

Calculus with Polar Curves

Tangents to polar curves : $r = f(\theta)$

↖ think of θ
as a parameter

$$\begin{cases} x = r \cos \theta = f(\theta) \cos \theta \\ y = r \sin \theta = f(\theta) \sin \theta \end{cases}$$

↗ parametrizes your polar curve!

We know that

$$\text{slope of tangent line} = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}$$

$$= \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

ex: $r = 1 + \sin \theta$ ← cardioid

(a) find the slope of tangent line at

$$\theta = \pi/3$$

(b) Find all pts. where the tangent line is horizontal / vertical.

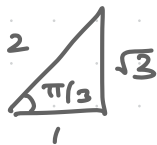
$$(a) \begin{cases} x = r \cos \theta = (1 + \sin \theta) \cos \theta \\ y = r \sin \theta = (1 + \sin \theta) \sin \theta \end{cases}$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\overbrace{\cos \theta \sin \theta + (1 + \sin \theta) \cos \theta}^{\text{product rule}}}{\underbrace{\cos \theta \cdot \cos \theta + (1 + \sin \theta) (-\sin \theta)}_{f' \cdot g + f \cdot g'}}$$

$$= \frac{2 \cos \theta \sin \theta + \cos \theta}{\cos^2 \theta - \sin^2 \theta - \sin \theta}$$

$$\left. \frac{dy}{dx} \right|_{\theta = \pi/3} = \frac{2 \cos(\pi/3) \sin(\pi/3) + \cos(\pi/3)}{\cos^2(\pi/3) - \sin^2(\pi/3) - \sin(\pi/3)}$$

$$= \frac{\cancel{2} (\cancel{1/2}) (\sqrt{3}/2) + (1/2)}{(1/2)^2 - (\sqrt{3}/2)^2 - (\sqrt{3}/2)}$$



$$= \frac{\left(\frac{\sqrt{3}+1}{2}\right)}{\frac{1}{4} - \frac{3}{4} - \frac{\sqrt{3}}{2}} = -1$$

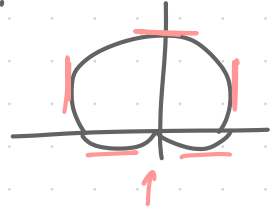
(b) horizontal or vertical tangents?

$$\frac{dy}{d\theta} = 0$$

(no vertical movement)

$$\frac{dx}{d\theta} = 0$$

(no horizontal movement)



from part (a)

$$\text{horiz: } \frac{dy}{d\theta} = 2 \cos \theta \sin \theta + \cos \theta = 0$$

$$\cos \theta (2 \sin \theta + 1) = 0$$



focus on θ
in $[0, 2\pi]$
because the
curve is
traced in
 $[0, 2\pi]$

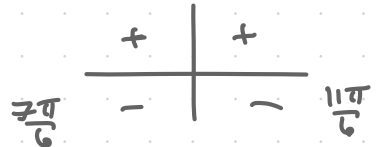
$$\cos \theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

or

$$2 \sin \theta + 1 = 0$$

$$\sin \theta = -\frac{1}{2}$$



$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\left[\frac{dy}{d\theta} = 0 \text{ at } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6} \right]$$

Need to also check $\frac{dx}{d\theta}$

$$\begin{aligned}\frac{dx}{d\theta} &= \cos^2\theta - \sin^2\theta - \sin\theta \\&= (1 - \sin^2\theta) - \sin^2\theta - \sin\theta \\&= 1 - 2\sin^2\theta - \sin\theta \\&= 1 - \underline{\sin\theta} - 2\sin^2\theta \quad \leftarrow \text{polynomial-ish} \\&= (1 - 2\sin\theta)(1 + \sin\theta) = 0\end{aligned}$$

$$1 - 2\sin\theta = 0$$

or

$$1 + \sin\theta = 0$$

$$\sin\theta = \frac{1}{2}$$

$$\sin\theta = -1$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{3\pi}{2}$$



$$\frac{dx}{d\theta} = 0 \quad \text{at} \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

horizontal tangents at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}$

vertical " at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

At $\theta = \frac{3\pi}{2}$, both $\frac{dy}{dx} = 0$ and $\frac{dx}{d\theta} = 0$

$\Rightarrow$ need to investigate further

$$\frac{dy}{dx} = \frac{0}{0}$$

$\leftarrow dy/d\theta$

$$\lim_{\theta \rightarrow \frac{3\pi}{2}^+} \left(\frac{dy}{dx} \right) = \lim_{\theta \rightarrow \frac{3\pi}{2}^+} \left(\frac{\cos \theta (2 \sin \theta + 1)}{(1 - 2 \sin \theta)(1 + \sin \theta)} \right)$$

$\nwarrow dx/d\theta$

$$\left| \frac{\cos(\frac{3\pi}{2}) (2 \sin(\frac{3\pi}{2}) + 1)}{(1 - 2 \sin(\frac{3\pi}{2})) (1 + \sin(\frac{3\pi}{2}))} = \frac{0 \cdot (-2 + 1)}{3 \cdot 0} = \frac{0}{0} \right|$$

$$= \lim_{\theta \rightarrow \frac{3\pi}{2}^+} \left(\frac{\cos \theta}{1 + \sin \theta} \cdot \frac{2 \sin \theta + 1}{1 - 2 \sin \theta} \right)$$

$$= \lim_{\theta \rightarrow \frac{3\pi}{2}^+} \left(\frac{\cos \theta}{1 + \sin \theta} \right) \cdot \lim_{\theta \rightarrow \frac{3\pi}{2}^+} \left(\frac{2 \sin \theta + 1}{1 - 2 \sin \theta} \right)$$

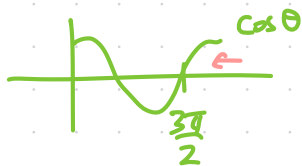
$\frac{0}{0}$, so use
L'Hospital's Rule

can just plug
in $\theta = \frac{3\pi}{2}$

$$\stackrel{LH}{=} \lim_{\theta \rightarrow \frac{3\pi}{2}^+} \left(\frac{-\sin \theta}{\cos \theta} \right) \cdot \left(\frac{2 \sin(\frac{3\pi}{2}) + 1}{1 - 2 \sin(\frac{3\pi}{2})} \right)$$

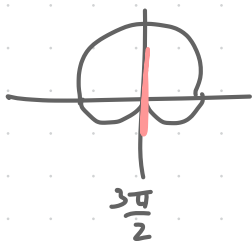
form $\frac{1}{0} = \pm \infty$

★ if $\cos \theta > 0$ as $\theta \rightarrow \frac{3\pi}{2}^+$ then limit is ∞
 if $\cos \theta < 0$ " " " limit is $-\infty$



$$= +\infty \cdot \left(\frac{-1}{3}\right) = -\infty$$

Get a vertical tangent at $\theta = \frac{3\pi}{2}$.



Areas Enclosed by Polar Curves

A sector of a circle of radius r :

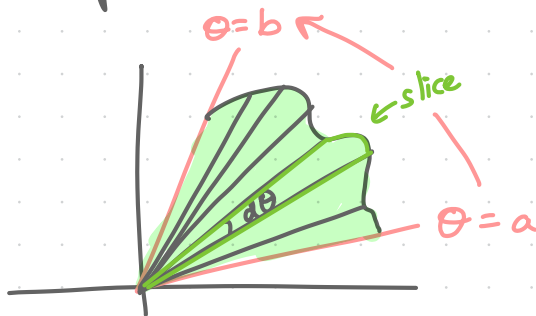


$$\text{Area of sector} = \left(\frac{\theta}{2\pi}\right)(\pi r^2)$$

$$A = \frac{1}{2} r^2 \theta$$

* θ in radians

A polar curve $r = f(\theta)$:



each slice is (approximately)
a sector of a circle
with angle $d\theta$:

$$dA = \frac{1}{2} r^2 d\theta$$

$$= \frac{1}{2} (f(\theta))^2 d\theta$$

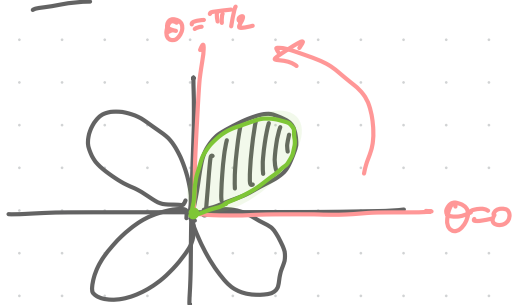
(* think of a ray from the origin, sweeping from a to b)

Area enclosed by $r = f(\theta)$:

$$A = \int_a^b dA = \int_a^b \frac{1}{2} r^2 d\theta$$

$$\text{OR } A = \int_a^b \frac{1}{2} f(\theta)^2 d\theta$$

ex: Find the area enclosed by $r = 2\sin(2\theta)$



compute area enclosed by 1 petal, then multiply by 4

$$A = \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} \frac{1}{2} (2\sin(2\theta))^2 d\theta$$

$$= 2 \int_0^{\pi/2} \sin^2(2\theta) d\theta$$

power reducing identity

$$\sin^2 \theta = \frac{1}{2} (1 - \cos(2\theta))$$

$\uparrow$
 2θ

$\uparrow$
 4θ

$$= 2 \int_0^{\pi/2} \frac{1}{2} (1 - \cos(2 \cdot 2\theta)) d\theta$$

$$= \int_0^{\pi/2} 1 - \cos(4\theta) d\theta$$

$$= \left(\theta - \frac{1}{4} \sin(4\theta) \right) \Big|_0^{\pi/2}$$

$$= \frac{\pi}{2} - \frac{1}{4} \sin(2\pi) - 0$$

$$= \pi/2$$

$$\text{total area : } 4\left(\frac{\pi}{2}\right) = 2\pi$$