

# Whitney Stratifications <sup>\*</sup>

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December 2022

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## Abstract

We study Whitney stratifications of subsets in Euclidean spaces. Examples are described as well as the methods to construct new stratifications from others via Cartesian product, intersection and union of manifolds in general position and transversal maps. Moreover, we introduce basic definitions and results of semialgebraic sets and the proof that these spaces admit a Whitney stratification with finitely many strata.

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<sup>\*</sup>Notes from a seminar presented in the Singularity Theory course of the Graduate Program in Mathematics of the ICMC-USP - Brazil.

# 1 Introduction

H. Whitney conjectured in the 1950's that smooth maps between a fixed pair of manifolds with the  $C^l$ -stability property were generic among the proper maps, that is, they formed a dense subset. R. Thom gave some counter-examples for this statement but conjectured that topological stability, that is,  $l = 0$ , satisfied this property. This latter was proved by J. Mather around 1970.

To deal with this question, R. Thom considered an intermediate between smooth and piecewise-linear categories of spaces, called regular stratified spaces, defined by Whitney previously. More specifically, this is used to control the behaviour of a smooth map on subsets of a given manifold. The two Thom's isotopy lemmas rely on this concept. Later, it was proved that a large class of sets could be stratified in this way. Here, we shall prove the existence for semialgebraic sets.

We begin with the basic definition of stratified sets and then introduce the regularity conditions. Also, we prove that this property is transferred under Cartesian product, union, and intersection operations on stratifications satisfying this condition. We also discuss the connections of strata considering the closures. In particular, this allows us to derive informations about the division of the set by dimensions. Finally, we briefly discuss some ideas to algebraic sets to motivate the main results for semialgebraic sets. Example are given throughout the text. The main text followed is [4], with some results, discussions, and references from other sources properly mentioned at each section. To begin with, we highlight the main idea and a convention.

**Idea.** *The main idea is to decompose a general subset, not necessarily a smooth manifold, into smooth pieces such that their behaviour is controlled by some regularity conditions on how they are connected.*

**Remark 1.1.** *The submanifolds considered here are Euclidean manifolds, that is, the smooth manifolds are supposed to be embedded into some euclidean space  $\mathbb{R}^n$ , for sufficiently large  $n$ . This is not crucial, since the definitions and arguments may be translated local charts, in the case of abstract manifolds.*

## 2 Whitney Stratified Sets

### Stratified Sets

We define the concept of a stratified set with some basic examples.

**Definition 2.1.** Let  $V$  be a subset of  $\mathbb{R}^n$ . A stratification  $\mathfrak{X}$  of  $V$  is a partition of  $V$  into disjoint smooth manifolds in  $\mathbb{R}^n$  without boundary (called the strata) which satisfies the local finiteness condition: every point in  $V$  has a neighbourhood which meet only finitely many strata.

**Example 2.1** (Parabola and a line). Let  $V = \{(x, y) \in \mathbb{R}^2 : x(y^2 - x) = 0\}$ . The two connected components of the  $y$ -axis without the origin and the parabola constitute a stratification for  $V$ .

**Example 2.2** (The axis). Let  $V = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ . Isolating the origin and considering the four connected components of the axis without this point, we obtain a stratification.

**Example 2.3** (Whitney Umbrella). Let  $V = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2z = 0\}$ . Cutting  $V$  along the  $z$ -axis, we divide it into two open surfaces and the  $z$ -axis, which gives a stratification. See figure 1.

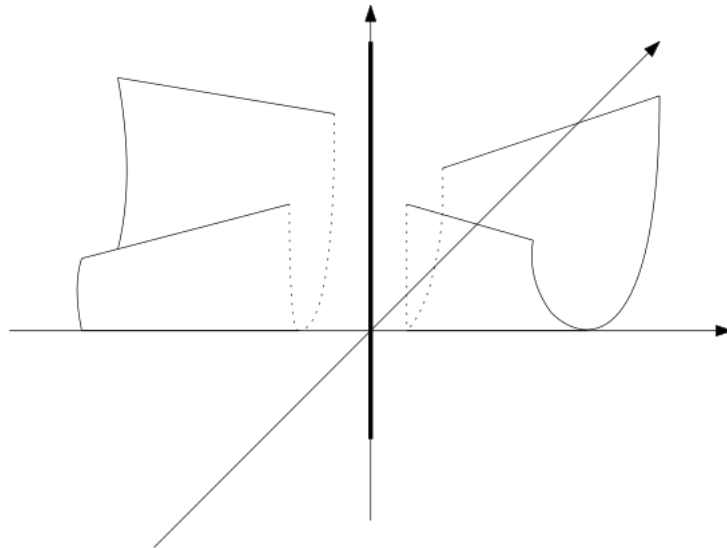


Figure 1: Stratification for Whitney's Umbrella

In general, algebraic sets always have a stratification, as we shall prove later. To conclude this section, we introduce a condition usually required to stratified sets and implied by the regularity condition, introduced in the next subsection.

**Definition 2.2** (Frontier Condition). *A stratification  $\mathfrak{X}$  satisfies the frontier condition if for any given pair of strata  $X, Y$  such that  $X \cap \bar{Y} \neq \emptyset$  it holds that  $X \subset \bar{Y}$ .*

Since we require disjointness for every pair of strata, the condition above being satisfied implies  $X \subset Y$  or  $X \subset \bar{Y} - Y$ . In other words, the condition also says that the frontier of a stratum is a union of strata. This is a first condition on how the strata are connected. The Whitney conditions concern to the same problem, as we shall see.

## Whitney Conditions

Although stratifications allow us to obtain smooth pieces of a general subset, they do not capture, in general, the singular behaviour. In particular, the topological setting along the connections. More specifically, in the previous examples, small neighbourhoods along the strata have similar topological nature, except in the origin. It suggests something different happens there. Indeed, the origin is exactly the singular point of the sets  $V$ . We would like to construct stratifications such that the points are *regularized* along the stratification in the sense that they will have constant topological figure on each stratum. This question is essentially related to how the strata are connected.

**Definition 2.3** (Whitney's Condition A). *Let  $Y, X$  smooth manifolds in  $\mathbb{R}^n$  and  $x \in X$  a point.*

1. *We say that  $Y$  and  $X$  satisfy the condition A at  $x$  if for every sequence of points  $(y_i) \subset Y$  converging to  $x$  and such that the sequence  $(T_{y_i}Y)$  of tangent spaces converges (in the Grassmannian space) to a subspace  $T \subset \mathbb{R}^n$  it holds that  $T_x X \subset T$ .*
2. *The manifolds  $X$  and  $Y$  satisfy the condition A if the previous implication holds for all  $x \in X$ .*

*We refer to this condition as A-regularity, Whitney A-regularity or Whitney's A condition.*

This is related to our previous examples since singular points were the locus of connection problems. However, we will see that it does not avoid the pathological topological behaviour.

**Example 2.4** (Condition A on the axis). Consider the union  $V$  of the axis and the stratification given in Example 2.1. The stratification satisfy the condition A. On the other hand, if we do not require disjointness and then define the stratification as the union of the axis this is not valid. Indeed, the limit lines of sequences on the axis are orthogonal.

**Example 2.5** (Condition A in general). In the space  $\mathbb{R}^3$  we may consider a curve tangent to a plane at a single point. In this situation, we have a pair satisfying A-regularity.

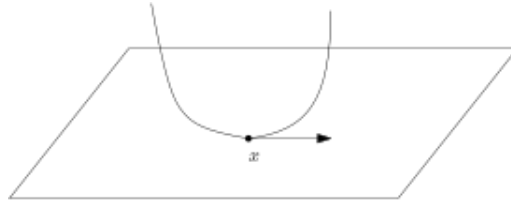


Figure 2: A-regularity between a curve and a plane

**Example 2.6** (Condition A on the parabola and the axis). Consider the Example 2.1. The A-regularity is verified in this pair. However, the pathological behaviour along each stratum remains. Therefore, we need a more refined condition.

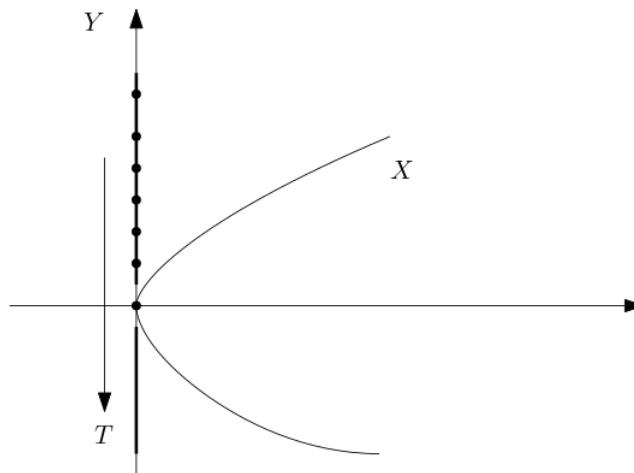


Figure 3: A-regularity for parabola and line

**Definition 2.4** (Whitney's Condition B). *Let  $Y, X$  smooth manifolds in  $\mathbb{R}^n$  and  $x \in X$  a point.*

1. We say that  $Y$  and  $X$  satisfy the condition  $B$  at  $x$  if for sequences of points  $(y_i) \subset Y$ ,  $(x_i) \subset X$  both converging to  $x$  and such that  $x_i \neq y_i$  for all  $i$ , the following condition holds: if the sequence  $(T_{y_i}Y)$  of tangent spaces converges (in a Grassmannian space) to a subspace  $T \subset \mathbb{R}^n$  and the sequence of lines  $(\overline{x_i y_i})$  converges (in the projective space) to a line  $L \subset \mathbb{R}^n$ , then  $L \subset T$ .
2. The manifolds  $Y$  and  $X$  satisfy the condition  $B$  if the previous implication holds for all  $x \in X$ .

We usually refer to this condition as regularity, Whitney regular, or Whitney's condition  $B$ .

The condition  $B$  is strong enough to imply condition  $A$ , as shows the next proposition. This was first noticed by Mather.

**Proposition 2.1** ([7], Proposition 2.4). *If the pair of manifolds  $Y$  and  $X$  satisfy the condition  $B$  at  $x \in X$ , then they satisfy the condition  $A$  at  $x$ .*

*Idea.* Let  $(y_i) \subset Y$  a sequence of points converging to  $x$  satisfying that the spaces  $T_{y_i}Y$  converges to  $T$ . It is enough to see that each direction  $l$  in  $T_x X$  may be approximated by a sequence of lines  $\overline{x_i y_i}$ , where  $(x_i) \subset X$  is a sequence of points distinct of  $y_i$  for each  $i$ , which converges to  $x$ . On the other hand, each  $l$  is a tangent vector of some curve  $X$ , therefore, a suitable choice of points in a suitable curve gives the sequence in  $X$ . Finally, condition  $B$  implies  $l \subset T$ . □

In other words, if  $Y$  is regular over  $X$  at  $x$ , then always that a sequence  $(y_i)$  in  $Y$  converges to  $x$  and  $(T_{y_i}Y)$  converges to  $T$ , it follows that  $T_x X \subset T$ . We translate now these conditions to stratification theory.

**Definition 2.5.**

1. A stratification  $\mathfrak{X}$  of a subset  $V \subset \mathbb{R}^n$  is an  $A$ -Whitney stratification if every stratum  $Y$  is  $A$ -regular over any other stratum  $X$ .
2. We say that the pair  $(V, \mathfrak{X})$  is an  $A$ -Whitney stratified set if  $\mathfrak{X}$  is an  $A$ -Whitney stratification of  $V$ .
3. If the condition  $A$  is replaced by the condition  $B$ , we say that the pair  $(V, \mathfrak{X})$  is a Whitney stratification.

**Example 2.7** (Condition B on the umbrella). Consider the Example 2.3. The stratification is not even  $A$ -regular. Indeed, consider a sequence of points in the  $y$ -axis converging to the origin. The (constant) sequence of tangent planes to the surface converges to the plane  $yx$ . On the other hand, this planes does not contain the  $z$ -axis. We can fix it by isolating the origin. See 4.

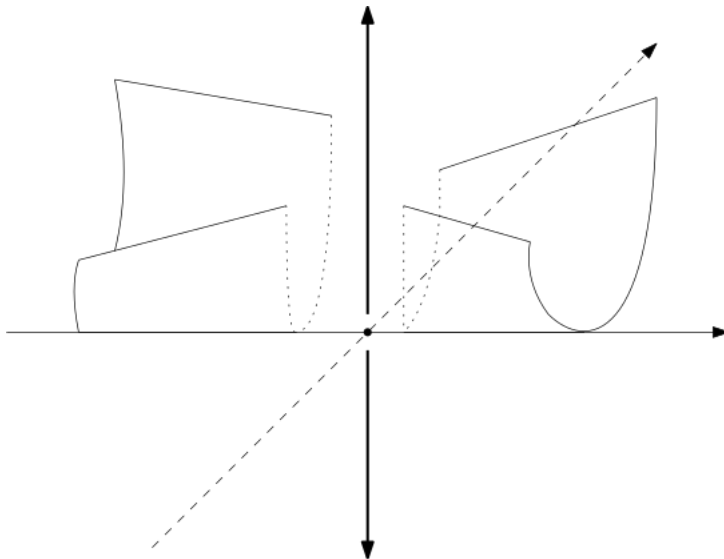


Figure 4:  $A$ -regularity for Whitney's Umbrella

**Example 2.8** (Circle and axis). A generalization of the previous examples is given by the set  $V = \{(x, y) \in \mathbb{R}^2 : xy(x^2 + y^2 - 1) = 0\}$ , where we must isolate each intersection of the circle with the other strata.

**Example 2.9** (The converse is not true). We have seen that the pairs in Examples 2.5 and 2.6 and satisfy  $A$ -regularity. On the other hand, they do not satisfy regularity. Indeed, we may verify this statement directly or by applying Proposition 2.1 considering the equality of dimensions. See Figure 5. Again, the condition is verified if we also isolate the origin also from the parabola.

**Example 2.10** (Spiral in the plane). Consider the curve  $Y$  characterized by the fact its tangent lines form a constant angle with radial direction. The curve does not contain the origin  $X$ , but any sequence of points  $(y_i)$  converging to  $X$  verifies that the tangent planes converge to the  $x$ -direction. In particular, the condition A is satisfied. On the other hand,

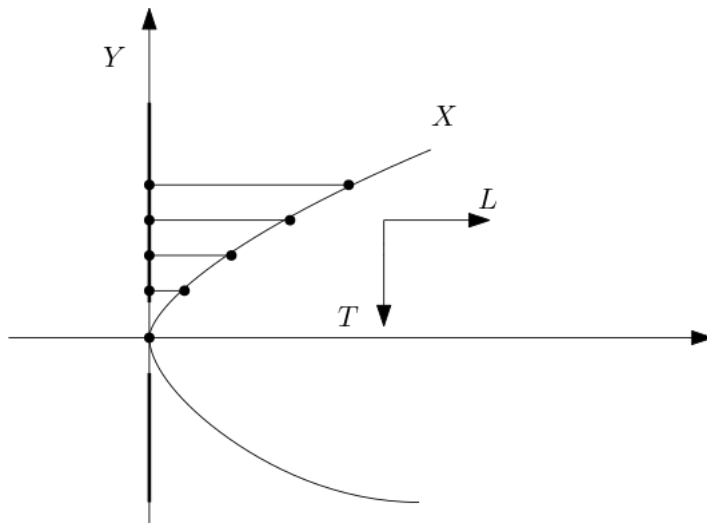


Figure 5:  $A$ -condition for parabola and line

since the origin is never reached, the lines converge to the radial direction. Therefore the pair is not regular.

The next theorem establishes the initial motivation to impose regularity condition, that is, the constant topological behaviour on each stratum.

**Theorem 2.1** (Thom-Mather, [6], Proposition 2.4). *Let  $(V, \mathfrak{X}) \subset \mathbb{R}^n$  be a Whitney stratified set. On each stratum  $X$  the topological figure is constant in the following meaning. Given two points  $x, y \in X$  there exist neighbourhoods  $U_x$  of  $x$  and  $U_y$  of  $y$  in  $V$  and a homeomorphism  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $h(U_x) = U_y$ .*

### 3 Constructing Regular Stratifications

#### Product Stratification

The first manner to obtain new stratification from others is by taking Cartesian products of manifolds in the Cartesian product of subsets.

**Definition 3.1.** *Let  $(V_i, \mathfrak{X}_i)$  be stratified sets of  $\mathbb{R}^{m_i}$ , where  $1 \leq i \leq n$ . The product stratification on  $V_1 \times \cdots \times V_n$  is given by strata of the form  $X_1 \times \cdots \times X_n$ , with  $X_i \in \mathfrak{X}_i$ . It is denoted by  $\mathfrak{X}_1 \times \cdots \times \mathfrak{X}_n$ .*

**Theorem 3.1** (Product Stratification). *If  $(V_i, \mathfrak{X}_i)$  is (an  $A$ )-Whitney stratified set for each, with  $1 \leq i \leq n$ , then the product stratification is a (an  $A$ )-Whitney stratification on  $V_1 \times \cdots \times V_n$ .*

*Proof.* Consider  $X = X_1 \times \cdots \times X_n$  and  $Y = Y_1 \times \cdots \times Y_n$ . Let  $(x_i) \subset X$  and  $(y_i) \subset Y$  sequences of points converging to  $x \in X$ . Suppose yet  $T_{y_i}Y$  converges to a subspace  $T$ . In details, we have that:

$$\begin{aligned} x_i &= (x_{i_1}, \dots, x_{i_k}) \longmapsto x \\ y_i &= (y_{i_1}, \dots, y_{i_k}) \longmapsto x \\ T_{y_i}Y &= T_{y_1}Y_1 \times \cdots \times T_{y_n}Y_n \longmapsto T \end{aligned}$$

The line  $L_i = \overline{x_i y_i}$  is decomposed by its projections on the Euclidean spaces  $\mathbb{R}^{m_i}$ , that is,  $L_i = \sum_j \pi_j(L_i)$ . We denote it by  $L_i = L_{i_1} \times \dots \times L_{i_n}$ . Note that this is a direct sum decomposition. By compactness, for each  $k$ , we may choose subsequences  $T_{y_{i_k}}Y_k$  (indexes still denoted by  $i$ ) such that  $T_{y_{i_k}}Y_k$  converges to  $T_k$  of dimension  $\dim(Y_k) = m_k$ . On the other hand, the convergence of the tangent spaces is in the direct product of the Grassmannians of subspaces with dimension  $m_i$  seen as a submanifold of the Grassmannian space of subspaces of dimension  $\sum_j m_j$ . Therefore,  $T = T_1 \times \cdots \times T_n$ . In details, we have that:

$$\begin{aligned} T_{y_1}Y_1 \times \cdots \times T_{y_n}Y_n &\longrightarrow T = T_1 \times \cdots \times T_n & T_{y_{i_k}} &\longrightarrow T_k \\ x_{i_k} &\longrightarrow x_k & y_{i_k} &\longrightarrow y_k \end{aligned}$$

Moreover, the same reasoning enable us to conclude that  $L_i$  converges to  $L$  in the form  $L = L_1 \times \cdots \times L_n$ . In this case, we consider a convergence in the direct product of the projective spaces (Grassmannians of dimension 1). Thus

$$\begin{aligned} L_i &\longrightarrow L & L_i &= L_{i_1} \times \dots \times L_{i_n} \\ L_{i_k} &\longrightarrow L_k & L &= L_1 \times \cdots \times L_n \end{aligned}$$

Now we apply the Whitney condition for each  $k$ , to conclude that  $L \subset T = T_1 \times \cdots \times T_n$ , because  $L_k \subset T_k$  for each  $k$ . For the condition A, we apply the same proof just removing the sequences of points (in  $X$ ) and lines.  $\square$

## Intersection and Union Stratifications

In general, the intersection of manifolds is not a manifold, as show the next example. Therefore, the construction of stratifications on the intersection of subsets is not direct. On

the other hand, the intersection of transversal manifolds is a manifold. In order to generalize for a finite number of manifolds, we introduce the concept of *general position*. We refer to [5, Section 1.5] for details.

**Example 3.1.** Let  $M = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$  and  $N = \{(x, y, z) \in \mathbb{R}^3 : z = xy\}$ . The set  $N$  is a manifold as the pre-image of zero by the function  $f = z - xy$ . It follows that  $M \cap N$  is the union of the axis  $x$  and  $y$ , which is not a manifold.

**Definition 3.2.** Let  $V$  be a vector space and  $H_1, \dots, H_r$  be subspaces of  $V$ . Then  $H_1, \dots, H_r$  are said to be in *general position* if for every sequence of integers  $1 \leq i_1 \leq \dots \leq i_s \leq r$  it holds that

$$\text{codim}(H_{i_1} \cap \dots \cap H_{i_s}) = \text{codim}(H_{i_1}) + \dots + \text{codim}(H_{i_s}).$$

**Proposition 3.1** (A characterization via transversality). *The subspaces  $H_1, \dots, H_r$  of  $V$  are in general position if and only if for any subset  $S \subset \{1, \dots, r\}$  and any  $k \notin S$  it holds that*

$$H_k \pitchfork \bigcap_{i \in S} H_i.$$

See, for example, [5, Exercise 3, Section 1.5].

**Corollary 3.1.** *Let  $H_1, \dots, H_r \subset V$  be subspaces in general position. If in addition for each  $i$ ,  $H_i$  is a subspace of a space  $W_i$ , then  $W_1, \dots, W_r$  are in general position.*

*Proof.* In the notation of the previous proposition, it is enough to see that

$$W_k + \bigcup_i W_i \supset H_k + \bigcup_i H_i = V,$$

because of transversality characterization. □

**Proposition 3.2.** *The subspaces  $H_1, \dots, H_r$  of  $V$  are in general position if and only if the projection map*

$$\pi : V \longrightarrow \bigoplus_{i=1}^r V/H_i$$

*is surjective.*

The next definition extends this concept for manifolds, which is again based on the definition of transversality.

**Definition 3.3.** Let  $X_1, \dots, X_r$  be denote manifolds in  $\mathbb{R}^n$ . The manifolds are said to be in general position if for each point  $x$  in the intersection it holds that  $T_x X_1, \dots, T_x X_n$  are in general position in  $\mathbb{R}^n$ .

As for transversal manifolds, the general position property allows us to conclude that the intersection is a manifold. As consequence, we will be able to derive an important property for the tangent space of the intersection which will be used in the main theorem of this section.

**Theorem 3.2** (Generality for Manifolds). Let  $X_1, \dots, X_n$  be in general position,  $S \subset \{1, \dots, r\}$  a subset of indexes of length  $s$ , and  $x \in X_1 \cap \dots \cap X_r$ .

1. It holds that the projection map below is surjective.

$$\pi : \mathbb{R}^n \longrightarrow \bigoplus_{j=1}^s \mathbb{R}^n / T_x X_{i_j}.$$

2. For each  $k \notin S$ ,  $\bigcap_{i \in S} T_x X_i \pitchfork T_x X_k$ .

3. For each  $k \notin S$ ,  $\bigcap_{i \in S} X_i \pitchfork X_k$  at  $x$ .

4. The intersection  $X_1 \cap \dots \cap X_n$  is a manifold whose tangent space satisfies

$$T_x(X_1 \cap \dots \cap X_r) = T_x X_1 \cap \dots \cap T_x X_r.$$

*Proof.* The two first items follow from the previous characterizations applied to the vector spaces  $T_x X_i$ , which are in general position. Here, we should note that for each point in the full intersection the partial intersections are *locally* manifolds, which follows from a well-known result of transversality (see [5, Section 1.5]). The second item and the previous remark implies the third item. Finally, the tangent spaces of the intersection of two transversal manifolds have a good behaviour. Using the third item, we may apply the property in [5, Exercise 3, Section 1.5] to obtain inductively the last statement.  $\square$

**Definition 3.4.**

1. Let  $(V_i, \mathfrak{X}_i)$  stratified sets of  $\mathbb{R}^n$ . We say that the stratifications are in general position if every  $n$ -tuple  $X_1, \dots, X_n$ , with  $X_i \in \mathfrak{X}_i$ , form a set of manifolds in general position.

2. We denote this stratification of  $V_1 \cap \dots \cap V_n$  by  $\mathfrak{X}_1 \cap \dots \cap \mathfrak{X}_n$ .

We remark that the previous definition is well posed by Theorem 3.2.

**Theorem 3.3** (Intersection Stratification). *Let  $(V_i, \mathfrak{X}_i)$  be (A-)Whitney stratified subsets of  $\mathbb{R}^n$ , where  $1 \leq i \leq n$ . If  $\mathfrak{X}_1, \dots, \mathfrak{X}_n$  are in general position, then  $\mathfrak{X}_1 \cap \dots \cap \mathfrak{X}_n$  is a (an A-)Whitney stratification of  $V_1 \cap \dots \cap V_n$ .*

*Proof.* We begin with regularity. Let  $X = X_1 \cap \dots \cap X_n$  and  $Y = Y_1 \cap \dots \cap Y_n$  be strata of the intersection stratification  $\mathfrak{X}_1 \cap \dots \cap \mathfrak{X}_n$ . As before, consider sequences  $(x_i), (y_i)$  both converging to  $x \in X$ ,  $(T_{y_i} Y)$  converging to  $T$ , and  $\overline{x_i y_i}$  to  $L$ . We want to verify that  $L \subset T$ . By compactness of the Grassmannian space, there exists, for each  $Y_k$  with  $1 \leq k \leq n$ , a convergent subsequence, still indexed by  $i$ , such that  $T_{y_i} Y_k$  converges to  $T_k$ . Here we extract subsequences from each factor of the intersection indexed by the suitable indexes to guarantee convergence for all of them simultaneously.

On the other hand, for each  $k$ ,  $Y_k$  is regular over  $X_k$ , thus  $L \subset T_k$  and hence  $L \subset \bigcap T_k$ . For each  $i$ , the tangent spaces  $T_{y_i} Y_k$  are in general position. By the properties discussed above, we have that:

$$T_{y_i} Y_1 \cap \dots \cap T_{y_i} Y_n = T_{y_i} (Y_1 \cap \dots \cap Y_n) = T_{y_i} Y.$$

Notice that the limits are not equal, since it is not necessarily true that the limit of the intersection is the intersection of the limits. However, for each  $k$ , we have that  $T_{y_i} Y \subset T_{y_i} Y_k$  for all  $i$ . Taking limits, we conclude that  $T \subset \bigcap T_k$ .

Since regularity implies A-regularity,  $T_k \supset T_x X_k$  for all  $k$ . Since the subspaces  $T_x X_k$  are in general position, applying Corollary 3.1, the spaces  $T_k$  are also in general position. By the item 1 of Theorem 3.2:

$$\dim(T) = \dim(T_{y_i} Y_1 \cap \dots \cap T_{y_i} Y_n) = \dim(T_x X_1 \cap \dots \cap T_x X_n) = \dim\left(\bigcap T_k\right),$$

for every index  $i$ . Therefore, it follows that  $T = \bigcap T_k \supset L$  and the regularity is verified. For condition A, note that  $T_x X_k \subset T_k$ . Therefore:

$$T_x X = \bigcap_k T_x X_k \subset \bigcap_k T_k = T,$$

as we prove below only under general position hypothesis. □

We deal now with stratifications of unions. Recall first that if we remove a closed subset of a manifold  $Z \subset \mathbb{R}^n$ , the result is a manifold, because according to the induced topology

we obtain an relative open set of  $Z$ . In particular, if  $W \subset \mathbb{R}^n$  is closed, then  $X - X \cap W$  is a manifold.

This leads to the following construction. Let  $(V_1, \mathfrak{X}_1) \dots, (V_n, \mathfrak{X}_n) \subset \mathbb{R}^n$  stratified closed sets. Suppose yet that the stratifications  $\mathfrak{X}_1, \dots, \mathfrak{X}_n$  are in general position, as in the Definition 3.4. We already know that the intersection  $X_1 \cap \dots \cap X_n$  of manifolds, where  $X_i \in \mathfrak{X}_i$ , is a manifold again. Then we have the following stratification, denoted by  $\mathfrak{X}_1 \cup \dots \cup \mathfrak{X}_n$  of the union  $V_1 \cup \dots \cup V_n$  and described by:

$$\left\{ X_i - (X_i \cap \bigcap_{j=1}^r V_j) : X_i \in \mathfrak{X}_i, 1 \leq r \leq n \right\} \cup \{X_1 \cap \dots \cap X_r : X_i \in \mathfrak{X}_i, 1 \leq r \leq n\}.$$

Since we are adding intersections of all lengths, the partition covers the whole set. We are able to conclude regularity for the union of two closed sets with this property.

**Theorem 3.4.** *Let  $(V_1, \mathfrak{X}_1), (V_2, \mathfrak{X}_2) \subset \mathbb{R}^n$  be (A-)Whitney stratified closed sets. Suppose that they are in general position. Then, the union stratification is a (an A-)Whitney stratification.*

*Proof.* We must verify regularity between the following types of pairs:

- $X_1 - X_1 \cap V_2$  and  $Y_1 - Y_1 \cap V_2$ , where  $X_1, Y_1 \in \mathfrak{X}_1$ : the (A-)regularity is a consequence of  $(V_1, \mathfrak{X}_1)$  being (A-)regular.
- $X_1 - X_1 \cap V_2$  and  $Y_2 - Y_2 \cap V_1$ , where  $X_1 \in \mathfrak{X}_1$  and  $Y_2 \in \mathfrak{X}_2$ : the regularity here has no meaning, since there are no sequence of points in one of them converging to a point in the other, because we are removing a closed set.
- $X_1 - X_1 \cap V_2$  and  $Y_1 \cap X_2$ , where  $X_1, Y_1 \in \mathfrak{X}_1$  and  $X_2 \in \mathfrak{X}_2$ : As before, there are no sequences of points for which we need to verify the conditions.
- $X_1 \cap X_2$  and  $Y_1 \cap Y_2$ , where  $X_i, Y_i \in \mathfrak{X}_i$ : this is a consequence of Theorem 3.3.

□

**Remark 3.1.** *One can consider an induction argument to obtain the same result for a finite union of regular stratified sets. However, in the induction step, we would also need a stronger condition: any subcollection of the sets are in general position. In particular, for any  $1 \leq i \leq n$ , the intersection of manifolds (each one from a different strata) of length  $i$  is again a manifold.*

## Transversal Stratification

A transversality condition on maps also induces stratifications on subsets of the source. Here we consider different ambient Euclidean spaces.

**Definition 3.5.** *Let  $(\mathfrak{X}', V')$  be a stratified subset of  $\mathbb{R}^m$ .*

1. *If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a smooth map transverse to every stratum  $X' \in \mathfrak{X}'$ , we say that  $f$  is transverse to  $\mathfrak{X}'$ .*
2. *The manifolds  $X = f^{-1}(X')$ , where  $X' \in \mathfrak{X}'$ , form a stratification  $\mathfrak{X}$  of  $f^{-1}(V')$  which is called the induced stratification.*

One notices that the fact of  $X$  is a manifold is a consequence of [5, Theorem, Section 1.5]. Local finiteness condition and completeness of decomposition is immediate.

**Theorem 3.5** (Transverse Stratification). *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  transverse to a (an  $A$ -)Whitney stratification  $\mathfrak{X}'$  of  $V' \subset \mathbb{R}^m$ . Then the stratified set  $(f^{-1}(V), \mathfrak{X})$  is a (an  $A$ -)Whitney stratified set.*

*Proof.* We begin with condition B. If  $\pi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  is the canonical projection and  $F : \mathbb{R}^n \rightarrow \text{graph}(f)$  is the diffeomorphism given by  $F(x) = (x, f(x))$ , then we may write  $f = \pi \circ F$ , where we denote the restriction of  $\pi$  by  $\pi$  itself. The induced stratification  $\mathfrak{X}$  of  $V = f^{-1}(V')$  induces a stratification  $\mathfrak{X}''$  on  $F(V) = V \times V'$  whose strata are of the form  $X'' = X \times X' = f^{-1}(X') \times X'$ . Now, it is enough to verify that this latter is Whitney.

Recall the following formulation of transversality in terms of  $\text{graph}(f)$ : the map is transversal to a given manifold  $Z$  if and only if  $\text{graph}(f)$  is transversal to  $\mathbb{R}^n \times Z$ . On the other hand, by Theorem 3.1, the stratification  $\mathbb{R}^n \times \mathfrak{X}'$  on  $\mathbb{R}^n \times V'$  is Whitney. Note that  $\text{graph}(f)$  is transversal to each stratum  $\mathbb{R}^n \times X'$  of this stratification. Moreover, the transversality between a pair of manifolds is a particular case of general position. By Theorem 3.3, the stratification

$$(\mathbb{R}^n \times \mathfrak{X}') \cap \text{graph}(f)$$

on the set  $(\mathbb{R}^n \times V') \cap \text{graph}(f) = f^{-1}(V') \times V' = V \times V'$  is Whitney. For condition A, the argument is the same, since the previous results for product and intersection stratification hold for this condition. This concludes the proof.  $\square$

The next figure gives an illustration of the stratification of  $V \times V'$  obtained as an intersection stratification.

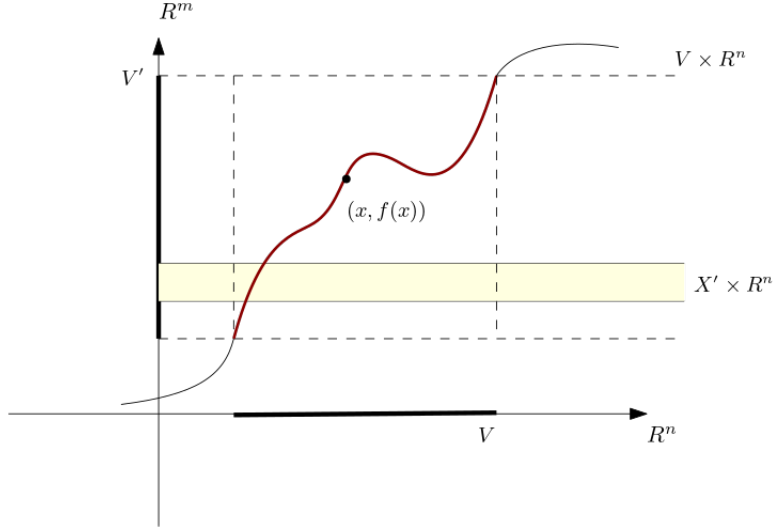


Figure 6: Transversal map inducing intersection stratification

**Corollary 3.2.** *Let  $(V, \mathfrak{X})$  a stratified set and  $U \subset \mathbb{R}^n$  an open set. Then  $\mathfrak{X}|_U$  is a stratification of  $U \cap V$ . Moreover, if  $\mathfrak{X}$  is (A)-regular, the same holds for  $\mathfrak{X}_U$ .*

*Proof.* Let  $i : U \rightarrow \mathbb{R}^n$  be the canonical inclusion. Since for each point  $x \in U$  the tangent spaces are the same, the map  $i$  is a submersion and then transversal to  $\mathfrak{X}$ . We have seen that it induces a stratification on  $i^{-1}(V) = U \cap V$ , given by the strata of the form  $i^{-1}(X)$ , where  $X \in \mathfrak{X}$ . It shows that each stratum is a restriction to  $U$  of some stratum in  $\mathfrak{X}$ . The statement about (A)-regularity also follows from the previous theorem.  $\square$

## Minimal and Canonical Stratifications

Rather than looking for a sufficiently large class of sets for which there exist Whitney stratifications, it may be useful to consider some notion of *uniqueness*. This section is devoted to this discussion. We start with a lemma which will be used to derive information about *filtrations*.

**Theorem 3.6** (Increasing Dimension). *Let  $X, Y$  be smooth manifolds and  $x \in X \cap \overline{Y - X}$ . Suppose that  $Y$  is regular over  $X$  at  $x$ . Then  $\dim(X) < \dim(Y)$ .*

*Proof.* Let  $(y_i)$  be a sequence in  $Y - X$  converging to  $x$ . By the compactness of the Grassmannian, we may suppose a subsequence of  $(T_{y_i}Y)$ , still indexed by  $i$ , and converging to a subspace  $T \subset \mathbb{R}^n$  of the same dimension as  $Y$ . Since the points lie on  $Y - X$ , there exists a

minimizing distance point  $x_i \in X$ . Now, again by the compactness of the Grassmannian, we may suppose  $(\overline{x_i y_i})$  a sequence of lines converging to a line  $L \subset T$ , by Whitney regularity. Since condition B implies condition A, we know that  $T_x X \subset T$ . Therefore,  $T_x X + L \subset T$ . But  $(\overline{x_i y_i})$  is orthogonal to  $T_{x_i} X$  for each  $i$ . We claim that the same holds for  $T_x X$  and  $L$ . Otherwise, for sufficiently large  $j$ , the line  $\overline{x_j y_j}$  does not connect  $y_i$  to a minimizing distance point, because the line would have a non trivial projection on  $T_x X$ . Therefore, the previous sum is direct and  $\dim(T_x X) = \dim(X) < \dim(Y)$ .  $\square$

**Example 3.2.** The Example 2.6 of the parabolla and the  $y$ -axis shows that Whitney's condition B is essential in the previous theorem, because the strata have same dimension. Indeed, the horizontal lines are the lines connecting the points  $y_i$  and the minimizing points  $x_i$ .

**Example 3.3.** In the Whitney stratification 2.7 of the Whitney umbrella, note that the origin is a point satisfying the hypothesis for every stratum  $Y$  and indeed it has the smallest strict dimension.

**Definition 3.6.**

1. Let  $(V, \mathfrak{X})$  be a stratified set. The associated filtration by dimension is the filtration  $(V_i)$  obtained by taking  $V_i$  to be the union of the strata of dimension less than or equal to  $i$ .
2. A partial order on the Whitney stratification is as follows: consider two stratifications  $\mathfrak{X}$  and  $\mathfrak{X}'$  of  $V$  and the associated filtrations  $(V_i)$  and  $(V'_i)$ . If there exists an integer  $i$  for which  $V_i \subset V'_i$  but  $V_j = V'_{j'}$  for all  $j > i$ , we say that  $\mathfrak{X} < \mathfrak{X}'$ .
3. If there exists a minimal element it is called a minimal Whitney stratification.

In other words, if  $\mathfrak{X} < \mathfrak{X}'$ , one have a refinement at the level of strata until some dimension and equality for the higher dimensions. One can ask about a refinement at the level of stratum, that is, when each stratum of  $\mathfrak{X}$  is contained in a stratum of  $\mathfrak{X}'$ . However, it is not true that maximal elements exist (see [4, p.14]). In a similar way, decomposing as much as we desire each stratum, minimal elements does not exist. This leads the definition above considering all strata until a given dimension for a partial order.

Now, Theorem 3.6 yields a corollary which details the relation between the different elements of a filtration of a Whitney stratified set. Roughly speaking, the filtration associated to a regular stratification is smoothly decomposed by dimensions and the manifolds of different dimensions are strictly separated from each other.

**Corollary 3.3.** *Let  $(V, \mathfrak{X})$  be a Whitney stratified set and  $(V_i)$  the associated filtration. Then the following conditions hold:*

1. *For each  $i$ ,  $V_i - V_{i-1}$  is empty or a manifold of dimension  $i$ .*
2. *For  $i > j$ ,  $V_i - V_{i-1} \cap \overline{V_j - V_{j-1}} = \emptyset$ .*

*Proof.* If  $V_i$  does not contain any stratum of dimension  $i$ , then  $V_i = V_{i-1}$ . Otherwise,  $V_i - V_{i-1}$  consists of strata of dimension exactly  $i$ . Since strata are disjoint,  $V_i - V_{i-1}$  is the disjoint union of manifolds, therefore, a manifold of dimension  $i$ . For the second item, let  $i > j$  and suppose that

$$(V_i - V_{i-1}) \cap \overline{(V_j - V_{j-1})} \neq \emptyset.$$

Since  $V_j - V_{j-1}$  does not contain any stratum of dimension  $i$ , we may rewrite the expression as

$$(V_i - V_{i-1}) \cap \overline{(V_j - V_{j-1}) - (V_i - V_{i-1})} \neq \emptyset.$$

By the regularity between the manifolds of  $\mathfrak{X}_i$ , we can apply Theorem 3.6 to obtain

$$\dim(V_j - V_{j-1}) = j > \dim(V_i - V_{i-1}) = i,$$

which is a contradiction. □

**Remark 3.2** (Frontier and Regularity Conditions). *We finish this section stating the non-trivial result that a regular stratification has the frontier condition satisfied for every pair of strata. See [9, Remark 1.2.4].*

## 4 Semialgebraic Sets

### Algebraic Sets

Recall that an algebraic set  $V \subset \mathbb{R}^n$  is defined as the common zeros of set of a finite collection  $f_1, \dots, f_n$  of polynomials. In this section, we introduce the notion of singular points for algebraic sets. We shall see that removing singular points we obtain smooth manifolds which turn out to be natural strata. The discussion here intends to motivate the analogous results to semialgebraic sets.

Let  $V$  be an algebraic set defined by the polynomials  $f_i$ ,  $1 \leq i \leq k$ . Consider the map:

$$F_V : \mathbb{R}^n \longrightarrow \mathbb{R}^k$$

$$x \longmapsto (f_1(x), \dots, f_k(x))$$

**Definition 4.1.**

1. A point  $p \in V$  is called *singular* if the map  $F_V$  does not have maximal rank at  $p$ .
2. The set of singular points is denoted by  $\Sigma(V)$ .
3. The points in  $V - \Sigma(V)$  are called *regular*.
4. The dimension of  $V$  is  $n - r$ , where  $r$  is the maximal rank of the map  $F_V$ .

**Remark 4.1.** An algebraic set may be defined by different sets of polynomials. Therefore, it remains to verify that the dimension does not depend on this choice. Nevertheless, this is true (see [1, Proposition 3.3.10]).

**Proposition 4.1.** Let  $V \subset \mathbb{R}^n$  be an algebraic set of dimension  $d$ . Then  $\Sigma(V)$  is also an algebraic set of strict lower dimension and  $V - \Sigma(V)$  is a smooth manifold of dimension  $n - d$ .

*Idea.* For the structure of manifold, the map  $F_V$  restricted to  $V - \Sigma(V)$  is a smooth map of constant rank  $d$ . Recall the Rank Theorem. Then the map is locally written in the following form:

$$\phi^{-1} \circ F_V \circ \psi(x_1, \dots, x_n) = (x_1, \dots, x_d, 0, \dots, 0),$$

where  $\phi, \psi$  are diffeomorphisms of open sets in  $\mathbb{R}^n$ . Therefore, the map is locally written as the zero locus  $x_1 = \dots = x_d = 0$ .

For the others statements, we discuss the basic ideas, for details, see [1, Proposition 3.3.14]. Note that  $\Sigma(V)$  is the disjoint union of the sets  $Z_i$  for which  $F_V$  has rank  $i$ , where  $1 \leq i < d$ . Each  $Z_i$  is given by a restriction of the entries of the Jacobian matrix  $J(F_V)$ , which are polynomials, since  $F_V$  is polynomial. Since the union of algebraic sets is again an algebraic set, we conclude that  $\Sigma(V)$  is an algebraic set. Note yet that we have imposed new restrictions keeping the same number of variables  $x_1, \dots, x_n$ . Considering the local form of the manifold structure given in the previous paragraph, we obtain that the dimension of  $\Sigma(V)$  necessarily decreases. □

Therefore, regular points of an algebraic set  $V$  are the points for which  $V$  is locally a manifold. Conversely, suppose  $U \cap V$  is a  $n - r$ -dimensional manifold. It is possible to show that this neighbourhood may be seen as the zero set of a collection of  $n - r$  polynomials, and the associated map  $F_V$  has rank  $r$  (see [1, Proposition 3.3.10]). The consequence will be that the definition in the context of semialgebraic sets coincides with the previous one. As a motivation, we prove the existence of stratifications for algebraic sets, since the general idea for semialgebraic sets will be similar.

**Proposition 4.2.** *Every algebraic set admits a stratification.*

*Proof.* Let  $V \subset \mathbb{R}^n$  be an algebraic set. We construct a filtration for  $V$  inductively as follows. Denote  $d = \dim(V)$  and fix  $V_{d-1} = \Sigma(V)$ . For each  $1 \leq i < d - 1$ , define:

$$V_{i-1} = \begin{cases} \Sigma(V_i) & \text{if } \dim(V_i) = i \\ V_i & \text{if } \dim(V_i) < i \end{cases}.$$

We know from the previous proposition that the dimension of the singular points is strictly lower. Thus, we are obtaining a partition such that  $V_{i-1}$  contains the strata with dimension bounded by  $i - 1$ . From the above construction it follows that the sets  $V_i - V_{i-1}$  are manifolds. Indeed, the desired stratification is given by these sets. To conclude, note that  $\bigcup_i (V_i - V_{i-1}) = V$  and that there are finitely many manifolds, since the dimension necessarily decreases.  $\square$

**Remark 4.2.** *We may construct a partition only taking the singular points for each step  $i$ :  $V_{i+1} = \Sigma(V_i)$ . However, it does not give a filtration, or more specifically, a well controlled division by dimensions.*

## Semialgebraic Sets

The main motivation to introduce semialgebraic sets is to obtain a class of objects with a morphism associated which preserves this class. This section is mainly based on [8, Section 1.1]. For example, the image of an algebraic set by an algebraic map is not necessarily an algebraic set, as shows the next example.

**Example 4.1.** Let  $V = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 1 = 0\}$ . The canonical projection on the first coordinate  $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}$  is an algebraic map, but  $\pi(V) = [-1, 1]$  is not algebraic, since every non-zero polynomial on one variable has finitely many roots.

This leads to the definition of semialgebraic sets.

**Definition 4.2.** A subset  $X \subset \mathbb{R}^n$  is semialgebraic if there exist finitely polynomials  $f_{ij}, g_{ij} \in \mathbb{R}[x_1, \dots, x_n]$  such that

$$X = \bigcup_i \bigcap_j \{x \in \mathbb{R} : f_{ij}(x) = 0; g_{ij}(x) > 0\}.$$

**Example 4.2.** An annulus is a semialgebraic set. For example, the annulus centered at the origin with rays 1 and 2 is defined by the following equations:

$$\begin{aligned} x^2 + y^2 - 2 &\leq 0 \\ x^2 + y^2 - 1 &\geq 0. \end{aligned}$$

**Example 4.3.** Define  $V = \{(x, y) \in \mathbb{R}^2 : y = e^x\}$ . It follows that  $V$  is not semialgebraic. Suppose the contrary. Since  $V$  is not open, there exists at least one nonzero polynomial  $f \in \mathbb{R}[x, y]$  such that  $f|_V = 0$ . Write  $f(x, y) = \sum a_{ij}x^i y^j$ . If there is no  $y$  factor in the decomposition of  $f$ , then  $f(x, e^x)$  is a polynomial on  $x$  identically null. We have the following description:

$$f(x, e^x) = \sum a_{ij}x^i e^{jx} \equiv 0.$$

Let  $(M, N)$  be the largest powers of the monomial  $x$  and the exponential in the decomposition above associated to the pair  $(i_1, j_1)$ . Dividing the expression by  $x^M e^{Nx}$  and making  $x \rightarrow \infty$ , we obtain that  $a_{i_1 j_1} = 0$ . We repeat this process to conclude that for all indexes  $j > 0$  for which there is a  $Y^j$  factor, it must hold that  $a_{ij} = 0$ . We conclude that  $f(x, y)$  is a polynomial only on  $x$  and since  $f(x, e^x) = f(x) = 0$  is null.

**Proposition 4.3** (Properties). *The following properties hold.*

1. An algebraic set is semialgebraic.
2. If  $X, Y \subset \mathbb{R}^n$  are semialgebraic, then  $X \cup Y$ ,  $X - Y$ , and  $X \cap Y$  are semialgebraic.
3. If  $X \subset \mathbb{R}^n$  and  $Y \subset \mathbb{R}^p$  are semialgebraic, then  $X \times Y \subset \mathbb{R}^{n+p}$  is semialgebraic.

The main result is the following.

**Theorem 4.1** (Tarski-Seidenberg). *Let  $\pi : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^p$  be the canonical projection. If  $X \subset \mathbb{R}^n \times \mathbb{R}^p$  is semialgebraic, then  $\pi(X)$  is also semialgebraic.*

**Corollary 4.1.** *If  $X$  is semialgebraic, then  $\overline{X}$ ,  $X^\circ$ , and  $\partial X$  are also semialgebraic.*

We notice that in the previous corollary, it is enough to prove the statement for the closure. Indeed, we have the following equalities:

$$\begin{aligned}\overline{X} &= X \cup \partial X \\ X^\circ &= \mathbb{R}^n - \overline{\mathbb{R}^n - X}\end{aligned}$$

Then, the conclusion follows by Proposition 4.3, item 2.

**Definition 4.3.** *Let  $V$  be semialgebraic. A mapping  $F : V \rightarrow \mathbb{R}^p$  is semialgebraic if  $\text{graph}(f) \subset \mathbb{R}^n \times \mathbb{R}^p$  is semialgebraic.*

**Corollary 4.2.** *Let  $X \subset \mathbb{R}^n$  be semialgebraic. If  $F : X \rightarrow \mathbb{R}^p$  is semialgebraic, then  $F(X) \subset \mathbb{R}^p$  is a semialgebraic.*

*Proof.* We have that  $\text{graph}(F) = (X, F(X))$ , which is semialgebraic by hypothesis. But  $\pi_p(\text{graph}(X)) = F(X)$  must be semialgebraic, by Tarski-Seidenberg Theorem.  $\square$

**Example 4.4.** Let  $D^2 \subset \mathbb{R}^2$  defined by the inequality  $x^2 + y^2 - 1 \leq 0$ . If  $F$  denotes the restriction to  $D^2$  of the canonical projection on the first coordinate, then  $\text{graph}(F)$  is the finite cylinder in  $\mathbb{R}^3$  defined by the following inequalities:

$$\begin{aligned}x^2 + y^2 - 1 &\leq 0 \\ z^2 - 1 &\leq 0\end{aligned}$$

**Example 4.5.** Consider  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  given by  $F(u, v) = (u, uv, v^2)$ . The image of  $F$  is the Whitney umbrella without the negative part of the  $z$ -axis. Recall that the Whitney umbrella is the zero set of  $f(x, y, z) = y^2 - x^2z$ . That is,  $\text{Im}(F)$  is defined by:

$$\begin{aligned}f(x, y, z) &= 0 \\ z &\geq 0\end{aligned}$$

**Example 4.6.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = e^x$ . We have seen in the Example 4.3 that its graph is not semialgebraic. It follows that  $f$  is not semialgebraic.

**Remark 4.3.** *Let  $F : X \rightarrow \mathbb{R}^n$  be semialgebraic. It follows that the domain is necessarily semialgebraic, because  $\pi(\text{graph}(f)) = X$ .*

The theory of semialgebraic sets allows us to derive when the orbits of a smooth Lie group action are embedded manifolds. This is particularly important to the actions of Mather reduced groups on jets spaces. For instance, recall that this a hypothesis of Mather's Lemma, which we have been used frequently.

**Proposition 4.4** (Appendix B.4, [3]). *Let  $\Psi : G \times M \longrightarrow M$  be a smooth Lie group action which is also a semialgebraic map. Then the orbits are submanifolds of  $M$ .*

*Proof.* We are supposing that  $G \times M$  is a semialgebraic set, according to the definition. By Tarski-Seindeberg Theorem, the projection  $\pi(G \times M) = G$  is a semialgebraic set, where the projection is on the first coordinate. For each point  $x \in M$  we shall prove that there exists a neighbourhood  $U_x$  of  $x$  in  $M$  such that  $U_x \cap G.x_0$  is a smooth manifold. Let  $x_0 \in M$  an arbitrary point and  $G.x_0$  its orbit. Since  $G$  and  $\{x_0\}$  are semialgebraic, it follows that  $G \times \{x_0\} \times \mathbb{R}^k$  is a semialgebraic set, where  $M \subset \mathbb{R}^k$ . On the other hand, the orbit may be obtained as:

$$G.x_0 = \pi(\text{graph}(\Psi) \cap G \times \{x_0\} \times \mathbb{R}^k).$$

Thus, the orbit is semialgebraic. By Theorem 4.2, item 1, the set of points for which  $G.x_0$  is locally a manifold is non-empty. Let  $x_1$  be a point with this property. Given any other point  $y \in G.x_0$ , there exists  $g_1 \in G$  such that  $\Psi(g_1, x_1) = y$ . But the restriction of  $\Psi$  to  $\{g_1\} \times M$  is a diffeomorphism. Therefore, the property is transferred to all  $G.x_0$ .  $\square$

**Lemma 4.1** (Appendix B.1, [3]). *Let  $V \subset \mathbb{R}^n$  be a semialgebraic set and  $F : X \longrightarrow \mathbb{R}^n$  a rational map. Then  $F$  is semialgebraic.*

*Proof.* Write  $F = (F_1, \dots, F_p)$ , where  $F_i = \phi_i(x)/\psi(x)$ , for polynomials  $\phi_i$  and  $\psi_i$ . Introduce for  $\mathbb{R}^n \times \mathbb{R}^p$  the coordinates  $(x_1, \dots, x_n, y_1, \dots, y_p)$ . Now, the set  $\text{graph}(F)$  is described by the relations defining  $X$  and the following equations:

$$\phi_i - \psi_i y_i = 0.$$

In other words,  $\text{graph}(F)$  consists of semialgebraic part, correspondent to  $X$ , and a purely algebraic part defined by the coordinate functions as above.  $\square$

**Corollary 4.3.** *Let  $\mathcal{R}^k \times J^k(n) \longrightarrow J^k(n)$  the action of the group  $\mathcal{R}^k$  on the space of  $k$ -jets given by*

$$(j^k(h), j^k f) \longmapsto j^k(h \circ f).$$

*Then the orbits are smooth manifolds.*

*Proof.* The action is a polynomial map and also a Lie group action on a smooth manifold. The conclusion follows by applying first Lemma 4.1 and then Proposition 4.4.  $\square$

## Singular and Bad Points

**Definition 4.4.** Let  $V \subset \mathbb{R}^n$  be a semialgebraic set.

1. A point  $p \in X$  is called  $d$ -regular if there exists a neighbourhood  $U \subset \mathbb{R}^n$  of  $p$  such that  $U \cap X$  is smooth  $d$ -dimensional manifold.
2. Suppose that  $X$  admits a regular point. The dimension of  $X$  is the maximal dimension among the regular points.
3. Assume yet that  $X$  has a regular point and  $\dim(X) = d$ . A singular point is a point which is not  $d$ -regular.

**Remark 4.4.** Note that a point may be both singular and  $d$ -regular for some  $d$ .

**Theorem 4.2.** Let  $X, Y \subset \mathbb{R}^n$  be semialgebraic sets.

1. The set of regular points of a semialgebraic set is non-empty and dense.
2. If  $X \subset Y$ , then  $\dim(X) \leq \dim(Y)$ .
3.  $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$ .
4.  $\dim(\overline{X}) = \dim(X)$ .

We notice that the notion of dimension coincides for algebraic sets. Therefore, the properties above hold in the previous context. Now, the analogous of Proposition 4.1 is the following result.

**Theorem 4.3.** The singular set  $\Sigma(X)$  of a semialgebraic set  $X$  is a semialgebraic set such that  $\dim(\Sigma(X)) < \dim(X)$ .

To obtain a stratification for any semialgebraic set, we proceed as for algebraic sets by excluding the set of singular and non-maximal regular points. To guarantee regularity, we shall exclude the points for which regularity fails. An important result shows that this set is also semialgebraic.

**Definition 4.5.**

1. Let  $Y, X$  be manifolds. We define the set  $B(Y, X)$  to be the set of points  $x$  in  $X$  such that  $Y$  is not regular over  $X$  at  $x$ . It is called the bad set.
2. Let  $A, B$  be semialgebraic sets. We denote

$$W(A, B) := \Sigma(A) \cup B(A - \Sigma(A), B - \Sigma(B)).$$

In the above definition, note that removing singular points, including the non maximal regular points, we indeed obtain smooth manifolds and the definition makes sense.

**Theorem 4.4.** *Let  $Y, X$  be semialgebraic manifolds. Then  $B(X, Y)$  is a semialgebraic set satisfying:*

$$\dim(B(X, Y)) < \dim(X).$$

**Corollary 4.4.** *Let  $A, B$  be semialgebraic sets. Then  $W(A, B)$  is a semialgebraic set such that  $\dim(W(A, B)) < \dim(A)$ .*

*Proof.* The set  $\Sigma(A)$  of singular points is semialgebraic as well as the bad set of the manifolds. Moreover, by Proposition 4.2, we have that:

$$\begin{aligned} \dim(W(A, B)) &= \max\{\dim(\Sigma(A)), \dim(B(A - \Sigma(A), B - \Sigma(B)))\} \\ &< \max\{\dim(A), \dim(A - \Sigma(A))\} \\ &= \dim(A). \end{aligned}$$

□

**Theorem 4.5.** *Let  $X \subset \mathbb{R}^n$  be a semialgebraic set. Then  $X$  admits a canonical Whitney stratification with finitely many strata.*

*Proof.* The stratification is as in the algebraic case. There, we remove the non regular points to obtain manifolds which turn out to form the stratification. We also consider a filtration by semialgebraic sets. Since we desire regularity, we must exclude points for which regularity fails. As before, the construction is by induction. Fix  $V_d = V$ . Then, for each  $i < d$ , we define:

$$V_{i-1} = \begin{cases} V_i & \text{if } \dim(V_i) < i \\ \overline{\left\{ \bigcup_{j=i+1}^d W(V_i, V_j - V_{j-1}) \right\}}^V & \text{if } \dim(V_i) = i. \end{cases}$$

We detail the following:

- $V_{i-1}$  is semialgebraic because each  $W$  consists of the union of singular points of a semialgebraic set and the bad set related to semialgebraic sets. Therefore, the union is semialgebraic as well as the closure in  $V$  by the previous properties. We observe that the closure in  $\mathbb{R}^n$  is semialgebraic and then the intersection with  $V$  yields a semialgebraic set and also the closure in relation to  $V$ .
- $V_i - V_{i-1}$  is a smooth manifold: recall that the closure of the union is the union of closures and that  $W(V_i, V_j - V_{j-1})$  contains  $\Sigma(V_i)$ . The other sets are all closed, so  $V_i - V_{i-1}$  is obtained by removing a closed set of the smooth manifold  $V_i - \Sigma(V_i)$ .
- The stratification is formed by the manifolds  $V_i - V_{i-1}$  which are in finitely many number, because the dimension necessarily decreases by Theorem 4.3.
- Let  $i > j$  and consider a pair  $V_i - V_{i-1}$  and  $V_j - V_{j-1}$ . Suppose the both non empty sets. Note that  $V_{i-1}$  contains all points for which the regularity fails:

$$W(V_i, V_j - V_{j-1}) = \Sigma(V_i) \cup B(V_i - \Sigma(V_i), V_j - \Sigma(V_j)).$$

- It remains to see that the stratification is canonical. For each  $i$ ,  $V_i - V_{i-1}$  is the only manifold of dimension  $i$  and by construction  $V_j - V_{j-1}$  is regular on it for every  $j > i$ , which shows that the stratification is canonical.

□

## 5 Further Remarks

We discuss further remarks on more advanced results mentioned before. Firstly, we would like to indicate how stratification theory may show up in the proof of Topological Stability Theorem.

- The notion of stratification is generalized to maps and vector fields on manifolds.
- According to D. Trotman [9], the two major results of the theory are: the existence of regular stratifications for analytic varieties whose strata are analytic manifolds, due to Whitney, and a stronger version of the theorem stating the invariance of the topological figure [9, Theorema 1.2.17].

- Thom also proved two important results, called Thom's Isotopy Lemmas, relating the behaviour of restrictions of smooth maps to strata. The second one has the following implication: Let  $M, N$  be smooth manifolds and  $F : M \times \mathbb{R} \rightarrow N \times \mathbb{R}$  a smooth map of the form  $F(x, t) = (f_t(x), t)$ . Under some conditions on the existence of stratifications for  $M$  and  $N$  and properties relating them and the map  $F$ , it is possible to conclude that  $f_t(x)$  is a topologically trivial family.

Secondly, we have seen that semialgebraic sets have some interesting properties in the smooth meaning. Indeed, more information can be derived about their structure.

- It is possible to show that any semialgebraic set has a finite number of connected components, which are also semialgebraic [1, Theorem 2.4.5].
- It is possible to show that any semialgebraic set has a decomposition into disjoint union of semialgebraic sets each of them semialgebraically diffeomorphic to an open hypercube [1, Theorem 2.3.6]. This leads to the definition of dimension as the maximal dimension among the hypercubes. It is possible to verify that this definition depends on the decomposition [1, Corollary 2.8.9], [2, Proposition 3.15]. An algebraic formulation also exists as an intrinsic property [1, Definition 2.8.1].
- The previous definition of dimension is not enough to guarantee the properties in Theorem 4.2. More specifically, consider the property 3. Intuitively, the closure must be obtained by relaxing the condition on the polynomials. This is not true in general as shows the next example. Define:

$$A = \{(x, y) : x^3 - x^2 - y^2 > 0\}$$

The set  $\bar{A}$  is described by the points  $(x, y)$  satisfying  $y^2 \leq x^3 - x^2$  and  $x \geq 1$ . Indeed, the origin satisfies  $x^3 - x^2 - y^2 \geq 0$ , but it is not contained in  $\bar{A}$ . However, in the mentioned decomposition, the closure is obtained as the union of the closures, and the dimension is preserved for each of them. Indeed, it means that the closure consists of the original set and extra pieces which do not increase the dimension [2, Corollary 3.8, Proposition 3.16].

- The previous decomposition also suggests that regular points are dense. Indeed, this is true, since we know that the singular set is a semialgebraic set with dimension strictly lower.

- The decomposition above suggests a kind of a stratification for semialgebraic sets. Moreover, for compact semialgebraic sets there exists a triangulation [2, Section 3.2].

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