The Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Note: You can only conclude *divergence* using the Divergence Test. If $\lim_{n\to\infty} a_n = 0$, the test is *inconclusive*.

The Geometric Series Test: This test only applies to geometric series, i.e. series of the form $\sum_{n=1}^{\infty} ar^{n-1}$. This series converges if $|r| < 1$ and diverges if $|r| \geq 1$.

Note: *If* the series converges, then we can compute the value using the formula $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

The p-Series Test: This test only applies to p-series, i.e. series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. A p-series converges if $p > 1$ and diverges if $p \leq 1$.

The Integral Test: Given $\sum a_n$, if $f(n)$ is a function such that $f(n) = a_n$ for each n and f is continuous, positive and decreasing, then

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \int_1^{\infty} f(x)\,dx \text{ converges.}$$

The (Direct) Comparison Test: Suppose $\sum a_n$, $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for each n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $b_n \leq a_n$ for each n, then $\sum a_n$ is also divergent.

The Limit Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad \text{where } c \text{ is positive and finite}$$

then either both series converge or both diverge.

Note: for both types of Comparison Test, good choices of series to compare to are p-series or geometric series.

The Alternating Series Test: This test applies only to alternating series, i.e. series that can be written $\sum(-1)^n b_n$ or $\sum(-1)^{n+1} b_n$ where $b_n > 0$. If

- $\lim_{n\to\infty} b_n = 0$, and
- $\{b_n\}$ is decreasing,

then the series is convergent.

Note: If one of the two conditions above is not true, Alternating Series Test is *inconclusive*. However, if the Alternating Series Test fails, the Divergence Test is a good idea to try next.

The Ratio Test: Suppose we have a series $\sum a_n$ and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

- If $L < 1$, the series is absolutely convergent (hence convergent).
- If $L > 1$, the series is divergent.
- If $L = 1$, the test is inconclusive.

1. Select the best test for determining whether the following series converge or diverge. Hint: try to eliminate the tests that do not apply. (You do not need to determine whether the series actually do converge or diverge.)

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n-1}}{(2n+1)!}$

   ~~A. p-series Test~~
   B. Divergence Test
   ~~C. Geometric Series Test~~
   D. Ratio Test

   Ratio Test is a good idea since the terms involve a factorial. The series is not a p-series or geometric series.

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{5n^3}{\sqrt{n^7-4}}$

   ~~A. Alternating Series Test~~
   ~~B. Divergence Test~~
   C. Ratio Test
   D. Limit Comparison Test

   Series is not alternating. Divergence Test fails. Limit Comparison is a good choice because the series is similar to a p-series. (Ratio test will be inconclusive.)

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n^2-2}$

   A. Alternating Series Test
   ~~B. Divergence Test~~
   ~~C. Integral Test~~
   D. Ratio Test

   Alternating Series Test makes the most sense. Divergence Test fails and Integral Test does not apply since terms are not positive. Ratio Test will be inconclusive.

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{10}{n(\ln(n))^2}$

   ~~A. Alternating Series Test~~
   ~~B. Divergence Test~~
   C. Integral Test
   D. Ratio Test

   Alt. Series Test does not apply. Div. Test fails. Integral Test is most appropriate. (Ratio Test will be inconclusive again.)

2. Find an example of each of the following or explain why such an example does not exist.

   (a) A sequence $\{a_n\}$ such that $\{a_n\}$ converges to 0, but $\sum_{n=1}^{\infty} a_n$ diverges.

   Let $a_n = \frac{1}{n}$. The seq. converges since $\lim_{n\to\infty} \frac{1}{n} = 0$.
   The series $\sum_1 \frac{1}{n}$ diverges.

   (b) A sequence $\{a_n\}$ such that $\{a_n\}$ diverges, but $\sum_{n=1}^{\infty} a_n$ converges.

   No such example exists by the Divergence Test

   (c) A sequence such that $\left|\frac{a_{n+1}}{a_n}\right| < 1$ for all n and the series $\sum_{n=1}^{\infty} a_n$ diverges.

   Consider $\sum_{n=1}^{\infty} 1/n$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{1/n+1}{1/n} = \frac{n}{n+1} < 1$
   for all n. Also this series diverges.

3. Suppose $a_n > 0$ for all n and the series $\sum_{n=1}^{\infty} a_n$ converges.

   (a) Must the series $\sum_{n=1}^{\infty} (-1)^n a_n$ converge? Explain why or why not.

   Yes since we know that $\sum_1 a_n = \sum_1 |(-1)^n a_n|$ converges,
   which means $\sum_1 (-1)^n a_n$ converges absolutely.

   (b) Why is it true that $a_n < 1$ for all n after a certain point?

   Since $\sum a_n$ converges, we know that $\lim_{n\to\infty} a_n = 0$.
   So the terms must get close to 0 after some point.

   (c) Show that $\sum_{n=1}^{\infty} a_n^2$ must converge. (Hint: use the Comparison Test)

   If $a_n < 1$ for all n after some point, then
   $a_n^2 < a_n$ " " " " " "
   Since $\sum_1 a_n$ converges, $\sum_1 a_n^2$ also converges
   by the Comparison Test.