The Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Note: You can only conclude *divergence* using the Divergence Test. If $\lim_{n\to\infty} a_n = 0$, the test is *inconclusive*.

The Geometric Series Test: This test only applies to geometric series, i.e. series of the form $\sum_{n=1}^{\infty} ar^{n-1}$. This series converges if $|r| < 1$ and diverges if $|r| \geq 1$.

Note: *If* the series converges, then we can compute the value using the formula $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

The p-Series Test: This test only applies to p-series, i.e. series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. A p-series converges if $p > 1$ and diverges if $p \leq 1$.

The Integral Test: Given $\sum a_n$, if $f(n)$ is a function such that $f(n) = a_n$ for each n and f is continuous, positive and decreasing, then

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \int_1^{\infty} f(x)\,dx \text{ converges.}$$

The (Direct) Comparison Test: Suppose $\sum a_n$, $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for each n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $b_n \leq a_n$ for each n, then $\sum a_n$ is also divergent.

The Limit Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad \text{where } c \text{ is positive and finite}$$

then either both series converge or both diverge.

Note: for both types of Comparison Test, good choices of series to compare to are p-series or geometric series.

The Alternating Series Test: This test applies only to alternating series, i.e. series that can be written $\sum(-1)^n b_n$ or $\sum(-1)^{n+1} b_n$ where $b_n > 0$. If

- $\lim_{n\to\infty} b_n = 0$, and
- $\{b_n\}$ is decreasing,

then the series is convergent.

Note: If one of the two conditions above is not true, Alternating Series Test is *inconclusive*. However, if the Alternating Series Test fails, the Divergence Test is a good idea to try next.

The Ratio Test: Suppose we have a series $\sum a_n$ and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

- If $L < 1$, the series is absolutely convergent (hence convergent).
- If $L > 1$, the series is divergent.
- If $L = 1$, the test is inconclusive.

1. Select the best test for determining whether the following series converge or diverge. Hint: try to eliminate the tests that do not apply. (You do not need to determine whether the series actually do converge or diverge.)

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n-1}}{(2n+1)!}$

   A. p-series Test
   B. Divergence Test
   C. Geometric Series Test
   D. Ratio Test

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{5n^3}{\sqrt{n^7-4}}$

   A. Alternating Series Test
   B. Divergence Test
   C. Ratio Test
   D. Limit Comparison Test

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n^2-2}$

   A. Alternating Series Test
   B. Divergence Test
   C. Integral Test
   D. Ratio Test

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{10}{n(\ln(n))^2}$

   A. Alternating Series Test
   B. Divergence Test
   C. Integral Test
   D. Ratio Test

2. Find an example of each of the following or explain why such an example does not exist.

   (a) A sequence $\{a_n\}$ such that $\{a_n\}$ converges to 0, but $\sum_{n=1}^{\infty} a_n$ diverges.

   (b) A sequence $\{a_n\}$ such that $\{a_n\}$ diverges, but $\sum_{n=1}^{\infty} a_n$ converges.

   (c) A sequence such that $\left|\frac{a_{n+1}}{a_n}\right| < \frac{1}{2}$ for all n and the series $\sum_{n=1}^{\infty} a_n$ diverges.

3. Suppose $a_n > 0$ for all n and the series $\sum_{n=1}^{\infty} a_n$ converges.

   (a) Must the series $\sum_{n=1}^{\infty} (-1)^n a_n$ converge? Explain why or why not.

   (b) Why is it true that $a_n < 1$ for all n after a certain point?

   (c) Show that $\sum_{n=1}^{\infty} a_n^2$ must converge. (Hint: use the Comparison Test)