide on sections of D .*
- (ii) *The Dorfman bracket fulfils the Leibniz rule on sections of D :*

$$[s, fs']_D = f[s, s']_D + \rho_*(s)(f) \cdot s' \text{ for all } f \in C^\infty(M) \text{ and } s, s' \in \Gamma(D),$$

where $\rho := \text{pr}_1$ (see (3.1)).

Proof. (i) Let $(X, \alpha), (Y, \beta) \in \Gamma(D)$. As D is a Lagrangian subbundle (see 3.4), we conclude that $\alpha(X) = -\beta(Y)$. Hence

$$\begin{aligned} \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + \frac{1}{2} d(\alpha(Y) - \beta(X)) &= \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + d\iota_Y \alpha \\ &= \mathcal{L}_X \beta - \iota_Y d\alpha. \end{aligned}$$

We have used Cartan's formula in the last equation.

(ii) Let $a_1 = (X, \alpha), a_2 = (Y, \beta) \in \Gamma(D)$ and $f \in C^\infty(M)$. Then,

$$\begin{aligned} [a_1, fa_2]_D &= [(X, \alpha), (fY, f\beta)]_D \\ &= ([X, fY], \mathcal{L}_X f\beta - \iota_{fY} d\alpha) \\ &= (f[X, Y] + X(f)Y, X(f)\beta + f\mathcal{L}_X \beta - f\iota_Y d\alpha) \\ &= f([X, Y], \mathcal{L}_X \beta - \iota_Y d\alpha) + X(f)(Y, \beta) \\ &= f[a_1, a_2]_D + (\text{pr}_1)_*(a_1)(f)(a_2). \end{aligned}$$

□

Definition 3.4. Let $D \subseteq \mathbb{T}M$ be an almost Dirac structure. Let Υ_D be a map defined by

$$\begin{aligned} \Upsilon_D : \Gamma(D) \times \Gamma(D) \times \Gamma(D) &\rightarrow C^\infty(M) \\ (a_1, a_2, a_3) &\mapsto \langle [a_1, a_2], a_3 \rangle. \end{aligned} \quad (3.7)$$

The map in (3.7) determines a section $\Upsilon_D \in \Gamma(\Lambda^3 D^*)$, which we call the *Courant tensor* of D .

For an almost Dirac structure $D \subseteq TM$, the map Υ_D indeed determines a tensor $\Upsilon_D \in \Gamma(\Lambda^3 D^*)$. We have to check that it is $C^\infty(M)$ -multilinear and alternating. Let $a_1, a_2, a_3 \in \Gamma(D)$ and $f \in C^\infty(M)$ and denote $a_1 = (X, \alpha)$, $a_2 = (Y, \beta)$, $a_3 = (Z, \gamma)$. On $\Gamma(D)$ the Courant bracket coincides with the Dorfman bracket (see Lemma 3.3). Therefore, we now compute:

$$\begin{aligned} \Upsilon_D(a_1, fa_2, a_3) &= \langle [a_1, fa_2]_D, a_3 \rangle \\ &= \langle (f[X, Y] + X(f)Y, \mathcal{L}_X f\beta - \iota_{fY} d\alpha), a_3 \rangle \\ &= \langle (f[X, Y] + X(f)Y, X(f)\beta + f\mathcal{L}_X \beta - f\iota_Y d\alpha), a_3 \rangle \quad (3.8) \\ &= \langle (f[X, Y], f\mathcal{L}_X \beta - f\iota_Y d\alpha), a_3 \rangle + X(f) \langle a_2, a_3 \rangle \\ &= f\Upsilon_L(a_1, a_2, a_3), \end{aligned}$$

where we used that $\langle a_2, a_3 \rangle = 0$. D is a Lagrangian subbundle. Therefore $\iota_Y \gamma = -\iota_Z \beta$, and hence

$$\begin{aligned} \Upsilon_D(a_1, a_2, a_3) &= \langle ([X, Y], \mathcal{L}_X \beta - \iota_Y d\alpha), (Z, \gamma) \rangle \\ &= \iota_{[X, Y]} \gamma + \iota_Z \mathcal{L}_X \beta - \iota_Z \iota_Y d\alpha \\ &= \mathcal{L}_X \iota_Y \gamma - \iota_Y \mathcal{L}_X \gamma + \iota_Z \mathcal{L}_X \beta - \iota_Z \iota_Y d\alpha \quad (3.9) \\ &= -(\iota_{[X, Z]} \beta - \iota_Y \mathcal{L}_X \gamma - \iota_Y \iota_Z d\alpha) \\ &= -\Upsilon_L(a_1, a_3, a_2). \end{aligned}$$

Combining the following facts: the natural pairing $\langle \cdot, \cdot \rangle$ is bilinear; the Courant bracket $[\cdot, \cdot]$ is bilinear and skew-symmetric; and Equations (3.8) and (3.9) hold, we conclude that Υ_D is alternating and $C^\infty(M)$ -multilinear. Hence it defines a tensor $\Upsilon_D \in \Gamma(\Lambda^3 D^*)$.

The Courant tensor provides a criterion for involutivity:

Proposition 3.5. *An almost Dirac structure $D \subseteq TM$ is involutive if and only if $\Upsilon_D \equiv 0$.*

Proof. " $\Rightarrow$ ": It holds that $D = D^\perp$. Involutivity of D then implies that $\Upsilon_D \equiv 0$.

" $\Leftarrow$ ": Now, suppose that $\Upsilon_D \equiv 0$. Let $a_1, a_2 \in \Gamma(D)$. We will show that $[[a_1, a_2]] \in \Gamma(D^\perp) = \Gamma(D)$, which will show the claim. Let $p \in M$ and $v \in D$ be arbitrary. Using the construction of the vector fields in Lemmas 2.15 and 2.16, there exists a section $a \in \Gamma(D)$ such that $a_p = v$. Since $p \in M$ and $v \in D$ were arbitrary, the identity

$$\langle [[a_1, a_2]], a \rangle = 0$$

implies that $[[a_1, a_2]] \in \Gamma(D^\perp)$. □

Remark 3.6. A Dirac Structure $D \subseteq TM$ inherits a Lie algebroid structure in a natural way:

Recall the canonical projection $\text{pr}_1: TM \oplus T^*M \rightarrow TM$. We define a Lie algebroid structure on D by taking the anchor map to be the restriction of pr_1 to D , and the bracket to be the restriction of the Courant bracket $[\cdot, \cdot]$ to sections of D , i.e.,

$$\rho := \text{pr}_1|_D : D \rightarrow TM$$

and

$$[\cdot, \cdot] := [[\cdot, \cdot]]|_{\Gamma(D)} : \Gamma(D) \times \Gamma(D) \rightarrow \Gamma(D).$$

Note that involutivity of D with respect to the Courant bracket ensures that the bracket $[\cdot, \cdot]$ is well-defined. $(D, \rho, [\cdot, \cdot])$ is a Lie algebroid. Indeed, this is a consequence of the Lemmas [3.3](#) and [3.2](#). According to Definition [2.18](#), the Lie algebroid D determines a characteristic foliation $\mathcal{F}(D)$. As we will see later, each leaf of $\mathcal{F}(D)$ carries a naturally induced presymplectic structure.

3.1 Examples of Dirac Structures

The main references for this chapter is [\[4\]](#) with examples adopted from [\[6\]](#). Here, it may be helpful to remind the fomulas of [Cartan-Calculus](#).

Example 3.7. Let M be a manifold and let $\omega \in \Omega^2(M)$ be a 2-form. As it is done for Riemannian metrics with the musical isomorphisms (see [\[16\]](#)) we define a bundle map $\omega^\sharp: TM \rightarrow T^*M$ as follows. For each $p \in M$ and each $v \in T_pM$ we define the covector $\omega^\sharp(v) \in T_p^*M$ by

$$\omega^\sharp(v)(w) = \omega_p(v, w), \quad \text{for all } w \in T_pM.$$

Let $X \in \mathfrak{X}(M)$. Then, we can define

$$\begin{aligned} \omega_*^\sharp(X): \mathfrak{X}(M) &\rightarrow C^\infty(M) \\ Y &\mapsto \omega(X, Y). \end{aligned}$$

As the map $\omega_*^\sharp(X)$ is $C^\infty(M)$ -linear, the *tensor characterization lemma* (see [\[16\]](#)) implies that $\omega_*^\sharp(X) \in \Omega^1(M)$. Furthermore, since $\omega_*^\sharp: \mathfrak{X}(M) \rightarrow \Omega^1(M)$ is $C^\infty(M)$ -linear, the *bundle homomorphism characterization lemma* (see [\[16\]](#)) implies that $\omega^\sharp: TM \rightarrow T^*M$ is a (smooth) bundle homomorphism.

Now, we take the graph of $\omega^\sharp$ and obtain a subbundle

$$D_\omega := \text{graph}(\omega^\sharp) = \{(X, \omega^\sharp(X)) \mid X \in TM\} \simeq TM \subseteq \mathbb{T}M.$$

Let $a_1 := (X, \omega_*^\sharp(X)), a_2 := (Y, \omega_*^\sharp(Y)), a_3 := (Z, \omega_*^\sharp(Z)) \in \Gamma(D_\omega)$. Then,

$$\begin{aligned} \langle (X, \omega_*^\sharp(X)), (Y, \omega_*^\sharp(Y)) \rangle &= \iota_X \omega_*^\sharp(Y) + \iota_Y \omega_*^\sharp(X) \\ &= \omega(X)(Y) + \omega(Y)(X) = 0. \end{aligned}$$

Because $\text{rank}(D_\omega) = \dim(M)$, we conclude that D_ω is an almost Dirac structure. Additionally, according to Proposition [3.3](#), we calculate the Courant tensor of D_ω :

$$\begin{aligned} \Upsilon_{D_\omega}(a_1, a_2, a_3) &= \left\langle ([X, Y], \mathcal{L}_X \omega^\sharp(Y) - \iota_Y d\omega^\sharp(X)), (Z, \omega_*^\sharp(Z)) \right\rangle \\ &= \iota_{[X, Y]} \iota_Z \omega + \iota_Z \mathcal{L}_X \iota_Y \omega - \iota_Z \iota_Y d\iota_X \omega \\ &= \iota_{[X, Y]} \iota_Z \omega + \iota_Z \mathcal{L}_X \iota_Y \omega - \iota_Z \iota_Y \mathcal{L}_X \omega + \iota_Z \iota_Y \iota_X d\omega \\ &= \underbrace{\iota_{[X, Y]} \iota_Z \omega + \iota_Z \iota_{[X, Y]} \omega}_{=0} + \iota_Z \iota_Y \iota_X d\omega \\ &= d\omega(X, Y, Z). \end{aligned}$$

We conclude that D_ω is a Dirac structure if and only if ω is presymplectic, i.e., $d\omega = 0$.

Remark 3.8. We will call a presymplectic form ω for which the map $\omega^\sharp$ from Example [3.7](#) is an isomorphism, a symplectic structure and the tuple (M, ω) a symplectic manifold. These notions are introduced in greater detail in the chapter on [Classical Mechanics](#).

Remark 3.9. Let $\omega \in \Omega^2(M)$ be a presymplectic structure on M . The Dirac structure D_ω induces a natural Lie algebroid structure (see Remark 3.6). In fact, we recover the tangent Lie algebroid (see Example 2.2).

Indeed, for $(X, \omega_*^\sharp(X)), (Y, \omega_*^\sharp(Y)) \in \Gamma(L_\omega)$ we obtain:

$$\begin{aligned} \llbracket (X, \omega_*^\sharp(X)), (Y, \omega_*^\sharp(Y)) \rrbracket &= ([X, Y], \mathcal{L}_X \iota_Y \omega - \iota_Y d\iota_X \omega) \\ &= ([X, Y], \mathcal{L}_X \iota_Y \omega - \iota_Y \iota_X d\omega - \iota_Y d\iota_X \omega) \\ &= ([X, Y], \mathcal{L}_X \iota_Y \omega - \iota_Y \mathcal{L}_X \omega) \\ &= ([X, Y], \omega_*^\sharp([X, Y])). \end{aligned}$$

Example 3.10. Let $\pi \in \mathfrak{X}^2(M)$ be a bivector field on M . Using the same arguments as in Example 3.7, we obtain a bundle map $\pi^\sharp: T^*M \rightarrow TM$. Concretely, for each $p \in M$ and each $\varphi \in T_p^*M$ we define the tangent vector $\pi^\sharp(\varphi) \in T_p M \simeq T_p^*M$ by

$$\pi^\sharp(\varphi)(\psi) = \pi_p(\varphi, \psi), \quad \text{for all } \psi \in T_p^*M, \quad (3.10)$$

where we used the isomorphism $\text{ev}: T_p M \xrightarrow{\sim} T_p^*M$, $v \mapsto (\varphi \mapsto \varphi(v))$.

Now, we take the graph of $\pi^\sharp$ and get a subbundle

$$D_\pi := \text{graph}(\pi^\sharp) = \{(\pi^\sharp(\alpha), \alpha) \mid \alpha \in T^*M\} \simeq T^*M \subseteq \mathbb{T}M.$$

Let $(\pi_*^\sharp(\alpha), \alpha), (\pi_*^\sharp(\beta), \beta) \in \Gamma(D_\omega)$ then

$$\begin{aligned} \langle (\pi_*^\sharp(\alpha), \alpha), (\pi_*^\sharp(\beta), \beta) \rangle &= \iota_{\pi_*^\sharp(\beta)} \alpha + \iota_{\pi_*^\sharp(\alpha)} \beta \\ &= \pi(\alpha, \beta) + \pi(\beta, \alpha) = 0. \end{aligned}$$

Because $\text{rank}(D_\pi) = \dim(M)$ we conclude that D_π is an almost Dirac structure.

We calculate the Courant tensor of D_π : Let $f_1, f_2, f_3 \in C^\infty(M)$. We denote $X_{f_i} := \pi_*^\sharp(df_i)$ and $a_i := (X_{f_i}, df_i)$, where $i \in \{1, 2, 3\}$. We conclude that

$$\begin{aligned} \Upsilon_{D_\pi}(a_1, a_2, a_3) &= \langle \llbracket (X_{f_1}, df_1), (X_{f_2}, df_2) \rrbracket, (X_{f_3}, df_3) \rangle \\ &= \langle ([X_{f_1}, X_{f_2}], \mathcal{L}_{X_{f_1}} df_2 - \iota_{X_{f_2}} ddf_1), (X_{f_3}, df_3) \rangle \\ &= df_3([X_{f_1}, X_{f_2}]) + \mathcal{L}_{X_{f_1}} df_2(X_{f_3}) \\ &= [X_{f_1}, X_{f_2}](f_3) + d(X_{f_1}(f_2))(X_{f_3}) \\ &= X_{f_1}(X_{f_2}(f_3)) - X_{f_2}(X_{f_1}(f_3)) + X_{f_3}(X_{f_1}(f_2)) \end{aligned} \quad (3.11)$$

By using Equation (3.10), first note that

$$\begin{aligned} \pi(df_1, df_2) &= \pi_*^\sharp(df_1)(df_2) \\ &= df_2(\pi_*^\sharp(df_1)) \\ &= df_2(X_{f_1}) \\ &= X_{f_1}(f_2). \end{aligned}$$

We denote $\{f_i, f_j\} := \pi(df_i, df_j)$, where $i, j \in \{1, 2, 3\}$. It follows that we can rewrite (3.11) as

$$\Upsilon_{D_\pi}(a_1, a_2, a_3) = \{f_1, \{f_2, f_3\}\} + \{f_2, \{f_1, f_3\}\} + \{f_3, \{f_1, f_2\}\}.$$

Therefore, because the Courant tensor is $C^\infty(M)$ -linear, Proposition 3.5 implies that D_π is a Dirac structure if and only if π is a Poisson structure, i.e., the induced bracket

$$\{\cdot, \cdot\} := \pi(d(\cdot), d(\cdot)): C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$$

satisfies the Jacobi identity.

Remark 3.11. Let $\pi \in \mathfrak{X}^2(M)$ be Poisson structure on M . The Dirac structure D_π induces a Lie algebroid structure (see Remark 3.6). In fact, we recover the cotangent Lie algebroid (see Example 2.3). Indeed, for $(\pi_*^\sharp(\alpha), \alpha), (\pi_*^\sharp(\beta), \beta) \in \Gamma(D_\pi)$, using that the anchor $\pi_*^\sharp$ is a Lie algebroid homomorphism, we obtain:

$$\begin{aligned} \llbracket (\pi_*^\sharp(\alpha), \alpha), (\pi_*^\sharp(\beta), \beta) \rrbracket &= ([\pi_*^\sharp(\alpha), \pi_*^\sharp(\beta)], \mathcal{L}_{\pi_*^\sharp(\alpha)}\beta - \iota_{\pi_*^\sharp(\beta)}d\alpha) \\ &= (\pi_*^\sharp([\alpha, \beta]_\pi), \iota_{\pi_*^\sharp(\alpha)}d\beta + d\iota_{\pi_*^\sharp(\alpha)}\beta - \iota_{\pi_*^\sharp(\beta)}d\alpha) \\ &= (\pi_*^\sharp([\alpha, \beta]_\pi), \iota_{\pi_*^\sharp(\alpha)}d\beta - \iota_{\pi_*^\sharp(\beta)}d\alpha + d\pi(\alpha, \beta)) \\ &= (\pi_*^\sharp([\alpha, \beta]_\pi), [\alpha, \beta]_\pi). \end{aligned}$$

Consequently, Dirac structures generalize both presymplectic and Poisson structures. We will revisit Poisson structures and symplectic structures (which are, in particular, presymplectic) later.

Let $F \subseteq TM$ be a vector subbundle, i.e., a regular distribution. We define the annihilator of F by

$$F^\circ := \{(x, \varphi) \in T^*M \mid \varphi(v) = 0, \text{ for all } v \in F_x\},$$

which is a subbundle of T^*M . Define $D := F \oplus F^\circ \subseteq \mathbb{T}M$. Clearly, D is an almost Dirac structure. Further, we obtain

Proposition 3.12. *Let $F \subseteq TM$ be a regular distribution. The almost Dirac structure $D = F \oplus F^\circ$ is a Dirac structure if and only if F is involutive.*

Proof. " $\Rightarrow$ ": Suppose D is integrable. Let $X, Y \in \Gamma(F)$ be arbitrary. For all $\alpha, \beta \in \Gamma(F^\circ)$,

$$\llbracket (X, \alpha), (Y, \beta) \rrbracket = ([X, Y], \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \frac{1}{2}d(\alpha(Y) - \beta(X))) \in \Gamma(D).$$

In particular, $[X, Y] \in \Gamma(F)$. Since X and Y were arbitrary, F is involutive.

" $\Leftarrow$ ": Now, suppose that F is involutive. Let $a_1 := (X, \alpha), a_2 := (Y, \beta), a_3 := (Z, \gamma) \in \Gamma(D)$. We compute:

$$\begin{aligned} \Upsilon_D(a_1, a_2, a_3) &= \langle \llbracket (X, \alpha), (Y, \beta) \rrbracket, (Z, \gamma) \rangle \\ &= \underbrace{\gamma([X, Y])}_{=0} + \iota_Z\mathcal{L}_X\beta - \iota_Z\mathcal{L}_Y\alpha \\ &= \iota_Z\mathcal{L}_X\beta + \iota_Zd\iota_X\beta - \iota_Z\mathcal{L}_Y\alpha \\ &= d\beta(X, Z) + \underbrace{Z(\beta(X))}_{=0} - d\alpha(Y, Z) \\ &= X(\beta(Z)) - Z(\beta(X)) - \beta([X, Z]) \\ &\quad - Y(\alpha(Z)) - Z(\alpha(Y)) - \alpha([Y, Z]) \\ &= 0. \end{aligned}$$

Each of the last terms vanishes because the brackets take values in F and $\alpha, \beta, \gamma \in \Gamma(F^\circ)$. $\square$

Therefore, according to the Frobenius Theorem, regular foliations are particular cases of Dirac structures.

3.2 Presymplectic Leaves

Here we follow [4]. Recall the situation described in Remark 3.6. Let $D \subseteq TM$ be a Dirac structure over M , which induces a natural Lie algebroid structure. The distribution

$$R := \text{pr}_1(D) \subseteq TM \quad (3.12)$$

defines the characteristic singular foliation $\mathcal{F}(D)$.

Lemma 3.13. *The annihilator of the distribution R in (3.12) fulfils*

$$R^\circ = D \cap T^*M.$$

Proof. Let $p \in M$. Then,

$$R_p^\circ = \{\alpha \in T^*M \mid \alpha(v) = 0 \text{ for all } v \in \text{pr}_1(D)_p \subseteq T_pM\}.$$

Let $\alpha \in R_p^\circ$. For every $(v, \beta) \in D_p$, we conclude that

$$\langle (0, \alpha), (v, \beta) \rangle = \alpha(v) + \beta(0) = 0.$$

Hence, $(0, \alpha) \in D_p^\perp = D_p$, and therefore $(0, \alpha) \in (D \cap T^*M)_p$. Conversely, if $(0, \alpha) \in (D \cap T^*M)_p$, then

$$0 = \langle (0, \alpha), (v, \beta) \rangle = \alpha(v)$$

for every $(v, \beta) \in D_p$. Consequently, $\alpha \in R_p^\circ$, proving the claim. $\square$

Let $N \subseteq M$ be a leaf of $\mathcal{F}(D)$. Let $p \in N$. Then, we define

$$\begin{aligned} \Omega_N|_p: T_pN \times T_pN &\rightarrow \mathbb{R} \\ (v, w) &\mapsto \alpha(w), \text{ where } (v, \alpha) \in D_p. \end{aligned} \quad (3.13)$$

Lemma 3.14. *The map in 3.13 is well-defined. Furthermore, by defining*

$$\begin{aligned} \Omega_N: N &\rightarrow \Lambda^2(T_p^*N) \\ p &\mapsto \Omega_N|_p, \end{aligned}$$

we obtain a presymplectic form $\Omega_N \in \Omega^2(N)$ on N .

Proof. Let $p \in N$ and $(v, \alpha), (v, \beta) \in D_p$. Then, $(0, \alpha - \beta) \in D_x$. By Lemma 3.13, it follows that

$$(\alpha - \beta)(w) = 0, \text{ for all } w \in T_pN = R_p.$$

It follows that (3.13) is well-defined.

Now, we show that $\Omega_N \in \Omega^2(N)$, i.e., Ω_N is smooth. The distribution R in (3.12) is integrable. The restricted map $\text{pr}_1|_N: D|_N \rightarrow TN$ is therefore a surjective bundle homomorphism. In particular, it has constant rank. The kernel of a vector bundle map

with constant rank is a vector bundle (see [16]). Hence, we obtain a split short exact sequence of vector bundles:

$$0 \longrightarrow \ker(\mathrm{pr}_1|_N) \longrightarrow D|_N \xrightarrow{\mathrm{pr}_1|_N} TN \longrightarrow 0$$

Therefore, there exists a bundle map $\sigma: TN \rightarrow D|_N$ such that $\mathrm{pr}_1|_N \circ \sigma = \mathrm{id}_{TN}$. Post-composing with $(\mathrm{pr}_2|_N)_*: \Gamma(D|_N) \rightarrow \Gamma(T^*N)$ yields a map

$$\hat{\sigma}_* := (\mathrm{pr}_2|_N)_* \circ \sigma_*: \Gamma(TN) \rightarrow \Gamma(T^*N).$$

Hence, we can define a map

$$\begin{aligned} \hat{\Omega}_N: \mathfrak{X}(N) \times \mathfrak{X}(N) &\rightarrow C^\infty(N) \\ (X, Y) &\mapsto \hat{\sigma}_*(X)(Y), \end{aligned} \tag{3.14}$$

which is $C^\infty(N)$ -linear. By the tensor characterization Lemma (see [16]) Ω_D is indeed smooth. Let $X \in \mathfrak{X}(N)$. Note that the vector field component of $\sigma_*(X)$ is precisely X because σ is a right inverse of the projection map $\mathrm{pr}_1|_N$.

It remains to show that Ω_N is closed. Let $X_1, X_2, X_3 \in \Gamma(TN)$. Using the same argument as above, there exists $\alpha_i \in \Gamma(T^*N)$ such that $a_i := (X_i, \alpha_i) \in \Gamma(D|_N)$ for all $i \in \{1, 2, 3\}$. By Proposition 3.5, the Courant Tensor $\Upsilon_{D|_N}$ vanishes. Hence,

$$0 = \Upsilon_{D|_N}(a_1, a_2, a_3) = \iota_{[X_1, X_2]}\alpha_3 + \iota_{X_3}\mathcal{L}_{X_1}\alpha_2 - \iota_{X_3}\iota_{X_2}d\alpha_1. \tag{3.15}$$

We compute:

$$\begin{aligned} d\Omega_N(X_1, X_2, X_3) &= \iota_{X_1}d(\Omega_N(X_2, X_3)) - \iota_{X_2}d(\Omega_N(X_1, X_3)) + \iota_{X_3}d(\Omega_N(X_1, X_2)) \\ &\quad + \Omega_N(X_3, [X_1, X_2]) - \Omega_N(X_2, [X_1, X_3]) + \Omega_N(X_1, [X_2, X_3]) \\ &= \iota_{X_1}d\iota_{X_3}\alpha_2 - \iota_{X_2}d\iota_{X_3}\alpha_1 + \iota_{X_3}d\iota_{X_2}\alpha_1 \\ &\quad + \iota_{[X_1, X_2]}\alpha_3 - \iota_{[X_1, X_3]}\alpha_2 + \iota_{[X_2, X_3]}\alpha_1 \\ &= \iota_{[X_1, X_2]}\alpha_3 + \iota_{X_3}\mathcal{L}_{X_1}\alpha_2 - \iota_{X_3}\iota_{X_2}d\alpha_1 \\ &= 0, \end{aligned}$$

where the last equality follows from (3.15). $\square$

In summary, given a leaf N of the characteristic foliation $\mathcal{F}(D)$ of a Dirac structure D , we obtain a presymplectic form $\Omega_N \in \Omega^2(N)$. Therefore, any Dirac structure carries a natural *presymplectic foliation*

$$\hat{\mathcal{F}}(D) = \{(N, \Omega_N) \mid N \in \mathcal{F}(D) \text{ leaf}\}.$$

Example 3.15. Let (M, ω) be a presymplectic manifold. Let $D_\omega \subseteq TM$ be the Dirac structure induced by ω , i.e.,

$$D_\omega = \{(X, \omega^\sharp(X)) \mid X \in TM\}.$$

Let $i: N \hookrightarrow M$ be a leaf of the foliation $\mathcal{F}(D_\omega)$. When M is connected, there is just one leaf, namely M itself. The presymplectic form Ω_N we have just constructed coincides with $\omega|_N$.

Indeed, since the foliation is induced by the characteristic distribution $\mathcal{D}(D_\omega)$ of D_ω , it holds that $\text{pr}_1(D|_N) = TN$. Furthermore, $\omega^\sharp(X) \in T_p^*N$, for all $X \in D_p$ and $p \in N$. Thus,

$$\begin{aligned}\Gamma_\omega|_N: TN &\rightarrow D|_N \\ X &\mapsto (X, \omega^\sharp(X))\end{aligned}$$

is a right inverse of the projection. Therefore, by using (3.14), it follows that the forms coincide.

Definition 3.16. Let $D \subseteq \mathbb{T}M$ be a Dirac structure over M . Let $\gamma: I \rightarrow M$ be a path on M and $H \in C^\infty(M)$. The path γ is called a *Hamiltonian curve on D for H* if $(\dot{\gamma}(t), dH_{\gamma(t)}) \in D$, for all $t \in I$.

Lemma 3.17. Let $D \subseteq \mathbb{T}M$ be a Dirac structure over M and $H \in C^\infty(M)$. Further, let $\gamma: I \rightarrow M$ be a Hamiltonian curve on D for H .

Then, γ stays on the presymplectic leaf $(N, \Omega_N) \in \hat{\mathcal{F}}(D)$, where N is the leaf through $\gamma(t_0)$, $t_0 \in I$. Further, it holds that

$$\iota_{\dot{\gamma}(t)}\Omega_N = d(H|_N)_{\gamma(t)} \quad \text{for all } t \in I.$$

Proof. By definition,

$$\dot{\gamma}(t) \in \rho(D)_{\gamma(t)} = \mathcal{F}(D)_{\gamma(t)}.$$

Thus, the tangent vector of γ lies, at any time $t \in I$, in the characteristic distribution $\rho(D)$. Therefore, γ determines an integral manifold (of dimension 1) of the distribution. Since integral manifolds cannot leave the maximal integral manifold we conclude that $\gamma(I) \subseteq N$, where N is the maximal integral manifold of the distribution, i.e., a leaf, through the initial point of γ .

Let $w \in T_{\gamma(t)}N$. Since $(\dot{\gamma}(t), dH_{\gamma(t)}) \in T_{\gamma(t)}N$, the definition of Ω_N (see (3.13)) yields

$$\Omega_N|_{\gamma(t)}(\dot{\gamma}(t), w) = dH_{\gamma(t)}(w).$$

Since w was arbitrary, this proves the claim. $\square$

Chapter 4

A Variational Approach on Dirac Structures

We mainly follow [6]. Let $D \subseteq \mathbb{T}M$ be a Dirac structure over M . According to Remark 3.6, every Dirac structures naturally carries the structure of a Lie algebroid with anchor map $\rho := \text{pr}_1|_D : D \rightarrow TM$. Therefore, we can talk about the horizontal cohomology of D .

Now, we define $\omega_D \in \Gamma(\Lambda^2 D^*)$ by

$$\omega_D((X, \alpha), (Y, \beta)) = \alpha(Y) - \beta(X), \quad (X, \alpha), (Y, \beta) \in \Gamma(D). \quad (4.1)$$

As D is isotropic, i.e., $D = D^\perp$, we conclude:

$$\omega_D((X, \alpha), (Y, \beta)) = 2\alpha(Y) = -2\beta(X). \quad (4.2)$$

Since $\ker(\rho|_p : D_p \rightarrow T_p M) = 0$, for every $p \in M$, Equation (4.2) implies that $\omega_D|_p$ is ρ -horizontal for every $p \in M$.

Moreover, $d_D \omega_D|_p$ is also ρ -horizontal for every $p \in M$. Indeed, let

$$\xi_1 := (X, \alpha), \xi_2 := (Y, \beta), \xi_3 := (Z, \gamma) \in \Gamma(D) \subseteq \mathfrak{X}(M) \oplus \Omega^1(M).$$

Recall that, for every $\alpha \in \Omega^1(M)$, the de Rham differential is given by

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]), \quad X, Y \in \mathfrak{X}(M).$$

Expanding the Lie algebroid differential (see (2.7)) yields

$$\begin{aligned} d_D \omega_D(\xi_1, \xi_2, \xi_3) &= \rho_*(\xi_1)(\omega_D(\xi_2, \xi_3)) - \rho_*(\xi_2)(\omega_D(\xi_1, \xi_3)) + \rho_*(\xi_3)(\omega_D(\xi_1, \xi_2)) \\ &\quad - \omega_D([\xi_1, \xi_2], \xi_3) + \omega_D([\xi_1, \xi_3], \xi_2) - \omega_D([\xi_2, \xi_3], \xi_1) \\ &= 2X(\beta(Z)) - 2Y(\alpha(Z)) + 2Z(\alpha(Y)) \\ &\quad - 2\mathcal{L}_X \beta(Z) + 2\iota_Y d\alpha(Z) - 2\beta([X, Z]) + 2\alpha([Y, Z]) \\ &= 2X(\beta(Z)) - 2Y(\alpha(Z)) + 2Z(\alpha(Y)) - 2Z(\beta(X)) \\ &\quad - 2d\beta(X, Z) + 2d\alpha(Y, Z) - 2\beta([X, Z]) + 2\alpha([Y, Z]) \\ &= 0. \end{aligned}$$

We can therefore define:

Definition 4.1. Let $D \subseteq \mathbb{T}M$ be a Dirac structure over M . The form

$$\omega_D \in \Gamma(\Lambda^2 D^*)^{hor},$$

defined in (4.1), is called the *natural horizontal two-cocycle* of the Dirac structure D .

Applying Lemma 2.40 to ω_D yields:

Lemma 4.2. Let $D \subseteq \mathbb{T}M$ be a Dirac structure over M . Suppose that $[\omega_D] = 0 \in H_{hor}^2(D)$. Then, $[(\omega_D)_N] = 0 \in H_{dR}^2(N)$ for any leaf $N \in \mathcal{F}(D)$.

Example 4.3. Let (M, ω) be a symplectic manifold (see Remark 3.8) with differential 2-form $\omega \in \Omega^2(M)$. According to Example 3.7, there is a Dirac structure

$$D_\omega := \text{graph}(\omega^\sharp) = \{(X, \omega^\sharp(X)) \mid X \in TM\} \simeq TM \subseteq \mathbb{T}M.$$

There is a Lie algebroid isomorphism $\text{pr}_1: D_\omega \xrightarrow{\sim} TM$ over M . TM is the tangent Lie algebroid from Example 2.2. Lemma 2.30, yields a cochain isomorphism

$$(\text{pr}_1)^*: \Omega^\bullet(M) \xrightarrow{\sim} \Gamma(\Lambda^\bullet D_\omega^*)^{hor}. \quad (4.3)$$

Recall that $\ker(\text{id}_{TM}: T_p M \rightarrow T_p M) = 0$, for every $p \in M$. Therefore $\Omega^\bullet(M) = \Gamma(\Lambda^\bullet T^* M)^{hor}$ (see Example 2.37). Hence, ω is ρ -horizontal and thus

$$[\omega] \in H_{hor}^2(TM) = H_{dR}^2(M)$$

defines a horizontal cohomology class.

Let $p \in M$ and $(X, \omega^\sharp(X)), (Y, \omega^\sharp(Y)) \in \Gamma(D_\omega)$. We obtain

$$(\text{pr}_1)^*(\omega)_p((X, \omega^\sharp(X)), (Y, \omega^\sharp(Y))) = \omega_p(X, Y).$$

Further, for the natural induced ρ -horizontal 2-cocycle of D_ω , there holds:

$$\omega_{D_\omega}((X, \omega^\sharp(X)), (Y, \omega^\sharp(Y))) = 2\omega^\sharp(X)(Y) = 2\omega(X, Y).$$

Using Corollary 2.31, (4.3) induces an isomorphism

$$H^\bullet(\text{pr}_1): H_{dR}^\bullet(M) \xrightarrow{\sim} H^\bullet(D_\omega)^{hor}.$$

Consequently, the class of ω corresponds to the class of ω_{D_ω} under $H^\bullet(\text{pr}_1)$.

Example 4.4. Let (M, π) be a Poisson manifold with Poisson structure $\pi \in \mathfrak{X}^2(M)$. According to Example 3.10, there is a Dirac structure

$$D_\pi := \text{graph}(\pi^\sharp) = \{(\pi^\sharp(\alpha), \alpha) \mid \alpha \in T^*M\} \simeq T^*M \subseteq \mathbb{T}M.$$

There is a Lie algebroid isomorphism $\text{pr}_2: D_\pi \xrightarrow{\sim} T^*M$ over M . T^*M is the cotangent Lie algebroid from Example 2.3. Lemma 2.30, yields a cochain isomorphism

$$(\text{pr}_2)^*: (\mathfrak{X}^\bullet(M))^{hor} \xrightarrow{\sim} \Gamma(\Lambda^\bullet D_\pi^*)^{hor}. \quad (4.4)$$

Now, let $p \in M$ and let $v \in \ker(\pi_p^\sharp: T_p^*M \rightarrow T_p M)$. Then, $\iota_v \pi_p = 0$. Moreover, $d_{T^*M} \pi = 0$, a fact that will be established in the section on **Poisson geometry**. Hence, π is ρ -horizontal. Consequently,

$$[\pi] \in H_{hor}^2(T^*M)$$

defines a horizontal cohomology class.

Let $p \in M$ and $(\pi_*^\sharp(\alpha), \alpha), (\pi_*^\sharp(\beta), \beta) \in \Gamma(D_\pi)$. We obtain

$$(\text{pr}_2)^*(\pi)_p((\pi_*^\sharp(\alpha), \alpha), (\pi_*^\sharp(\beta), \beta)) = \pi_p(\alpha, \beta).$$

Further, for the natural induced ρ -horizontal 2-cocycle of D_π , there holds:

$$\omega_{D_\pi}((\pi_*^\sharp(\alpha), \alpha), (\pi_*^\sharp(\beta), \beta)) = 2\alpha(\pi_*^\sharp(\beta)) = 2\pi(\alpha, \beta).$$

Using Corollary 2.31, (4.4) induces an isomorphism

$$H^\bullet(\text{pr}_2): H_\pi^\bullet(M) \xrightarrow{\sim} H^\bullet(D_\pi).$$

Consequently, the class of π corresponds to the class of ω_{D_π} under $H^\bullet(\text{pr}_2)$.

Definition 4.5. Let $D \subseteq \mathbb{T}M$ be a Dirac structure over M . Denote the bundle projection of $\mathbb{T}M$ by $\pi: \mathbb{T}M \rightarrow M$. Let $\gamma: I \rightarrow M$ be a path on M .

- (i) A path $\zeta: I \rightarrow D$ on D is called a *Dirac path* if the tangent component coincides with the speed of the associated base path $\pi(\zeta): I \rightarrow M$ at each point, i.e.,

$$\text{pr}_1(\zeta(t)) = \frac{d}{dt}\pi(\zeta(t)) \quad \text{for all } t \in I.$$

- (ii) A path $\zeta: I \rightarrow D$ on D is called a *Dirac path over γ* if the tangent component coincides with the speed of the path γ at each point, i.e.,

$$\text{pr}_1(\zeta(t)) = \dot{\gamma}(t) \quad \text{for all } t \in I.$$

Theorem 4.6. Let $D \subseteq \mathbb{T}M$ be a Dirac structure over M and $H \in C^\infty(M)$ a Hamiltonian function. Furthermore, let $\gamma: I \rightarrow M$ be a path on M .

Assume that the horizontal 2-cocycle ω_D admits a primitive $\theta \in \Gamma(D^*)^{\text{hor}}$. Then, the following statements are equivalent:

- (i) The path γ is a Hamiltonian curve.
(ii) All Dirac paths $\zeta: I \rightarrow D$ over γ are critical points among the Dirac paths with the same end points of the following functional:

$$\zeta \mapsto \int_I (\theta_{\gamma(t)}(\zeta(t)) + H(\gamma(t))) dt. \quad (4.5)$$

- (iii) There exists a Dirac path $\zeta: I \rightarrow D$ over γ , and the latter is the critical point among Dirac paths with the same end points of the functional in (4.5).

Proof. (ii) $\Leftrightarrow$ (iii) Let $\zeta, \zeta': I \rightarrow D$ be two Dirac paths over γ . Then $\text{pr}_1(\zeta - \zeta') = 0$. As $\rho = \text{pr}_1$ is the anchor map of D , we have $\theta(\zeta - \zeta') = 0$, because θ is horizontal. Therefore $\theta(\zeta)$ does not depend on the choice of the Dirac path over γ and we get the equivalence between (ii) and (iii).

(i) $\Leftrightarrow$ (ii) Using Lemma 3.17 it follows that we can restrict the functional in (4.5) to a fixed leaf N containing γ . Further, using Lemma 2.40 it follows that we can restrict ω_D to the leaf N and get a corresponding exact 2-form $\omega_N \in \Omega^2(N)$ on N , where the primitive is given by $\theta_N \in \Omega^1(N)$. As ω_N is exact, it is, in particular, presymplectic. The proof therefore reduces to showing that the following statements are equivalent:

- (i) $\omega_*^\#(dH_\gamma) = \dot{\gamma}$.
- (ii) The curve $\gamma: I \rightarrow N$ is a critical point of the functional

$$\gamma \mapsto \int_I (\theta_N|_{\gamma(t)}(\dot{\gamma}(t)) + H(\gamma(t))) dt, \quad (4.6)$$

among curves with fixed endpoints.

For symplectic manifolds, this is a classical result that will be discussed later in the context of [Classical Mechanics](#).

Take a small variation

$$\begin{aligned} v: I \times (-\epsilon, \epsilon) &\rightarrow N \\ (t, s) &\mapsto v(t, s) \end{aligned}$$

of γ with fixed endpoints.

For all $(t, s) \in I \times (-\epsilon, \epsilon)$, we denote

$$v_s: I \rightarrow N, \quad t \mapsto v(t, s)$$

and analogously

$$v_t: (-\epsilon, \epsilon) \rightarrow N, \quad s \mapsto v(t, s).$$

We obtain a vector field $Y: I \rightarrow TN$ along γ by setting

$$Y(t) = \left. \frac{\partial}{\partial s} \right|_{s=0} v \in T_{\gamma(t)}N \quad t \in I.$$

Define the following (smooth) function on N :

$$\begin{aligned} \gamma^* \mathcal{L}_Y \theta_N: I &\rightarrow \mathbb{R} \\ t &\mapsto \frac{\partial}{\partial t} \theta_N|_{\gamma(t)}(Y(t)) + d\theta_N(Y(t), \dot{\gamma}(t)). \end{aligned} \quad (4.7)$$

The global coordinates t and s on the manifold $V = I \times (-\epsilon, \epsilon)$ induce coordinate vector fields $\partial_t, \partial_s \in \mathfrak{X}(V)$. Let $f \in C^\infty(I \times (-\epsilon, \epsilon))$ and $(\hat{t}, \hat{s}) \in I \times (-\epsilon, \epsilon)$. The isomorphism

$$T_{(\hat{t}, \hat{s})}V \simeq V$$

yields

$$\partial_t|_{(\hat{t}, \hat{s})} f = \frac{\partial}{\partial t} f(\hat{t}, \hat{s}) \quad \text{and} \quad \partial_s|_{(\hat{t}, \hat{s})} f = \frac{\partial}{\partial s} f(\hat{t}, \hat{s}).$$

The variation v yields a pullback on differential forms, i.e., a map

$$v^*: \Omega^k(N) \rightarrow \Omega^k(V).$$

Consider now the provided $\theta_N \in \Omega^1(N)$. Since the coordinate vector fields on vector spaces commute, i.e., $[\partial_t, \partial_s] = 0$, we obtain

$$dv^* \theta_N(\partial_t, \partial_s) = \partial_t v^* \theta_N(\partial_s) - \partial_s v^* \theta_N(\partial_t). \quad (4.8)$$

But

$$\begin{aligned} v^* d\theta_N(\partial_t, \partial_s)_{(t,0)} &= d\theta_N|_{\gamma(t)} (\partial_t v_0(t), \partial_s|_0 v_s(t)) \\ &= d\theta_N|_{\gamma(t)} (\dot{\gamma}(t), Y(t)) \end{aligned} \quad (4.9)$$

$$\begin{aligned} \partial_t v^* \theta_N(\partial_s)_{(t,0)} &= \partial_t|_{(t,0)} \theta_N|_{v_s(t)} (\partial_s v_s(t)) \\ &= \frac{\partial}{\partial t} \theta_N|_{\gamma(t)} (\partial_s|_0 v_s(t)) \\ &= \frac{\partial}{\partial t} \theta_N|_{\gamma(t)} (Y(t)) \end{aligned} \quad (4.10)$$

and similarly

$$\partial_s v^* \theta_N(\partial_t)_{(t,0)} = \partial_s|_0 \theta_N|_{v_s(t)} (\partial_t v_s(t)) \quad (4.11)$$

Combining the Equations (4.8), (4.9), (4.10), and (4.11) yields

$$\frac{\partial}{\partial s} \Big|_{s=0} \theta_N|_{v_s(t)} \left(\frac{\partial v_s}{\partial t}(t) \right) = \gamma^* \mathcal{L}_Y \theta_N(t) \quad (4.12)$$

Further, we compute:

$$\begin{aligned} \frac{\partial}{\partial s} \Big|_{s=0} H(v_s(t)) &= d(H \circ v_s(t))_0 \\ &= dH_{v_0(t)} \circ d(v_s(t))_0 \\ &= dH_{\gamma(t)}(Y(t)) \end{aligned} \quad (4.13)$$

The endpoints of v are fixed. Therefore, $Y(0) = Y(1) = 0$ and thus

$$\int_I \gamma^* \mathcal{L}_Y \theta_N(t) dt = \int_I d\theta_N(Y(t), \dot{\gamma}(t)) dt. \quad (4.14)$$

By applying Equations (4.12), (4.13), (4.14), and the Leibniz integral rule to interchange the integration and partial derivative, we can finally make the variational principle in (4.6) explicit:

$$\frac{\partial}{\partial s} \Big|_{s=0} \int_I \left(\theta_N|_{v_s(t)} \left(\frac{\partial v_s}{\partial t}(t) \right) + H(v_s(t)) \right) dt \quad (4.15)$$

$$\begin{aligned} &= \int_I \frac{\partial}{\partial s} \Big|_{s=0} \left(\theta_N|_{v_s(t)} \left(\frac{\partial v_s}{\partial t}(t) \right) + H(v_s(t)) \right) dt \\ &= \int_I \left(\gamma^* \mathcal{L}_Y \theta_N(t) + dH_{\gamma(t)}(Y(t)) \right) dt \\ &= \int_I \left(d\theta_N(Y(t), \dot{\gamma}(t)) + dH_{\gamma(t)}(Y(t)) \right) dt. \end{aligned} \quad (4.16)$$

We can now complete the proof. Suppose that γ is a critical point of the functional in (4.6), i.e., the expression in (4.15) vanishes for every variation with fixed endpoints. Consequently, the expression in (4.16) vanishes, where Y is arbitrary apart from vanishing at the endpoints. Since $d\theta_N = \omega_N$, it follows from the fundamental lemma of the calculus of variations that $\omega_*^\sharp(dH_\gamma) = \dot{\gamma}$. Conversely, if $\omega_*^\sharp(dH_\gamma) = \dot{\gamma}$, then the preceding argument can be read in reverse to conclude that γ is a critical point of the functional in (4.6). $\square$

Remark 4.7. If there exists a vector field $\hat{Y} \in \Gamma(TN)$ such that $Y = \hat{Y} \circ \gamma$, then the pullback $\gamma^*: \Omega^1(N) \rightarrow \Omega^1(I)$ yields

$$\gamma^* \mathcal{L}_{\hat{Y}} \theta_N = \left(\frac{\partial}{\partial t} \theta_N|_{\gamma(t)} (Y(t)) + d\theta_N(Y(t), \dot{\gamma}(t)) \right) dt \in \Omega^1(I).$$

This justifies the notation in [\(4.7\)](#).

Chapter 5

A Variational Approach: Classical Mechanics

Definition 5.1. Let $\omega \in \Omega^2(M)$ be a differential 2-form on M . We call ω a *presymplectic structure* if it is closed, i.e., $d\omega = 0$. We call the tuple (M, ω) a *presymplectic manifold*. Further, we call ω nondegenerate if the induced bundle map $\omega^\sharp: TM \rightarrow T^*M$ from Example 3.7 is a bundle isomorphism. A nondegenerate presymplectic form ω is called a *symplectic structure* and the tuple (M, ω) a *symplectic manifold*.

5.1 The Canonical Symplectic Form on the Cotangent Bundle

We will now present that the cotangent bundle naturally has the structure of a symplectic manifold (see [16]). Let Q be a manifold. Consider the cotangent bundle

$$T^*Q = \coprod_{p \in M} \{p\} \times T_p^*Q$$

of Q with projection $\pi: T^*Q \rightarrow Q$. Let $(p, \varphi) \in T^*Q$. The differential of π at (p, φ) is the linear map $d\pi_{(p, \varphi)}$. Dualizing yields the linear map

$$d\pi_{(p, \varphi)}^*: T_p^*Q \rightarrow T_{(p, \varphi)}^*(T^*Q).$$

We define the differential 1-form $\tau \in \Gamma(T^*(T^*Q)) = \Omega^1(T^*Q)$ by

$$\tau_{(p, \varphi)} = d\pi_{(p, \varphi)}^* \varphi \in T_{(p, \varphi)}^*(T^*Q) \quad (5.1)$$

for all $(p, \varphi) \in T^*Q$. The map $\tau: T^*Q \rightarrow T^*(T^*Q)$ defined in this way is indeed a 1-form:

Proposition 5.2. *The map τ defined above is smooth and therefore a 1-form. Further, the map $\omega := -d\tau$ is a symplectic structure on the manifold T^*Q .*

Proof. Let $(U, (x^i))$ be a coordinate chart on an open subset $U \subseteq Q$. Set $n := \dim(Q)$. The coordinates induce natural coordinates $(U, (x^i, \xi_i))$ on $T^*U \subseteq T^*Q$ ([16]). Let $(p, \varphi) \in T^*U$. Remind that the according coordinate representation is given by

$$\begin{aligned} \hat{p} &= (x^1(p), \dots, x^n(p)) \\ \hat{\varphi} &= \sum_{i=1}^n \xi_i(\varphi) dx^i. \end{aligned}$$

In terms of these coordinates, $\pi: T^*Q \rightarrow Q$ reads as $\pi(x, \xi) = x$, and it follows that $d\pi^*(dx^i) = dx^i$. By using (5.1), the coordinate expression for τ reads

$$\begin{aligned}\tau_{(x, \xi)} &= d\pi_{(x, \xi)}^* \left(\sum_{i=1}^n \xi_i dx^i \right) \\ &= \sum_{i=1}^n \xi_i dx^i.\end{aligned}$$

It follows that τ is smooth in these coordinates and therefore smooth on all of Q because the same can be done for all coordinate charts.

Now, we define $\hat{\omega} := -d\tau$. In the natural coordinates $(U, (x^i, \xi_i))$, $\hat{\omega}$ takes the form

$$\hat{\omega} = \sum_{i=1}^n dx^i \wedge d\xi_i.$$

Let $p \in U$. Then, T_p^*Q is spanned by $\{v_1, \dots, v_n, w_1, \dots, w_n\}$ with $v_i := \partial_{x^i}|_p$ and $w_i := \partial_{\xi_i}|_p$, where $i \in \{1, \dots, n\}$. Note that

$$\hat{\omega}_p(v_i, v_j) = \hat{\omega}_p(w_i, w_j) = 0, \quad \hat{\omega}_p(v_i, w_j) = -\hat{\omega}_p(w_j, v_i) = \delta_{ij},$$

for all $i, j \in \{1, \dots, n\}$. Now, let $X \in \ker(\hat{\omega}_p^\sharp) \subseteq T_p^*Q$. Hence,

$$\hat{\omega}(X, w) = 0 \text{ for all } w \in T_p^*Q.$$

The basis yields a representation of X by

$$X = \sum_{i=1}^n \lambda_i v_i + \mu_i w_i,$$

where $\lambda_i, \mu_i \in \mathbb{R}$ for all $i \in \{1, \dots, n\}$.

It follows that

$$\hat{\omega}_p(X, v_i) = \lambda_i \text{ and } \hat{\omega}_p(X, w_i) = -\mu_i \text{ for all } i \in \{1, \dots, n\}.$$

Consequently, $X = 0$ and hence, by a dimension argument, $\hat{\omega}$ is nondegenerate. Furthermore, $\hat{\omega}$ is closed as it is exact. Consequently $\hat{\omega}$ is symplectic. $\square$

Definition 5.3 (tautological 1-form). The 1-form τ defined in (5.1) is called the *tautological 1-form*. The induced natural symplectic form $\omega = -d\tau$ endows T^*Q with the structure of a symplectic manifold. ω is called the *canonical symplectic form on T^*Q* .

Remark 5.4. The symplectic form on U in Proposition 5.2 corresponds to the *standard symplectic form* ω_{std} on $\mathbb{R}^{2n}$ given by

$$\omega_{\text{std}} = \sum_{i=1}^n dx^i \wedge dy^i$$

with standard coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ on $\mathbb{R}^{2n}$.

Remark 5.5. A fact from linear algebra states that the rank of any alternating bilinear form is even. This implies that symplectic forms can only exist on even-dimensional manifolds.

5.2 Hamilton's Equations in Phase Space

Before introducing a first mechanical example, we review some basic facts from symplectic geometry that follow from the defining properties of a symplectic form. Our presentation of these facts follows [23]. Let (M, ω) be a symplectic manifold. The bundle isomorphism $\omega^\sharp: TM \rightarrow T^*M$ induces an isomorphism on sections

$$\omega_*^\sharp: \mathfrak{X}(M) \rightarrow \Omega^1(M).$$

Let $H \in C^\infty(M)$ be a function on M . The isomorphism $\omega_*^\sharp$ yields a unique vector field $X_H \in \mathfrak{X}(M)$ characterized by

$$dH = \omega(X_H, \cdot) = \iota_{X_H} \omega.$$

We call X_H the *Hamiltonian vector field* of H .

Proposition 5.6. *Let (M, ω) be a symplectic manifold and $H \in C^\infty(M)$. Then, H is constant along each integral curve of X_H .*

Proof. We have

$$X_H(H) = dH(X_H) = \omega(X_H, X_H) = 0.$$

Therefore, H is constant on the flow lines of X_H . $\square$

Proposition 5.7. *Let (M, ω) be a symplectic manifold. Let $X \in \mathfrak{X}(M)$ be a vector field on M with flow φ_X . Then the following are equivalent:*

(i) $(\varphi_X^t)^* \omega = \omega$ for all t where the flow is defined.

(ii) $\mathcal{L}_X \omega = 0$

For the first condition we also say that φ_X preserves the symplectic form ω .

Proof. (i) $\Rightarrow$ (ii) In terms of the flow the Lie derivative of ω with respect to X calculates as

$$(\mathcal{L}_X \omega)_p = \frac{d}{dt} \Big|_{t=0} [(\varphi_X^t)^* \omega]_p = \lim_{t \rightarrow 0} \frac{(d\varphi_X^t)_p^* \omega_{\varphi_X^t(p)} - \omega_p}{t} = 0,$$

for all $p \in M$.

(ii) $\Rightarrow$ (i) First, note that $(\varphi_X^0)^* \omega = (\text{id}_M)^* \omega = \omega$.

$$\begin{aligned} \frac{d}{dt} [(\varphi_X^t)^* \omega]_p &= \frac{d}{ds} \Big|_{s=0} [(\varphi_X^{s+t})^* \omega]_p \\ &= (\varphi_X^t)^* \frac{d}{ds} \Big|_{s=0} [(\varphi_X^s)^* \omega]_p \\ &= (\varphi_X^t)^* \underbrace{(\mathcal{L}_X \omega)_p}_{=0} \\ &= 0, \end{aligned}$$

for all $p \in M$. Therefore, $(\varphi_X^t)^* \omega = \omega$ for all t where the flow is defined. $\square$

Definition 5.8. Let (M, ω) be a symplectic manifold. A vector field $X \in \mathfrak{X}(M)$ is called *symplectic* if $\mathcal{L}_X \omega = 0$, or equivalently, if its flow preserves the symplectic form ω .

Proposition 5.9. *Let (M, ω) be a symplectic manifold and $H \in C^\infty(M)$. Then, the Hamiltonian vector field X_H of H is symplectic, i.e., it preserves the symplectic form ω .*

Proof. We have

$$\begin{aligned} \mathcal{L}_{X_H} \omega &= \underbrace{d\iota_{X_H} \omega}_{=dH} + \underbrace{\iota_{X_H} d\omega}_{=0} \\ &= ddH = 0 \end{aligned}$$

□

In Hamiltonian mechanics the function $H \in C^\infty(M)$ on a symplectic manifold (M, ω) often works as the Energy of a system. Since the Hamiltonian vector field of H preserves H , Proposition 5.6 encodes Conservation of Energy. We call the triple (M, ω, H) a *Hamiltonian system*. The physical motion is then described by an integral curve of X_H through a point $p \in M$.

Example 5.10 [25]. Consider the simple pendulum. This mechanical system consists of a massless, rigid rod of length l , which is fixed at one end. At the other end is a bob of mass m , which can oscillate in the vertical plane. Assume that the only external force acting on the system is gravity, which is constant and points vertically downwards. We assume the gravitational acceleration to be normalized to $g = 1$. The position of the mass is uniquely characterized by the angle θ between the rod and the vertical direction.

A bob of mass m on a massless rigid rod

The configuration space is therefore the manifold S^1 . Let ξ be the coordinate along the fibres of T^*S^1 induced by θ . In this setting, ξ is called the generalized momentum conjugate to θ . It can be shown that the system's kinetic energy is given by

$$K(\theta, \xi) = \frac{1}{2ml^2} \xi^2$$

and its potential energy by

$$V(\theta) = ml(1 - \cos(\theta)).$$

The system's total energy is therefore encoded by the Hamiltonian function

$$H(\theta, \xi) = (K + V)(\theta, \xi) = \frac{1}{2ml^2} \xi^2 + ml(1 - \cos(\theta)) \in C^\infty(T^*S^1).$$

We now use the fact that every cotangent bundle naturally carries a symplectic structure. Let ω be the canonical symplectic form on T^*S^1 . The Hamiltonian vector field $X_H \in \mathfrak{X}(T^*S^1)$ of H is defined implicitly by

$$dH = \iota_{X_H} \omega.$$

We can describe the system's dynamics then by determining the flow of X_H . One can show that a curve $\gamma(t) = (\theta(t), \xi(t))$ is an integral curve of X_H if and only if

$$\begin{aligned}\dot{\theta} &= \frac{1}{ml^2} \xi \\ \dot{\xi} &= -ml \sin(\theta).\end{aligned}$$

By combining these equations, we obtain the well-known differential equation for the simple pendulum (gravitational acceleration has been normalized to $g = 1$):

$$\ddot{\theta} + \frac{1}{l} \sin(\theta) = 0.$$

When we follow this idea in the general case, we can derive Hamilton's equations.

We do already know the local expression of the canonical symplectic structure on a cotangent bundle (Proposition 5.2). Hence, locally we may equivalently work on $\mathbb{R}^{2n}$ equipped with its standard symplectic form ω_{std} (see Remark 5.4). We denote the coordinates of $\mathbb{R}^{2n}$ by $(q^1, \dots, q^n, p^1, \dots, p^n)$. Therefore, the standard symplectic form reads

$$\omega_{\text{std}} = \sum_{i=1}^n dq^i \wedge dp^i.$$

Let $H \in C^\infty(\mathbb{R}^{2n})$ be a Hamiltonian function and $X_H \in \mathfrak{X}(\mathbb{R}^{2n})$ its corresponding Hamiltonian vector field. Then, we obtain Hamilton's equations by solving for the integral curves of X_H :

Proposition 5.11 [25]. *The curve $\gamma: \mathbb{R} \supseteq I \rightarrow \mathbb{R}^{2n}$, written as $\gamma(t) = (q(t), p(t))$, is an integral curve of X_H if and only if it satisfies Hamilton's equations*

$$\begin{cases} \frac{dq^i}{dt}(t) = \frac{\partial H}{\partial p^i} \\ \frac{dp^i}{dt}(t) = -\frac{\partial H}{\partial q^i}. \end{cases} \quad (5.2)$$

Proof. First, observe that

$$dH = \sum_{i=1}^n \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p^i} dp^i.$$

We write

$$X_H = \sum_{i=1}^n f_i \frac{\partial}{\partial q^i} + g_i \frac{\partial}{\partial p^i}$$

with $f_i, g_i \in C^\infty(\mathbb{R}^{2n})$, where $i \in \{1, \dots, n\}$. The equation $dH = \iota_{X_H} \omega_{\text{std}}$ then yields

$$X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} \right). \quad (5.3)$$

One checks that X_H indeed satisfies $dH = \iota_{X_H} \omega_{\text{std}}$, and by uniqueness we obtain the Hamiltonian vector field. Hamilton's equations then follow by expanding the condition for an integral curve,

$$\dot{\gamma}(t) = X_H|_{\gamma(t)},$$

using (5.3) and writing $\gamma(t) = (q(t), p(t))$. $\square$

5.3 The Principle of Least Action in Phase Space

Let Q be the configuration manifold with phase space T^*Q . By Proposition 5.2 (T^*Q, ω) is naturally a symplectic manifold. Recall Example 4.3. The symplectic form ω corresponds to the natural horizontal 2-cocycle ω_{D_ω} . Since $\omega = -d\tau$ is exact, ω_{D_ω} admits a horizontal primitive, namely

$$\theta = \text{pr}_1^*(-2\tau),$$

where $\text{pr}_1^*: \Omega^\bullet(M) \xrightarrow{\sim} \Gamma(\Lambda^\bullet D_\omega^*)^{hor}$ is the isomorphism from Example 4.3.

Therefore, Theorem 4.6 applies. Suppose that the dynamics are determined by a Hamiltonian function $H \in C^\infty(T^*Q)$ representing the energy of the system. Although not every system admits such a description, we restrict ourselves to this case. Theorem 4.6 then recovers the classical *Principle of Least Action* (see [1]), while at the same time revealing it as a special case of the more general variational principle from Chapter 4.

Theorem 5.12 (The Principle of Least Action in Hamiltonian Mechanics) [1]. *Let*

$$\gamma: I \rightarrow T^*Q, \quad t \mapsto (q(t), p(t))$$

be a solution of Hamilton's equations in (5.2). Then, γ is a critical point of the functional

$$\gamma \mapsto \int_I pdq - Hdt$$

with respect to all variations for which the configuration endpoints remain fixed, i.e.,

$$q(t_0) = q_0, \quad q(t_1) = q_1.$$

Chapter 6

A Variational Approach: Poisson Geometry

6.1 Poisson Cohomology

The Schouten-Bracket

Here we follow [7]. We briefly review the basic notions, omitting most details. As the theory is entirely dual to that of differential forms, much of what follows should be familiar. We therefore present only the essential definitions. Let $k \in \mathbb{N}_{>0}$. We denote

$$\mathfrak{X}^k(M) := \Gamma(\Lambda^k TM)$$

and set $\mathfrak{X}^0(M) := C^\infty(M)$. Note that vector fields coincide with multivector fields of degree 1, i.e., $\mathfrak{X}(M) = \mathfrak{X}^1(M)$. We call elements of $\mathfrak{X}^k(M)$ *multi vectorfields of degree k on M* . Just as differential forms act on vector fields, a multi vectorfield $\vartheta \in \mathfrak{X}^k(M)$ can be identified with a $C^\infty(M)$ -multilinear alternating map

$$\vartheta: \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k \text{ times}} \rightarrow C^\infty(M).$$

Just as vector fields can be identified with derivations on $C^\infty(M)$, we can identify a multivector field $\vartheta \in \mathfrak{X}^k(M)$ with a multiderivation

$$\begin{aligned} \mathcal{L}_\vartheta: C^\infty(M) \times \cdots \times C^\infty(M) &\rightarrow C^\infty(M) \\ (f_1, \dots, f_k) &\mapsto \vartheta(df_1, \dots, df_k). \end{aligned}$$

Multiderivation means that $\mathcal{L}_\vartheta$ is alternating and a derivation with respect to each argument. Note that, when dealing with a vector field $X \in \mathfrak{X}^1(M)$, the Lie derivative of a function $f \in C^\infty(M)$ is the case $k = 1$ of multiderivations:

$$\mathcal{L}_X(f) = X(df) = df(X) = X(f).$$

Therefore, there is no ambiguity in the use of $\mathcal{L}$.

Remark 6.1. A multiderivation of degree $k = 2$ is a biderivation

$$\{\cdot, \cdot\}: C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M).$$

This map is skew-symmetric and $\mathbb{R}$ -bilinear, and a derivation in each argument. In particular, it satisfies the *Leibniz identity* in the first argument, i.e.,

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \quad \text{for all } f, g, h \in C^\infty(M),$$

and due to skew-symmetry also in the second argument.

Definition 6.2. A multivector field $\pi \in \mathfrak{X}^2(M)$ of degree 2 is called a *Poisson structure* if the induced bracket

$$\{f, g\} = \pi(df, dg), \quad f, g \in C^\infty(M)$$

satisfies the Jacobi identity. We call such a bracket *Poisson bracket*. Further, we call the tuple (M, π) a *Poisson manifold*.

Definition 6.3 (Schouten bracket). Let $\vartheta \in \mathfrak{X}^{k+1}(M)$ and $\zeta \in \mathfrak{X}^{l+1}(M)$. The *Schouten-bracket* of ϑ and ζ is the unique multivector field $[\vartheta, \zeta] \in \mathfrak{X}^{k+l+1}(M)$ satisfying

$$\mathcal{L}_{[\vartheta, \zeta]} = \mathcal{L}_\vartheta \circ \mathcal{L}_\zeta - (-1)^{kl} \mathcal{L}_\zeta \circ \mathcal{L}_\vartheta,$$

where

$$\begin{aligned} & \mathcal{L}_\vartheta \circ \mathcal{L}_\zeta(f_1, \dots, f_{k+l+1}) \\ &:= \sum_{\sigma \in S_{[k, l+1]}} \text{sgn}(\sigma) \mathcal{L}_\vartheta(f_{\sigma(1)}, \dots, f_{\sigma(k)}, \mathcal{L}_\zeta(f_{\sigma(k+1)}, \dots, f_{\sigma(k+l+1)})), \end{aligned} \quad (6.1)$$

$f_1, \dots, f_{k+l+1} \in C^\infty(M)$. $S_{[k, l+1]}$ denotes the set of all $(k, l+1)$ -shuffles. We use the convention $\mathcal{L}_f = f$ and $\mathcal{L}_f \circ \mathcal{L}_\theta = 0$, for all $f \in C^\infty(M)$ and $\theta \in \mathfrak{X}^k(M)$, $k \in \mathbb{N}_0$.

Example 6.4. Let $\pi \in \mathfrak{X}^2(M)$ be a bivector field on M with corresponding biderivation $\{\cdot, \cdot\}$. A symmetry argument yields $\frac{1}{2}\mathcal{L}_{[\pi, \pi]} = \mathcal{L}_\pi \circ \mathcal{L}_\pi$. Further, let $f_1, f_2, f_3 \in C^\infty(M)$. We apply (6.1) and obtain:

$$\begin{aligned} \mathcal{L}_\pi \circ \mathcal{L}_\pi(f_1, f_2, f_3) &= \sum_{\sigma \in S_{[1, 2]}} \text{sgn}(\sigma) \pi(df_{\sigma(1)}, \pi(df_{\sigma(2)}, df_{\sigma(3)})) \\ &= \{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\}. \end{aligned}$$

Therefore, it follows

Proposition 6.5. A bivector field $\pi \in \mathfrak{X}^2(M)$ on M is a Poisson structure if and only if $[\pi, \pi] = 0$.

Remark 6.6. The Schouten bracket extends the notion of the Lie bracket $[\cdot, \cdot]_{\mathfrak{X}}$ to multivector fields. We have

$$[X, Y]_{\mathfrak{X}} = [X, Y], \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

So there is no ambiguity with the Lie bracket. In fact, the Schouten bracket is $\mathbb{R}$ -bilinear and satisfies a graded version of skew-symmetry, a graded version of the Leibniz identity, and a graded version of the Jacobi identity, dda. For details, we refer to [7]. We just mention how the Schouten bracket interacts with the Lie bracket and the graded Leibniz identity, because this will be useful later. For $X, Y \in \mathfrak{X}(M)$ and $\vartheta \in \mathfrak{X}^{k+1}(M)$:

$$[\vartheta, [X, Y]] + [X, [\vartheta, Y]] + [Y, [\vartheta, X]] = 0. \quad (6.2)$$

Further, for $\vartheta \in \mathfrak{X}^{k+1}(M)$, $\zeta \in \mathfrak{X}^{l+1}(M)$ and $\tau \in \mathfrak{X}^{m+1}(M)$,

$$[\vartheta, \zeta \wedge \tau] = [\vartheta, \zeta] \wedge \tau + (-1)^{k(l+1)} \zeta \wedge [\vartheta, \tau]. \quad (6.3)$$

The Poisson Differential

Consider the cotangent Lie algebroid $(T^*M, [\cdot, \cdot]_\pi, \pi^\#)$. The Lie algebroid cohomology of T^*M coincides with the Poisson cohomology of M (see Example 2.20). Using the Schouten-bracket, we can now present a more explicit description of Poisson cohomology.

Lemma 6.7. *The Lie algebroid differential $d_{T^*M}: \Gamma(\Lambda^\bullet TM) \rightarrow \Gamma(\Lambda^{\bullet+1} TM)$ coincides with the so called Poisson differential*

$$\begin{aligned} d_\pi: \mathfrak{X}^\bullet(M) &\rightarrow \mathfrak{X}^{\bullet+1}(M) \\ \vartheta &\mapsto [\pi, \vartheta]. \end{aligned}$$

Proof. For $f \in C^\infty(M)$, we get $[\pi, f] = -\pi^\#(df)$. Now, consider the case when $k > 0$. Let $\vartheta \in \mathfrak{X}^{k+1}(M)$, and $f_1, \dots, f_{k+2} \in C^\infty(M)$. With $\sigma \in S_{[1, k+1]}$, we obtain

$$\mathcal{L}_\pi \circ \mathcal{L}_\vartheta(f_1, \dots, f_{k+1}) = \text{sgn}(\sigma) \pi^\#(df_{\sigma(1)}) (\vartheta(df_{\sigma(2)}, \dots, df_{\sigma(k+1)})).$$

Further, for $f, g \in C^\infty(M)$, note that $[df, dg]_\pi = -d\pi(df, dg)$. Therefore, with $\sigma \in S_{[k, 2]}$:

$$\begin{aligned} \mathcal{L}_\vartheta \circ \mathcal{L}_\pi(f_1, \dots, f_{k+1}) &= \text{sgn}(\sigma) \vartheta(df_{\sigma(1)}, \dots, df_{\sigma(k)}, -[df_{\sigma(k+1)}, df_{\sigma(k+2)}]_\pi) \\ &= (-1)^{k+1} \text{sgn}(\sigma) \vartheta([df_{\sigma(1)}, df_{\sigma(2)}]_\pi, df_{\sigma(3)}, \dots, df_{\sigma(k+2)}). \end{aligned}$$

By comparing this result with the Lie algebroid differential given in (2.7), the claim follows. Looking at (2.7), the $(1, k+1)$ -shuffles correspond to the omission of one input, while the $(k, 2)$ -shuffles correspond to the omission of two inputs. $\square$

Definition 6.8. Let (M, π) be a Poisson manifold with associated Poisson bracket $\{\cdot, \cdot\}$.

1. For $f \in C^\infty(M)$, we define $X_f := \pi_*^\#(df) = \{f, \cdot\} \in \mathfrak{X}(M)$ and call it the *Hamiltonian vector field* of f . We denote the space of Hamiltonian vector fields on (M, π) by $\mathfrak{X}_{\text{ham}}(M, \pi)$.
2. We call a vector field $X \in \mathfrak{X}(M)$ a *Poisson vector field* if

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}, \quad \text{for all } f, g \in C^\infty(M).$$

We denote the space of Poisson vector fields on (M, π) by $\mathfrak{X}_\pi(M)$.

3. We call a function $C \in C^\infty(M)$ a *Casimir function* if

$$\{C, f\} = 0, \quad \text{for all } f \in C^\infty(M).$$

We denote the space of Casimir functions, or alternatively Casimirs, on (M, π) by $\text{Cas}(M, \pi)$.

Remark 6.9. The anchor map of a Lie algebroid is a Lie algebroid morphism. Hence, using (5.2), the Hamiltonian and Poisson vector fields on a Poisson manifold (M, π) form a Lie subalgebra of $\mathfrak{X}(M)$. Further, we have that

$$\mathfrak{X}_{\text{ham}}(M, \pi) \subseteq \mathfrak{X}_\pi(M) \subseteq \mathfrak{X}(M). \quad (6.4)$$

As we have already seen, the Poisson differential fulfils $d_\pi(f) = [\pi, f] = -X_f$ for each $f \in C^\infty(M)$. Further, note that Casimir functions $C \in C^\infty(M)$ are precisely those functions for which its Hamiltonian vector field vanish, i.e., $X_C \equiv 0$. Using the definition of the Schouten bracket, provided in Definition 6.3, one may check that a vector field $X \in \mathfrak{X}(M)$ is Poisson if and only if $[\pi, X] = 0$. This fact can be used to verify the first inclusion in (6.4). These results provide a characterization of low degree Poisson cohomology modules:

Corollary 6.10. *Let (M, π) be a Poisson manifold. Then, it holds:*

1. $H_\pi^0(M) = \text{Cas}(M, \pi)$
2. $H_\pi^1(M) = \frac{\mathfrak{X}_\pi(M)}{\mathfrak{X}_{\text{ham}}(M, \pi)}$

6.2 The Symplectic Foliation of a Poisson Manifold

Here, we follow [15]. Let (M, π) be a Poisson manifold. In chapter 3, we have already seen that the Lie algebroid structure induced from the Dirac structure $D_\pi = \{(\pi^\sharp(\alpha), \alpha) \mid \alpha \in T^*M\}$ carries a presymplectic foliation. The characteristic distribution of D_π is given by

$$\mathcal{D}(D_\pi) = \text{Im}(\pi^\sharp).$$

By the existence of charts, we know that the cotangent space T_p^*M of M at $p \in M$ is spanned by the set $\{df_p \mid f \in C^\infty(M)\}$. Therefore, with Definition 6.8, it follows that the characteristic distribution is spanned by the Hamiltonian vector fields, i.e.,

$$\mathcal{D}(D_\pi)|_p = \{X_f|_p \mid f \in C^\infty(M)\}, \quad (6.5)$$

for all $p \in M$. This implies that the integral curves of any Hamiltonian vector field stay inside a leaf of the characteristic foliation $\mathcal{F}(D_\pi)$. Let $N \in \mathcal{F}(D_\pi)$ be a leaf. By (3.13), the presymplectic form on N is given, for each $q \in N$, by

$$\omega_N|_q(\pi_q^\sharp(\alpha), \pi_q^\sharp(\beta)) = \alpha(\pi_q^\sharp(\beta)) = -\pi_q(\alpha, \beta), \quad (6.6)$$

for all $\alpha, \beta \in T_q^*M$. Actually, ω_N is a symplectic structure on N .

Lemma 6.11 [22]. *Let (M, π) be a Poisson manifold and $N \in \mathcal{F}(D_\pi)$ a leaf. The presymplectic form ω_N , defined in (6.6), is nondegenerate. In other words, (N, ω_N) is a symplectic manifold.*

Proof. Let $v \in \ker(\omega_N|_z : T_z N \rightarrow T_z^* N)$, i.e.,

$$\omega_N|_z(v, w) = 0, \quad \text{for all } w \in T_z N. \quad (6.7)$$

Equation (6.5) implies that there is a Hamiltonian vector field $X_f = \pi^\sharp(df)$, for some $f \in C^\infty(M)$, such that $X_f|_z = v$. Consequently, (6.7) can be rewritten as

$$-\pi(df, dg)_z = X_f(g)_z = 0, \quad \text{for all } g \in C^\infty(M).$$

Equivalently, $X_f|_z = v_z = 0$. Since $T_z M$ is finite-dimensional, $\omega_N|_z$ is bijective, proving the claim. $\square$

In conclusion, any Poisson structure carries a natural symplectic foliation

$$\hat{\mathcal{F}}(D_\pi) = \{(N, \omega_N) | N \in \mathcal{F}(D_\pi) \text{ leaf}\}.$$

The proof of the [Stefan-Sussmann Theorem](#) shows that every symplectic leaf of a Poisson manifold admits a geometric description in terms of Hamiltonian vector fields. As $\mathfrak{X}_{\text{ham}}(M, \pi) \subseteq \text{Im}(\pi_*^\sharp)$ generate the characteristic distribution of D_π , we can characterize a leaf $N \subseteq M$ through a point $p \in M$ by

$$q \in N \iff \exists t \in \mathbb{R}, \exists f_1, \dots, f_k \in C^\infty(M) \text{ such that } \varphi_{X_{f_1}}^t \circ \dots \circ \varphi_{X_{f_k}}^t(p) = q, \quad (6.8)$$

where $\varphi_{X_f}^t$ denotes the flow of the Hamiltonian vector field X_f , for any $f \in C^\infty(M)$. This coincides with the definition of an orbit (or symplectic leaf) of a Poisson manifold provided in [\[7\]](#) or [\[15\]](#).

Lemma 6.12 [\[15\]](#). *Let (M, π) be a Poisson manifold. A function $f \in C^\infty(M)$ is a Casimir function if and only if f is constant on each symplectic leaf.*

Proof. " $\Rightarrow$ ": Let $C \in C^\infty(M)$ be a Casimir function. Let $X_f \in \mathfrak{X}(M)$ be a Hamiltonian vector field. Therefore, $0 = \{C, f\} = -X_f(C)$. Therefore, C is constant on the flow lines of X_f . Using [\(6.8\)](#), it follows that C is constant on each leaf.

" $\Leftarrow$ ": Let $C \in C^\infty(M)$ be constant on each leaf. Then, due to [\(6.8\)](#), we conclude that $X_f(C) = 0$, for all $f \in C^\infty(M)$. Consequently C is a Casimir function. $\square$

Casimir functions can be used to characterize symplectic leaves. In the spirit of the *constant-rank level set theorem* (see [\[16\]](#)), this yields

Proposition 6.13 [\[15\]](#). *Let (M, π) be a Poisson manifold with $\dim(M) = n$. Let $U \subseteq M$ be a nonempty open subset, and let $f_1, \dots, f_m \in C^\infty(U)$ such that:*

1. *The rank of π is constant on U and equal to $n - m$.*
2. *The functions $f_1, \dots, f_m$ are Casimirs of $\pi|_U$.*
3. *The differentials $\{df_1|_p, \dots, df_m|_p\} \subseteq T_p^*M$ are independent, for all $p \in U$.*

Then, the symplectic foliation of the restriction of π to U coincides with the foliation defined by $F := (f_1, \dots, f_m): U \rightarrow \mathbb{R}^m$. Note that the foliation $\mathcal{F}$, which is induced by F , is given by the level sets of F , i.e., $\mathcal{F} = \{F^{-1}(q) \mid q \in \mathbb{R}^m\}$.

Proof. As $\pi|_U$ has constant rank, any leaf of its symplectic foliation has dimension $\text{rank}(\pi) = n - m$. Since F has constant maximal rank, the leaves of its foliation $\mathcal{F}$ on U also have dimension $n - m$. By [Lemma 6.12](#), any symplectic leaf is contained in a leaf of $\mathcal{F}$. The foliations coincide, as the dimensions coincide. $\square$

6.3 Examples of Poisson Structures Admitting Primitives

Our goal now is to find an example where [Theorem 4.6](#) applies. In this section, we are dealing with the cotangent Lie algebroid with anchor $\rho = \pi^\sharp$. Therefore, instead of talking about $\pi^\sharp$ -horizontal forms, we just talk about horizontal ones. Let (M, π) be a Poisson

manifold with Dirac structure D_π and induced cotangent Lie algebroid $(T^*M, [\cdot, \cdot]_\pi, \pi^\sharp)$ (see Remark 3.11). We have $[\pi, \pi] = 0$, i.e., $d_{T^*M}\pi = 0$. Therefore, the element $\pi \in \mathfrak{X}^2(M)$ defines a class in $H_\pi^2(M)$. This completes the discussion of Example 4.4. In the same Example, we already discussed that the horizontal 2-form on D_π corresponds to π . Therefore, in order to apply Theorem 4.6, we need a horizontal primitive in Poisson cohomology. We start with the following Example from [6]. The starting point of the ideas of this section were Examples 1.16 and 1.17 from this reference.

Example 6.14. Let $M = \mathbb{R}^2$ with Poisson structure $\pi = x^2\partial_x \wedge \partial_y \in \mathfrak{X}^2(M)$. Let $E = x\partial_x \in \mathfrak{X}^1(M)$. Then, one verifies that $d_{T^*M}E = [\pi, E] = \pi$. Hence, E is a primitive of π in Poisson cohomology. But is it a primitive in the horizontal cohomology? In order to check this, we need E to be horizontal. We can use $C^\infty(M)$ -linearity of $\pi^\sharp$ and a bump function argument to show that $E_p \in \text{Im}(\pi_p^\sharp)$, for all $p \in M$. The following Lemma then shows that E is horizontal.

Lemma 6.15. *Let (M, π) be a Poisson manifold and let $A = T^*M$ be the corresponding cotangent Lie algebroid. Let $E \in \mathfrak{X}(M) = \Gamma(A^*)$ be a multivector field of degree 1 such that $d_A E$ is horizontal. Then, E is horizontal if and only if it is tangent to the characteristic distribution $\mathcal{D}(A)$ of A , i.e., $E_p \in \mathcal{D}(A)_p$ for all $p \in M$.*

Proof. " $\Rightarrow$ ": Let E be horizontal. Let $p \in M$. Denote $S := \pi_p^\sharp$ for the map $\pi_p^\sharp: T_p^*M \rightarrow T_pM$. Using the isomorphism $T^{**}M \simeq TM$, one may verify that S coincides with its dual map S^* under this isomorphism. Further, recall that $\ker(S)^\perp = \text{Im}(S^*)$. Because E is horizontal, we conclude that $E_p \in \text{Im}(S)$. But $\text{Im}(\pi_p^\sharp) = T_pN$, where N is the leaf of $\mathcal{F}^{\mathcal{D}(A)}$ through p . As p was arbitrary, E is tangent to $\mathcal{D}(A)$.

" $\Leftarrow$ ": Let $p \in M$ and $E_p \in \text{Im}(\pi_p^\sharp)$. Note that

$$\text{Im}(\pi_p^\sharp) = \{\pi(df_p, \cdot) \in T_pM \mid f \in C^\infty(M)\}.$$

Hence, there is $f \in C^\infty(M)$ with $E_p = \pi(df_p, \cdot)$. Let $v \in \ker(\pi_p^\sharp)$. Then,

$$\iota_v E_p = \pi(df_p, v) = -\pi_p^\sharp(v)(df_p) = 0.$$

$d_A E_p$ is horizontal by assumption. As p was arbitrary, it follows that E is horizontal. $\square$

We now consider a special class of Poisson structures for which π admits a (not necessarily horizontal) primitive. These are linear Poisson structures, which are classical examples of Poisson structures. They arise from Lie algebras, and consequently provide a rich source of examples, in particular, there is a good chance of finding a horizontal primitive. Here, we follow [7] and used the examples from [13], [15], and [18] for our investigation.

We start with an $\mathbb{R}$ -vector space. As usual, we denote the space of linear functionals $\varphi: V \rightarrow \mathbb{R}$ (i.e., the dual vector space to V) as $V^* \subseteq C^\infty(V)$.

Definition 6.16. A Poisson bracket

$$\{\cdot, \cdot\}: C^\infty(V) \times C^\infty(V) \rightarrow C^\infty(V)$$

on V is called a *linear Poisson bracket* if it restricts to linear functions, i.e.,

$$\{V^*, V^*\} \subseteq V^*.$$

A Poisson structure which corresponds to a linear Poisson bracket will be called a *linear Poisson structure*.

Proposition 6.17 [7]. *There is a 1-to-1 correspondence*

$$\left\{ \begin{array}{l} \text{linear Poisson brackets } \{\cdot, \cdot\} \\ \text{on a space } V = \mathfrak{g}^* \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{l} \text{Lie algebra structures } [\cdot, \cdot] \\ \text{on the dual vector space } \mathfrak{g} = V^* \end{array} \right\}$$

determined by

$$\{\mathrm{ev}_u, \mathrm{ev}_v\} = \mathrm{ev}_{[u,v]}, \quad \text{for all } u, v \in \mathfrak{g}, \quad (6.9)$$

where

$$\begin{aligned} \mathrm{ev}: \mathfrak{g} &\xrightarrow{\sim} \mathfrak{g}^{**} \subseteq C^\infty(\mathfrak{g}^*) \\ v &\mapsto (\varphi \mapsto \varphi(v)) \end{aligned} \quad (6.10)$$

is the evaluation map.

Proof. Let $\mathfrak{g}$ be a vector space. Suppose that there is a linear Poisson bracket $\{\cdot, \cdot\}$ on $\mathfrak{g}^*$. Let $u, v \in \mathfrak{g}^*$. As $\{\cdot, \cdot\}$ is linear, the element $\{\mathrm{ev}_u, \mathrm{ev}_v\} \in \mathfrak{g}^{**}$ corresponds to an element $[u, v] \in \mathfrak{g}$ under the isomorphism in (6.10). The bracket $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, defined as such, is alternating, bilinear, and satisfies the Jacobi identity because $\{\cdot, \cdot\}$ also has these properties. Hence, we get a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$.

For the other direction, let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra. Let $f, g \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$. The isomorphism $T_\xi^* \mathfrak{g}^* \simeq \mathfrak{g}$ yields that the differential $T_\xi^* \mathfrak{g}^* \ni df_\xi: T_\xi \mathfrak{g}^* \rightarrow \mathbb{R}$ can be viewed as an element of $\mathfrak{g}$. Therefore, we get $[df_\xi, dg_\xi] \in \mathfrak{g}$. There is a natural pairing on $\mathfrak{g}$:

$$\langle \cdot, \cdot \rangle: \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}, \quad (v, \xi) \mapsto \xi(v) \quad (6.11)$$

using the isomorphism (6.10), we obtain:

$$\begin{aligned} \{\cdot, \cdot\}: C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) &\rightarrow C^\infty(\mathfrak{g}^*) \\ (f, g) &\mapsto (\xi \mapsto \langle \xi, [df_\xi, dg_\xi] \rangle). \end{aligned} \quad (6.12)$$

It defines a Poisson bracket on $\mathfrak{g}^*$ because $\{\cdot, \cdot\}$ inherits the necessary conditions from $[\cdot, \cdot]$. It is a *linear* Poisson bracket, because $\mathrm{Im}(\mathrm{ev}) = \mathfrak{g}^{**}$. $\square$

Remark 6.18. The preceding proposition also implies that the evaluation map defines an embedding of Lie algebras

$$\mathrm{ev}: (\mathfrak{g}, [\cdot, \cdot]) \hookrightarrow (C^\infty(\mathfrak{g}^*), \{\cdot, \cdot\}).$$

The Poisson bracket $\{\cdot, \cdot\}$ corresponds to a bivector field π on $\mathfrak{g}^*$. Therefore, there is a map

$$\begin{aligned} C^\infty(\mathfrak{g}^*) &\rightarrow \mathfrak{X}_{\mathrm{ham}}(\mathfrak{g}^*, \pi) \\ f &\mapsto \{f, \cdot\} = X_f. \end{aligned}$$

Postcomposing this map with the evaluation map yields a $\mathfrak{g}$ -action on the manifold $\mathfrak{g}^*$:

$$\begin{aligned} A: (\mathfrak{g}, [\cdot, \cdot]) &\rightarrow (\mathfrak{X}(\mathfrak{g}^*), [\cdot, \cdot]) \\ v &\mapsto X_{\mathrm{ev}_v} \end{aligned}$$

which is precisely the coadjoint $\mathfrak{g}$ -action ad^* on $\mathfrak{g}^*$.

In summary, every Lie algebra $\mathfrak{g}$ induces a natural linear Poisson bracket on $\mathfrak{g}^*$. We call the corresponding Poisson structure $\pi_{\mathfrak{g}}$ the *Lie–Poisson structure* on $\mathfrak{g}^*$.

At this point, it makes sense to briefly review some fundamental facts concerning the coadjoint action on a Lie algebra. Further details can be found, for instance, in [7] and [16]. Let G be a Lie group. For any $g \in G$, there is a diffeomorphism $c_g: G \rightarrow G$, $x \mapsto gxg^{-1}$, which is called the conjugation map. Taking the differential at the neutral element $e \in G$ yields the adjoint map $\text{Ad}_g = d(c_g)_e: \mathfrak{g} \rightarrow \mathfrak{g}$.

Now, let $g \in G$. We obtain a unique element $\text{Ad}_g^*(\xi) \in \mathfrak{g}^*$, for any element $\xi \in \mathfrak{g}^*$, defined by

$$\text{Ad}_g^*(\xi)(v) = \langle \xi, \text{Ad}_{g^{-1}}(v) \rangle, \quad \text{for all } v \in \mathfrak{g},$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing on $\mathfrak{g}$ (see (6.11)). This defines the coadjoint G -action on $\mathfrak{g}^*$, namely $\text{Ad}^*: G \rightarrow \text{GL}(\mathfrak{g}^*)$. Finally, we can define the coadjoint $\mathfrak{g}$ -action on $\mathfrak{g}^*$ by

$$\begin{aligned} \text{ad}^*: \mathfrak{g} &\rightarrow \mathfrak{X}(\mathfrak{g}^*) \\ v &\mapsto \text{ad}_v^*, \end{aligned}$$

where the vector field ad_v^* is given by

$$(\text{ad}_v^*)_{\xi} := \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\exp(tv)}^*(\xi), \quad \xi \in \mathfrak{g}^*.$$

The following facts hold:

- For any $v, w \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, it holds

$$\text{ad}_v^*(\xi)(w) = -\xi([v, w]). \quad (6.13)$$

- Let $\xi \in \mathfrak{g}^*$. The coadjoint orbit

$$\mathcal{O}_{\xi} = \{\text{Ad}_g^*\xi \mid g \in G\} \subseteq \mathfrak{g}^*$$

through ξ is an immersed submanifold of $\mathfrak{g}^*$.

- Let $\xi \in \mathfrak{g}^*$, the coadjoint G -action on $\mathfrak{g}^*$ induces a transitive action on $\mathcal{O}_{\xi}$:

$$\begin{aligned} \mathcal{A}: G \times \mathcal{O}_{\xi} &\rightarrow \mathcal{O}_{\xi} \\ (h, \text{Ad}_g^*\xi) &\mapsto \text{Ad}_{hg}^*\xi. \end{aligned}$$

- Let $\xi \in \mathfrak{g}^*$ and $\eta \in \mathcal{O}_{\xi}$. The tangent space of $\mathcal{O}_{\xi}$ at η fulfils

$$T_{\eta}\mathcal{O}_{\xi} \simeq \{\text{ad}_v^*\eta \mid v \in \mathfrak{g}\} \subseteq \mathfrak{g}^*.$$

Proposition 6.19 [15]. *Let G be a connected Lie group with Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$. The symplectic leaves of the Lie–Poisson structure $\pi_{\mathfrak{g}}$ on $\mathfrak{g}^*$ are the coadjoint orbits of G .*

Proof. Let $f, g \in C^{\infty}(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$. We can rewrite (6.12) as

$$\begin{aligned} \{f, g\}(\xi) &= \langle \xi, [df_{\xi}, dg_{\xi}] \rangle \\ &= -\langle \text{ad}_{df_{\xi}}^*\xi, dg_{\xi} \rangle. \end{aligned}$$

It follows that

$$\pi^{\sharp}(df)_{\xi} = \{f, \cdot\}(\xi) = -\text{ad}_{df_{\xi}}^*\xi \in \mathfrak{X}(\mathfrak{g}^*).$$

Therefore, using that $df_\xi \in T_\xi^* \mathfrak{g}^* \simeq \mathfrak{g}$:

$$\{\pi^\sharp(df)_\xi \mid f \in C^\infty(\mathfrak{g}^*)\} = \{\text{ad}_v^* \xi \mid v \in \mathfrak{g}\} = T_\xi \mathcal{O}_\xi. \quad (6.14)$$

The characteristic distribution $\mathcal{D}(\mathfrak{g}^*)$ of $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ is given at each $\xi \in \mathfrak{g}^*$ by

$$\mathcal{D}(\mathfrak{g}^*)_\xi = \text{Im}(\pi^\sharp(\cdot)_\xi).$$

Since the G -action on the coadjoint orbits is transitive, for every $\eta \in \mathcal{O}_\xi$ there exists $g \in G$ such that $\eta = g.\xi$. By using (6.14), we conclude

$$D(\mathfrak{g}^*)_\eta = T_\eta \mathcal{O}_\eta = T_\eta \mathcal{O}_{g.\xi} = T_\eta \mathcal{O}_\xi.$$

Because the coadjoint orbits of G are connected, they form integral manifolds of the characteristic distribution, proving the claim. $\square$

Definition 6.20 [15]. Let (M, π) be an n -dimensional Poisson manifold and $(U, (x^i))$ a coordinate chart. As in the case of differential forms, π admits the local expression

$$\pi|_U = \frac{1}{2} \sum_{i,j=1}^n \{x^i, x^j\} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

We call the matrix $X = (\{x^i, x^j\})_{i,j}$ the *Poisson matrix* of π with respect to the coordinates (x^i) . Note that X is a matrix with entries in $C^\infty(U)$.

Remark 6.21. When we evaluate the Poisson matrix of π with respect to coordinates (x^i) on U at a point $p \in U$, we obtain an $\mathbb{R}$ -valued matrix representation of the skew-symmetric bilinear map

$$\pi_p: T_p^* M \times T_p^* M \rightarrow \mathbb{R}$$

with respect to the basis $(dx^i|_p)$ of $T_p^* M$.

Definition 6.22. Let V be an n -dimensional vector space with global coordinates (x^i) . Let $E \in \mathfrak{X}(V)$ be the vector field defined by

$$E = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i}.$$

We call E the *Euler vector field* on V .

Let π be a linear Poisson structure on an n -dimensional vector space V . The following lemma shows that the Euler vector field is always a primitive of π . Since π is a *linear* Poisson structure, it admits a particularly simple representation. By Proposition 6.17, (V, π) corresponds to a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$. Let $\{e_i\}$ be a basis of $\mathfrak{g}$. Then, for $i, j \in \{1, \dots, n\}$,

$$[e_i, e_j] = \sum_{k=1}^n c_k^{ij} e_k,$$

where $c_k^{ij} \in \mathbb{R}$ are called the structure constants of $\mathfrak{g}$ with respect to the basis $\{e_i\}$. Let (x^i) be the induced linear coordinates on $V = \mathfrak{g}^*$. Proposition 6.17 then implies that π is given by

$$\{x^i, x^j\} = \sum_{k=1}^n c_k^{ij} x^k,$$

for all $i, j \in \{1, \dots, n\}$.

Lemma 6.23. *Let π be a linear Poisson structure on a vector space $M = V$ with global coordinates (x^i) . Then, the Euler vector field $E \in \mathfrak{X}(M)$ is a primitive of π . In particular, $[\pi] = 0 \in H_\pi^2(M)$.*

Proof. By Definition 6.20:

$$\pi = \frac{1}{2} \sum_{i,j=1}^n \{x^i, x^j\} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

Since π is linear, we have

$$\{x^i, x^j\} = \sum_{k=1}^n c_k^{ij} x^k$$

where $c_k^{ij} \in \mathbb{R}$ and $i, j \in \{1, \dots, n\}$. Using the graded Leibniz identity (see (6.3)), we conclude that

$$\left[\pi, x^i \frac{\partial}{\partial x^i} \right] = [\pi, x^i] \wedge \frac{\partial}{\partial x^i} + (-1)^{1 \cdot 0} x^i \left[\pi, \frac{\partial}{\partial x^i} \right],$$

for each $i \in \{1, \dots, n\}$. Now, applying the definition of the Schouten bracket (see Section 6.1), we obtain that

$$\begin{aligned} \sum_{i=1}^n \left([\pi, x^i] \wedge \frac{\partial}{\partial x^i} \right) &= \sum_{i=1}^n \left(\sum_{k=1}^n \{x^k, x^i\} \frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial x^i} \right) \\ &= 2\pi. \end{aligned} \tag{6.15}$$

Further, by Schwarz's theorem, we conclude that

$$\begin{aligned} \left[\pi, \frac{\partial}{\partial x^i} \right] &= - \sum_{k,l=1}^n \frac{1}{2} \left(\frac{\partial}{\partial x^i} \{x^k, x^l\} \right) \frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial x^l} \\ &= - \sum_{k,l=1}^n \frac{1}{2} c_i^{kl} \frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial x^l}. \end{aligned}$$

Summing over $i \in \{1, \dots, n\}$ leads to

$$\begin{aligned} \sum_{i=1}^n \left(x^i \left[\pi, \frac{\partial}{\partial x^i} \right] \right) &= - \sum_{k,l=1}^n \frac{1}{2} \left(\sum_{i=1}^n c_i^{kl} x^i \right) \frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial x^l} \\ &= -\pi. \end{aligned} \tag{6.16}$$

We combine the results from (6.15) and (6.16), and the claim follows when we recall that the Schouten bracket is bilinear:

$$d_\pi E = [\pi, E] = \sum_{i=1}^n \left[\pi, x^i \frac{\partial}{\partial x^i} \right] = 2\pi - \pi = \pi.$$

□

It turns out that the converse of Lemma 6.23 is also true. Let π be a Poisson structure on a vector space V and suppose that the Euler vector field E on V is a primitive of π . Then, π is a linear Poisson structure ([7]).

We now have all the pieces needed to construct an example illustrating Theorem 4.6. Let $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ be a Lie-Poisson structure. As discussed previously, the horizontal 2-form on the Dirac structure $D_{\pi_{\mathfrak{g}}}$, considered as a Lie algebroid, corresponds to $\pi_{\mathfrak{g}}$. By Lemma 6.23, the Poisson bivector $\pi_{\mathfrak{g}} \in \mathfrak{X}^2(\mathfrak{g}^*)$ always admits a primitive in Poisson cohomology, namely the Euler vector field E , i.e., $d_{\pi_{\mathfrak{g}}}E = \pi_{\mathfrak{g}}$. As $\pi_{\mathfrak{g}}$ is horizontal, it remains to check that E is tangent to the characteristic distribution $D(\mathfrak{g}^*)$ (see Lemma 6.15). Proposition 6.13 provides a geometric description of the leaves of the induced foliation. As there are several examples of Lie groups with corresponding Lie-Poisson structures (see, for example, [13], [18]), this provides a promising source of examples.

We will now present an example in which the Euler vector field is already tangent to the foliation.

Example 6.24 (Open book foliation).

Step 1: The Lie group and associated Lie algebra

Let $n \in \mathbb{N}_{>0}$. Consider the Lie groups $SO(n)$ with matrix multiplication, and $\mathbb{R}^n$ with addition. The group $SO(n)$ acts on $\mathbb{R}^n$ by

$$\begin{aligned} \tau: SO(n) &\rightarrow \text{Aut}(\mathbb{R}^n) \\ R &\mapsto (x \mapsto R(x)). \end{aligned}$$

We can therefore construct the semidirect product of Lie groups, which is itself a Lie group (see, for example, [22]):

$$SE(n) := \mathbb{R}^n \rtimes_{\tau} SO(n) = \left\{ \begin{bmatrix} 1 & 0 \\ x & A \end{bmatrix} \mid A \in SO(n), x \in \mathbb{R}^n \right\}.$$

The Lie group $SE(n)$ is called the *special Euclidean group*. Its group multiplication is given by matrix multiplication. The according Lie algebra is

$$\mathfrak{se}(n) = \left\{ \begin{bmatrix} 0 & 0 \\ b & B \end{bmatrix} \mid B \in \mathfrak{so}(n), b \in \mathbb{R}^n \right\}, \quad (6.17)$$

where the Lie bracket $[\cdot, \cdot]_{\mathfrak{se}}$ of $\mathfrak{se}$ is provided by the matrix commutator. Recall that the Lie algebra of the special orthogonal group $SO(n)$ is

$$\mathfrak{so}(n) = \left\{ A \in \mathbb{R}^{n \times n} \mid A = -A^{\top} \right\}.$$

We will now consider the case when $n = 2$, for which the above representation becomes

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

Computing the matrix commutator of two arbitrary elements shows that the Lie algebra $(\mathfrak{se}(2), [\cdot, \cdot]_{\mathfrak{se}})$ is isomorphic to the Lie algebra $(\mathbb{R}^3, [\cdot, \cdot])$, where

$$[v, w] = \text{pr}_{xy}(v \times w). \quad (6.18)$$

Here, $\times$ denotes the vector (cross) product on $\mathbb{R}^3$, and $\text{pr}_{xy}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ denotes the projection onto the xy -plane embedded in $\mathbb{R}^3$.

Step 2: The corresponding Lie-Poisson structure

By Proposition 6.17, there is a corresponding Poisson structure $(\mathbb{R}^{3*}, \{\cdot, \cdot\})$. Let $e_1, e_2, e_3 \in \mathbb{R}^3$ be the standard basis of $\mathbb{R}^3$. We obtain linear coordinates $(x^i: \mathfrak{g}^* \rightarrow \mathbb{R})$ on the manifold $\mathfrak{g}^* = (\mathbb{R}^3)^* \simeq \mathbb{R}^3$ by defining

$$\begin{aligned} x^i &:= \text{ev}_{e_i}: \mathfrak{g}^* \rightarrow \mathbb{R} \\ \xi &\mapsto \xi(e_i), \end{aligned}$$

for each $i \in \{1, 2, 3\}$ ([15]). Further, note that (6.9) implies $\{x^i, x^j\} = \text{ev}_{[e_i, e_j]}$ for all $i, j \in \{1, 2, 3\}$. Using (6.18), we can derive the global Poisson matrix:

$$X = \begin{bmatrix} 0 & 0 & x_1 \\ 0 & 0 & x_2 \\ -x_1 & -x_2 & 0 \end{bmatrix}.$$

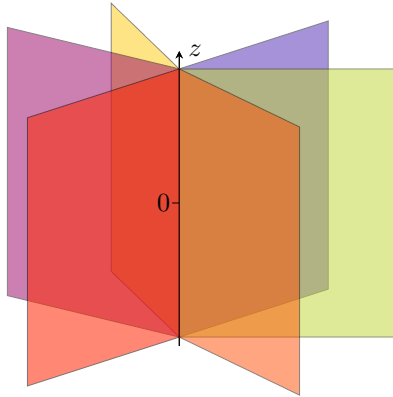
From now on, we adopt the more convenient notation $x = x^1, y = x^2$ and $z = x^3$.

Step 3: The Geometry of the foliation

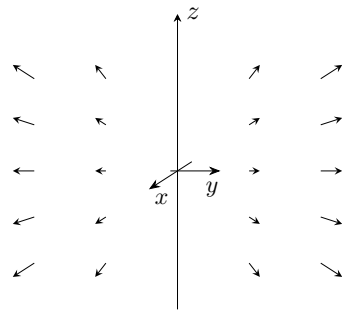
We now apply the geometric method, presented in Proposition 6.13, to find out the symplectic leaves. The matrix X has $\text{rank}(X) = 2$ on $U = \mathbb{R}^3 \setminus \{x = y = 0\}$. Further,

$$\ker(X) = \text{span} \left\{ \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \right\}.$$

We want to apply Proposition 6.13. We seek a function $F: U \rightarrow \mathbb{R}$ such that $\nabla F \parallel \ker(X)$ ([21]). Unfortunately, this is not possible. But we can find functions $F_i: U_i \rightarrow \mathbb{R}$ defined on open subsets $U_i \subseteq U$ such that $\nabla F_i \parallel \ker(X)$, where $i \in \{1, 2\}$, and such that $U_1 \cap U_2 = U$. This means that the gradient ∇F_i of F_i is tangent to the cylinders around the z -axis. According to Proposition 6.13, the level sets of F_i constitute the foliation. From Calculus we know that the gradient of a scalar function is perpendicular to the level sets. Consequently, we obtain the open book foliation:



(a) Open book foliation of $\mathbb{R}^3 \setminus \{x = y = 0\}$



(b) Euler vector field on $\mathbb{R}^3 \cap \{x = 0\}$

The Euler vector field is tangent to the foliation. Therefore, we obtain an example where Theorem 4.6 applies.

Remark 6.25. Let $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ be a Lie-Poisson structure. A natural question is whether $\pi_{\mathfrak{g}}$ admits a primitive other than the Euler vector field E . This is an interesting question

because when the Euler vector field does not fit in the context of Theorem 4.6 it may still be possible to obtain an example. By Corollary 6.10, we have

$$H_{\pi}^1(M) = \frac{\mathfrak{X}_{\pi}(M)}{\mathfrak{X}_{\text{ham}}(M, \pi)}.$$

Therefore, another primitive $X \in \mathfrak{X}(\mathfrak{g}^*)$ of $\pi_{\mathfrak{g}}$ in Poisson cohomology satisfies

$$E - X \in \mathfrak{X}_{\pi}(M).$$

Hence, in order to obtain tangency to the foliation, this leads to the question whether it is possible to find a Poisson vector field that can be added to the Euler vector field such that the resulting vector field is tangent to the foliation. This is a question that could be considered in future projects.

Suppose the Lie algebra $\mathfrak{g}$ is semisimple and compact. Then, the leaves are compact. But the flow of the Euler vector field goes to infinity. Therefore, E cannot be tangent to the leaves. Furthermore, in the semisimple and compact case, we have $H_{\pi_{\mathfrak{g}}}^1(\mathfrak{g}^*) = 0$. Consequently, Poisson vector fields coincide with Hamiltonian vector fields. By construction, Hamiltonian vector fields are tangent to the foliation; therefore, we cannot adjust the Euler vector field, and this class of examples fails. Hence, this provides a first obstruction to the existence of such a modification of the Euler vector field.

Appendix A

Vector Bundles

Definition A.1 (vector bundle) [16]. A (smooth) vector bundle of rank k over a manifold M is a manifold E together with a surjective smooth map (called projection) $\pi: E \rightarrow M$ satisfying

- (i) For each $p \in M$, the fiber $E|_p := E_p := \pi^{-1}(p)$ over p is endowed with the structure of a k -dimensional real vector space.
- (ii) For each $p \in M$, there exists a neighborhood U of p in M and a diffeomorphism $\Phi_U: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ (called a (smooth) local trivialization of E over U), satisfying
 - $\text{pr}_U \circ \Phi_U = \pi$ (where $\text{pr}_U: U \times \mathbb{R}^k \rightarrow U$ is the projection onto U)
 - for each $q \in U$, the restriction $\Phi|_{E_q}$ is a vector space isomorphism between E_q and $\{q\} \times \mathbb{R}^k \simeq \mathbb{R}^k$.

The pair (Φ_U, U) is also called a bundle chart.

Lemma A.2 [16]. Let $\pi: E \rightarrow M$ be a vector bundle over the manifold M with local trivialization $\Phi_U: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$. Let $i \in \{1, \dots, k\}$. Denote with e_i the i -th standard basis vector of $\mathbb{R}^k$. Further we define maps

$$\begin{aligned} \sigma_i: U &\rightarrow E \\ p &\mapsto \Phi_U^{-1}(p, e_i) = \Phi_U^{-1} \circ \tilde{e}_i(p) \end{aligned}$$

where $\tilde{e}_i: U \rightarrow U \times \mathbb{R}^k$, $p \mapsto (p, e_i)$ is the section on the product bundle $U \times \mathbb{R}^k$ induced by the i -th standard basis vector.

Then, $\sigma_1, \dots, \sigma_k$ are local sections on E over U and form a basis of E_p at each $p \in U$, which we will call a local frame for E over U . In particular the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi_U} & U \times \mathbb{R}^k \\ \pi \swarrow & & \nwarrow \text{pr}_1 \\ \sigma_i \searrow & & \swarrow \tilde{e}_i \\ & U & \end{array}$$

Proof. Let $i \in \{1, \dots, k\}$ and $p \in U$ arbitrary. Smoothness of σ_i follows from smoothness of Φ_U . Furthermore, by $\text{pr}_1 \circ \Phi_U = \pi$ (see Definition A.1):

$$\pi \circ \sigma_i(p) = \pi \circ \Phi_U^{-1}(p, e_i) = \text{pr}_1(p, e_i) = p$$

Therefore σ_i is a section on U . Φ_U restricts to an isomorphism

$$\Phi_U|_{E_p} : E_p \xrightarrow{\sim} \{p\} \times \mathbb{R}^k.$$

Since $\Phi_U(\sigma_i(p)) = (p, e_i)$, the identification $\{p\} \times \mathbb{R}^k \simeq \mathbb{R}^k$ yields that $(\sigma_i(p))$ is a basis of E_p . We call (σ_i) the *local frame associated with Φ_U* . $\square$

Lemma A.3 [16]. *Let $\pi: E \rightarrow M$ be a vector bundle over M . Let $\Phi_U: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ be a local trivialization with associated local frame (σ_i) .*

Then, for any $\xi \in \Gamma(E)$, there are k smooth functions $f_i: U \rightarrow \mathbb{R}$ such that

$$\xi|_U = \sum_{i=1}^k f_i \sigma_i.$$

The f_i , $i \in \{1, \dots, k\}$ are called the component functions of ξ

Proof. Since (σ_i) forms a local frame over U ,

$$\xi(p) = \sum_{i=1}^k f_i(p) \sigma_i(p),$$

for each $p \in U$. This defines k functions $f_i: U \rightarrow \mathbb{R}$. As ξ and Φ_U are smooth, $\Phi_U \circ \xi$ is smooth on U . Further

$$\begin{aligned} \Phi_U \circ \xi(p) &= \Phi_U|_p \left(\sum_{i=1}^k f_i \sigma_i \right) \\ &= \sum_{i=1}^k f_i \Phi_U|_p (\sigma_i) \\ &= (p, (f_1(p), \dots, f_k(p))). \end{aligned}$$

Hence, $f_i: U \rightarrow \mathbb{R}$ are smooth for all $i \in \{1, \dots, k\}$. $\square$

Definition A.4 (dual bundle) [16]. Let $\pi: E \rightarrow M$ be a vector bundle over M . We define the *dual bundle to E* to be the bundle E^* with

$$E^* = \coprod_{p \in M} E_p^* = \bigcup_{p \in M} \{(p, \psi) \mid \psi \in E_p^*\}$$

and projection

$$\begin{aligned} \pi^*: E^* &\rightarrow M \\ (p, \psi) &\mapsto p, \end{aligned}$$

where E_p^* denotes the dual vector space to E_p .

Definition A.5 (product and sum of vector bundles) [14]. Let $\pi: E \rightarrow M$, $\pi': E' \rightarrow M$ be vector bundles over M . Let $\eta: F \rightarrow N$ be a vector bundle over N .

(i) The vector bundle

$$E \oplus_M E' = \coprod_{p \in M} (E_p \oplus E'_p)$$

over M with projection

$$\begin{aligned}\bar{\pi}: E \oplus_M E' &\rightarrow M \\ (E_p \oplus E'_p) \ni (e, e') &\mapsto p\end{aligned}$$

is called the *Whitney sum* or *direct sum* of the bundles E and E' . When the base manifold is clear we also write $E \oplus E'$ instead of $E \oplus_M E'$.

(ii) The set $E \times F$ with projection

$$\begin{aligned}\bar{\pi}: E \times F &\rightarrow M \times N \\ (e, f) &\mapsto (\pi(e), \eta(f))\end{aligned}$$

is called the *direct product* of the bundles E and F and is a vector bundle over $M \times N$.

Definition A.6 (bundle homomorphism) [16]. Let $\pi: E \rightarrow M$ and $\pi': E' \rightarrow N$ be vector bundles over M and N , respectively. A map $F: E \rightarrow E'$ is called a bundle homomorphism if there exists a map $f: M \rightarrow N$, such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \pi \downarrow & \# & \downarrow \pi' \\ M & \xrightarrow{f} & N \end{array} \quad (\text{A.1})$$

commutes and

$$F|_{E_p}: E_p \rightarrow E'_{f(p)}$$

is linear for each $p \in M$. One also uses the notation $F_p := F|_p := F|_{E_p}$. For the relationship $\pi' \circ F = f \circ \pi$ is also expressed by saying that F *covers* f .

In the case that F covers the identity map of M , i.e., $f = \text{id}_M$, we call F a bundle homomorphism *over* M .

The following two remarks are general facts from vector bundle theory ([27]). In particular, Proposition 20.14 in [27] can serve as a starting point for a more detailed discussion.

Remark A.7. Let $F: E \rightarrow E'$ be a bundle homomorphism over M . Then there is a dual bundle homomorphism $F^*: E'^* \rightarrow E^*$ defined fibrewise by

$$F^*|_{E'^*} := (F|_{E_p})^* \quad \text{for each } p \in M.$$

Remark A.8. Let $F: E \rightarrow E'$ be a bundle homomorphism over M . For $k \in \mathbb{N}_0$, F induces a bundle homomorphism $\Lambda^k F: \Lambda^k E \rightarrow \Lambda^k E'$ over M defined fibrewise by

$$F|_{\Lambda^k E_p} := \Lambda^k(F|_{E_p}) \quad \text{for each } p \in M.$$

Let $\pi: E \rightarrow M$ be a vector bundle over M . Recall the space of sections

$$\Gamma(E) = \{s: M \rightarrow E \mid \pi \circ s = \text{id}_M\}.$$

It follows that

Remark A.9 [16]. Let $F: E \rightarrow E'$ be a bundle homomorphism covering M . It induces a $C^\infty(M)$ -linear map on sections of E and E' :

$$\begin{aligned} F_*: \Gamma(E) &\rightarrow \Gamma(E') \\ \sigma &\mapsto F \circ \sigma \end{aligned}$$

Proposition A.10 [17]. Consider a short exact sequence of vector bundles over M

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0,$$

that is, an assembly of three vector bundles E', E, E'' over M and two vector bundle homomorphisms $F': E' \rightarrow E$ and $F'': E \rightarrow E''$ such that

$$0 \rightarrow E'_p \xrightarrow{F'_p} E_p \xrightarrow{F''_p} E''_p \rightarrow 0$$

is a short exact sequence of vector spaces, for each $p \in M$. Then there is a right inverse $\sigma'': E'' \rightarrow E$ to F'' and a left inverse $\sigma': E \rightarrow E'$ to F .

Definition A.11 (pullback bundle) [14]. Let $\pi: E \rightarrow N$ be a vector bundle over N . Let $f: M \rightarrow N$ be a (smooth) map between M and N . The *pullback bundle* f^*E of E by f defined by

$$f^*E := \{(p, e) \in M \times E \mid \pi(e) = f(p)\} \subseteq M \times E \quad (\text{A.2})$$

with projection

$$\bar{\pi}: f^*E \rightarrow M, (p, e) \mapsto p.$$

The definition of the pullback bundle implies that the following diagram commutes:

$$\begin{array}{ccc} f^*E & \xrightarrow{\text{pr}_2} & E \\ \bar{\pi} \downarrow & \# & \downarrow \pi \\ M & \xrightarrow{f} & N \end{array}$$

i.e., pr_2 is a bundle homomorphism covering f .

With this definition in mind we can now improve Remark A.9:

Lemma A.12. Let $F: E \rightarrow E'$ be a bundle homomorphism covering $f: M \rightarrow N$.

(a) f induces a $C^\infty(M)$ -linear map:

$$\begin{aligned} f^*: \Gamma(E') &\rightarrow \Gamma(f^*E') \\ \sigma &\mapsto [M \ni p \mapsto (p, (\sigma \circ f)(p))] \end{aligned}$$

(b) F induces a $C^\infty(M)$ -linear map

$$\begin{aligned} F_*: \Gamma(E) &\rightarrow \Gamma(f^*E') \\ \sigma &\mapsto [M \ni p \mapsto (p, (F \circ \sigma)(p))] \end{aligned}$$

(c) The map

$$\begin{aligned} C^\infty(M) \otimes \Gamma(E') &\rightarrow \Gamma(f^*E') \\ g \otimes \sigma &\mapsto gf^*(\sigma) \end{aligned}$$

is an isomorphism of $C^\infty(M)$ -modules (see [17]). The tensor product is over $C^\infty(N)$. The $C^\infty(N)$ -module structure on $C^\infty(M)$ is defined by

$$\begin{aligned} C^\infty(N) \times C^\infty(M) &\rightarrow C^\infty(M) \\ (g, g') &\mapsto (g \circ f)g'. \end{aligned}$$

In particular, $\Gamma(f^*E')$ is generated by $\text{Im}(f^*)$ as a $C^\infty(M)$ -module.

Note that (b) indeed covers Remark A.9, as $\text{id}_M^*E' \simeq E'$.

Remark A.13. Let E be a vector bundle over M . A fact from vector bundle theory states that the $C^\infty(M)$ -module $\Gamma(E)$ of sections of E is finitely generated ([20]). Now, consider the situation of Lemma A.12c, i.e., let E' be a vector bundle over N and $f: M \rightarrow N$ a map between M and N . It follows that every $\alpha \in \Gamma(f^*E')$ admits a finite representation

$$\alpha = \sum_{i=1}^n c_i f^*(\alpha_i)$$

where $c_i \in C^\infty(M)$ and $\alpha_i \in \Gamma(E')$ for $i \in \{1, \dots, n\}$.

Proposition A.14 [3]. Let $f: E \rightarrow E'$ be a surjective bundle homomorphism over M . Then, for all $x \in M$ there are bundle charts (Φ_U, U) of E and (Ψ_U, U) of E' , such that F takes the form of a projection under these trivializations. That is,

$$\bar{F}_U := (\Psi_U \circ F|_U \circ \Phi_U^{-1})$$

takes the form

$$\begin{array}{ccc} E_U & \xrightarrow{F|_U} & E'_U \\ \wr \downarrow \Phi_U & \# & \Psi_U \downarrow \wr \\ U \times \mathbb{R}^m & \xrightarrow{\bar{F}_U} & U \times \mathbb{R}^k \\ (u, (v_1, \dots, v_m)) & \mapsto & (u, (v_1, \dots, v_k)) \end{array}$$

Proposition A.15 [27]. Let $F: E \rightarrow E'$ be a vector bundle homomorphism covering $f: M \rightarrow N$ as in Definition A.6. Then, there is a unique bundle map $\varphi_F: E \rightarrow f^*E'$ over M such that the following diagram commutes:

$$\begin{array}{ccccc} E & & & & \\ & \searrow F & & & \\ & & f^*E' & \xrightarrow{\text{pr}_2} & E' \\ & \swarrow \pi & \downarrow \bar{\pi} & \# & \downarrow \pi' \\ & & M & \xrightarrow{f} & N. \end{array} \quad (\text{A.3})$$

From a Category Theory point of view we say that $(f^*E', \bar{\pi}, \text{pr}_2)$ is the limit of the diagram $M \xrightarrow{f} N \xleftarrow{\pi'} E'$.

Proof. Suppose the diagram in (A.3) commutes. Since the maps $\bar{\pi} = \text{pr}_1$ and pr_2 provide the entries, it follows that $\varphi_F(e) = (\pi(e), F(e))$ for all $e \in E$. So, if this map exists, it is unique.

As F covers f (see (A.1)), we conclude that $(\pi(e), F(e)) \in f^*E'$, for all $e \in E$. Hence φ_F exists. It is $\mathbb{R}$ -linear on each fibre as F is. Further, it is smooth as F and π are. $\square$

The following Lemma is the same as Proposition A.15 adjusted for the Lie algebroid case.

Lemma A.16. *Let $F: A_1 \rightarrow A_2$ be a Lie algebroid morphism covering $f: M_1 \rightarrow M_2$ such that $f^!A_2$ is a well-defined Lie algebroid. Then there is a unique Lie algebroid morphism $\varphi_F: A_1 \rightarrow f^!A_2$ over M_1 such that the following diagram commutes:*

$$\begin{array}{ccc}
 A_1 & & \\
 \swarrow \varphi_F & \searrow F & \\
 f^!A_2 & \xrightarrow{\text{pr}_1} & A_2 \\
 \downarrow \rho_{f^!A_2} & \# & \downarrow \rho_2 \\
 TM_1 & \xrightarrow{df} & TM_2
 \end{array}
 \quad (A.4)$$

Proof. Suppose the diagram (A.4) commutes. Since the maps $\rho_{f^!A_2} = \text{pr}_1$ and pr_2 provide the entries, it follows that $\varphi_F(a) = (F(a), \rho_1(a))$ for all $a \in A_1$. So, if this map exists, it is unique.

As F is a Lie algebroid morphism we conclude with (2.9) that $(F(a), \rho_1(a)) \in f^!A_2$, for all $a \in A_1$. Hence φ_F exists.

Furthermore, φ_F is a Lie algebroid morphism. Indeed, first, note that φ_F is smooth as F and ρ_1 are. Recall the definition of the projection map $\pi_{f^!A_2}$ (see 2.19). φ_F preserves fibers as ρ_1 does. Commutativity of the diagram 2.9, i.e., $\rho_{f^!A_2} \circ \varphi_F = \rho_1$, holds by definition of $\rho_{f^!A_2}$.

It remains to show that φ_F is compatible with the brackets. Since φ_F is a bundle homomorphism covering the identity id_{M_1} , Equation 2.12 simplifies and it remains to show that

$$\varphi_{F*} \circ [\alpha, \beta]_1 = [\varphi_{F*} \circ \alpha, \varphi_{F*} \circ \beta]_{f^!A_2}, \text{ for all } \alpha, \beta \in \Gamma(A_1).$$

So, let $\alpha, \beta \in \Gamma(A_1)$ with representations

$$\begin{aligned}
 F_*(\alpha) &= \sum_i c_i f^*(\alpha_i) \\
 \text{and } F_*(\beta) &= \sum_j c'_j f^*(\beta_j),
 \end{aligned}$$

which exist by Remark A.13. The sums are finite. Using that F is a Lie algebroid morphism

and combining [2.1](#), [2.12](#), and [2.21](#), we obtain:

$$\begin{aligned}
 \varphi_{F*} \circ [\alpha, \beta]_1 &= (F_*([\alpha, \beta]_1), \rho_{1*}([\alpha, \beta]_1)) \\
 &\stackrel{1.1}{=} \left(\sum_{i,j} c_i c'_j f^*([\alpha_i, \beta_j]_2) + \sum_j \rho_*(\alpha)(c'_j) f^*(\beta_j) \right. \\
 &\quad \left. - \sum_i \rho_*(\beta)(c_i) f^*(\alpha_i), [\rho_{1*}(\alpha), \rho_{1*}(\beta)] \right) \\
 &\stackrel{1.7}{=} \left[\left(\sum_i c_i f^*(\alpha_i), \rho_*(\alpha) \right), \left(\sum_j c'_j f^*(\beta_j), \rho_*(\beta) \right) \right]_{f^! A} \\
 &= [\varphi_{F*} \circ \alpha, \varphi_{F*} \circ \beta]_{f^! A_2}
 \end{aligned}$$

□

Appendix B

Calculations Regarding the Dorfman and Courant Brackets

This small section is following [12].

Lemma B.1. *Let $a_1, a_2 \in \Gamma(\mathbb{T}M)$, then*

$$[[a_1, a_2]] = [a_1, a_2]_D - (0, \frac{1}{2}d\langle a_1, a_2 \rangle).$$

Proof. Let $a_1, a_2 \in \Gamma(\mathbb{T}M)$ with $a_1 = (X, \alpha)$ and $a_2 = (Y, \beta)$. We calculate

$$\begin{aligned} \mathcal{L}_X \beta - \iota_Y d\alpha - \frac{1}{2}d\langle (X, \alpha), (Y, \beta) \rangle &= \mathcal{L}_X \beta - \iota_Y d\alpha - \frac{1}{2}d(\iota_Y \alpha + \iota_X \beta) \\ &= \mathcal{L}_X \beta - \iota_Y d\alpha - d\iota_Y \alpha + \frac{1}{2}d\iota_Y \alpha - \frac{1}{2}d\iota_X \beta \\ &= \text{pr}_2([a_1, a_2]). \end{aligned}$$

□

Lemma B.2. *The Dorfman bracket fulfills the Jacobi identity.*

Proof. Let $a, b, c \in \Gamma(\mathbb{T}M)$ with $a = (X, \alpha)$, $b = (Y, \beta)$ and $c = (Z, \gamma)$. Then,

$$\begin{aligned} [[a, b]_D, c]_D + [b, [a, c]_D]_D &= [([X, Y], \mathcal{L}_X \beta - \iota_Y d\alpha), (Z, \gamma)]_D \\ &\quad + [(Y, \beta), ([X, Z], \mathcal{L}_X \gamma - \iota_Z d\alpha)]_D \\ &= ([X, Y, Z] + [Y, [X, Z]], \underbrace{\mathcal{L}_{[X, Y]} \gamma - \iota_Z d\mathcal{L}_X \beta}_{\text{Jacobi eq. for commutator}} \\ &\quad + \iota_Z d\iota_Y d\alpha + \mathcal{L}_Y \mathcal{L}_X \gamma - \mathcal{L}_Y \iota_Z d\alpha - \iota_{[X, Z]} d\beta) \\ &= ([X, [Y, Z]], \mathcal{L}_X \mathcal{L}_Y \gamma - \iota_Z d\mathcal{L}_X \beta - \iota_{[X, Z]} d\beta \\ &\quad + \iota_Z d\iota_Y d\alpha + \underbrace{\iota_Z \iota_Y dd\alpha}_{=0} - \mathcal{L}_Y \iota_Z d\alpha) \\ &= ([X, [Y, Z]], \mathcal{L}_X \mathcal{L}_Y \gamma - \iota_Z d\mathcal{L}_X \beta - \mathcal{L}_X \iota_Z d\beta \\ &\quad + \iota_Z \mathcal{L}_X d\beta + \iota_Z \mathcal{L}_Y d\alpha - \mathcal{L}_Y \iota_Z d\alpha) \\ &= ([X, [Y, Z]], \mathcal{L}_X \mathcal{L}_Y \gamma - \mathcal{L}_X \iota_Z d\beta - \iota_{[Y, Z]} d\alpha) \\ &= [(X, \alpha), ([Y, Z], \mathcal{L}_Y \gamma - \iota_Z d\beta)]_D \\ &= [a, [b, c]_D]_D \end{aligned}$$

□

Lemma B.3. (i) Let $\mu \in \Gamma(T^*M)$ be a closed 1-form. Then,

$$[(0, \mu), a]_D = 0, \quad \text{for all } a \in \Gamma(TM)$$

(ii) For $a, b, c \in \Gamma(TM)$ it holds:

$$[[a, b], c] = [[a, b]_D, c]_D - \frac{1}{2}d\langle [a, b], c \rangle$$

Proof. (i) Let $a \in \Gamma(TM)$ with $a = (X, \alpha)$, then

$$[(0, \mu), (X, \alpha)]_D = ([0, X], \mathcal{L}_0\alpha - \iota_X d\mu) = 0.$$

(ii) Let $a, b, c \in \Gamma(TM)$. Lemma B.1 and part (i) imply

$$\begin{aligned} [[a, b], c] &= [[a, b]_D, c]_D - (0, \frac{1}{2}d\langle [a, b], c \rangle) \\ &= [[a, b]_D, c]_D - [(0, \frac{1}{2}d\langle a, b \rangle), c]_D - (0, \frac{1}{2}d\langle [a, b], c \rangle) \\ &= [[a, b]_D, c]_D - \frac{1}{2}d\langle [a, b], c \rangle. \end{aligned}$$

□

The Jacobiator of the Courant bracket is not zero. However, the following holds.

Proposition B.4. Let $a, b, c \in \Gamma(TM)$. Then,

$$\text{Jac}_{[\cdot, \cdot]}(a, b, c) = \frac{1}{6}(d\langle [a, b], c \rangle + \text{c.p.}),$$

where 'c.p.' stands for cyclic permutations of the preceding expression.

Proof. Recall that the Courant bracket is the antisymmetrization of the Dorfman bracket, i.e.,

$$[a, b] = \frac{1}{2}([a, b]_D - [b, a]_D), \quad \text{for all } a, b \in \Gamma(TM). \quad (\text{B.1})$$

Now, let $a, b, c \in \Gamma(TM)$. By expanding the expression, one verifies that

$$0 = -[c, [a, b]_D]_D - [b, [a, c]_D]_D + [a, [b, c]_D]_D + [c, [b, a]_D]_D + \text{c.p.} \quad (\text{B.2})$$

The 'c.p.' stands for adding the cyclic permutation of the whole preceding expression. To keep track of every term, we indicate which of the preceding equations we use.

$$\begin{aligned} \text{Jac}_{[\cdot, \cdot]}(a, b, c) &= [[a, b], c] + \text{c.p.} \\ &\stackrel{(\text{B.1})}{=} \frac{1}{2}[[a, b]_D, c]_D - \frac{1}{2}[[b, a]_D, c]_D + \text{c.p.} \\ &\stackrel{(\text{B.1})}{=} \frac{1}{4}([a, b]_D, c]_D - [c, [a, b]_D]_D - \underbrace{[b, a]_D, c]_D}_{\text{Lemma B.2}} + [c, [b, a]_D]_D + \text{c.p.}) \\ &= \frac{1}{4}([a, b]_D, c]_D - [c, [a, b]_D]_D - [b, [a, c]_D]_D + [a, [b, c]_D]_D + [c, [b, a]_D]_D + \text{c.p.}) \\ &\stackrel{(\text{B.2})}{=} \frac{1}{4}([a, b]_D, c]_D + \text{c.p.}) \end{aligned}$$

Using Lemma B.3, we obtain

$$\begin{aligned} \text{Jac}_{[\cdot, \cdot]}(a, b, c) &= \frac{1}{4}([a, b]_D, c]_D + \text{c.p.}) \\ &= \frac{1}{4}([a, b], c] + \frac{1}{2}d\langle [a, b], c \rangle + \text{c.p.}) \\ &= \frac{1}{4}(\text{Jac}_{[\cdot, \cdot]}(a, b, c) + \frac{1}{2}d\langle [a, b], c \rangle + \text{c.p.}) \end{aligned}$$

Rearranging then yields

$$\frac{3}{4}\text{Jac}_{\llbracket \cdot, \cdot \rrbracket}(a, b, c) = \frac{1}{8}(d\langle \llbracket a, b \rrbracket, c \rangle + \text{c.p.}),$$

proving the claim. □

Appendix C

Useful Facts from Differential Geometry

Proposition C.1 [8]. *Let M be a manifold. For any $X, Y \in \mathfrak{X}(M)$ there holds the Cartan Calculus:*

$$\begin{aligned}d \circ d &= 0 \\ \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X &= \mathcal{L}_{[X, Y]} \\ \iota_X \circ \iota_Y + \iota_Y \circ \iota_X &= 0 \\ d \circ \mathcal{L}_X - \mathcal{L}_X \circ d &= 0 \\ \mathcal{L}_X \circ \iota_Y - \iota_Y \circ \mathcal{L}_X &= \iota_{[X, Y]} \\ \iota_X \circ d - d \circ \iota_X &= \mathcal{L}_X\end{aligned}$$

Proposition C.2 [16]. *Let M be a manifold and $X, Y \in \mathfrak{X}(M)$. Let φ be the flow of X . For any (t_0, p) in the domain of φ , the following holds:*

$$\left. \frac{d}{dt} \right|_{t=t_0} d(\varphi^{-t})_{\varphi^t(p)}(Y_{\varphi^t(p)}) = d(\varphi^{-t_0}) \left((\mathcal{L}_X Y)_{\varphi^{t_0}(p)} \right).$$

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