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Modified Backstepping Algorithm for Plants under Mismatched Disturbances and Varying Time-Delay*

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Abstract— **The paper deals with an algorithm, formed from the synthesis of a control system based on a backstepping algorithm and an auxiliary loop algorithm, for the compensation of mismatched disturbances depending on the unknown varying time-delay. The proposed algorithm guarantees the stabilization of the plant with the required accuracy. We also compare the proposed algorithm with the backstepping algorithm and the auxiliary loop algorithm. The simulation results illustrate the efficiency and robustness of the suggested control system.**

I. Introduction

The paper is devoted to the synthesis of a robust algorithm for plants under mismatched nonlinear parametric uncertainties and external disturbances with varying time-delay. The backstepping algorithm is one of the most effective and widely-used algorithms [1, 2]. According to [3], the backstepping algorithms provide a high quality of transients without increase of the amplitude of a control signal and these algorithms have low dynamic order. The first backstepping algorithm was proposed in [4]. There are results for control of plants with time delay.

In [5] for nonlinear systems with delay a new technique of stability and robustness analysis was proposed. It relies on the construction of functional which make it possible to establish estimates of the solutions different from, but very similar to, estimates of input-to-state stability. The approach applies to systems which do not have a globally Lipchitz vector field and are not necessarily locally exponentially stable.

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In [6] the state feedback control problem is addressed for a class of nonlinear time-delay systems. The time delays appear in all state variables of the nonlinear system, which brings a challenging issue for controller design. With the help of a backstepping method, a memoryless state feedback controller is designed, which does not need the precise knowledge of time delays. It is rigorously proved that the closed-loop system is asymptotically stable.

The auxiliary loop algorithm belongs to algorithm with disturbances compensation, where main idea consist of representation of uncontrollable disturbances as a new known function. Currently, they have been many propositions for disturbances compensation, for example [2, 7, 8]. Among of them the auxiliary loop algorithm is one of the effective algorithm. The auxiliary loop algorithm was first proposed by [2] for control of a continuous-time plant under matching parametric uncertainties and external disturbance. The idea of this method consists in implementing an auxiliary loop with desired parameters parallel to the plant. The difference between the output of the plant and the output of the auxiliary loop gives a function which depends on parametric and external disturbances. This function gives the control law that guarantees required accuracy of the control system. The proposed algorithm provides the tracking by the output of the plant of the reference output with the required accuracy. However, mismatched disturbances could make system unstable.

However, there are not papers devoted to compensation of mismatched disturbances with unknown varying time-delay. In this paper we consider a modified backstepping algorithm with disturbances compensation (MBADC). This algorithm is a generalization of the results shown in [1] and [2] for robust control of nonlinear plants under mismatched parametric uncertainties and external disturbances depending on unknown varying time-delay. It is assumed that the state vector of the plant is available for measurement. The proposed algorithm guarantees stabilization of the plant state vector with the required accuracy.

The paper is organized as follows. The problem statement is presented in Section 2. The MBADC for control of nonlinear plants under mismatched perturbations is proposed in Section 3. In Section 4 the efficiency of MBADC is illustrated by modeling of an unstable nonlinear plant. Also, in Section 4 the comparison of the simulation results for the MBADC, the

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backstepping algorithm and auxiliary loop algorithm are presented. Concluding remarks are given in Section 5. Appendix A gives the proof of the MBADC.

II. Problem Statement

Consider the plant model with varying time-delay in the form

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + \varphi_1\big(x_2(t), ..., x_n(t), \\ &x_2(t-\tau_{12}(t)), ..., x_n(t-\tau_{1n}(t)), f, t\big), \\ \dot{x}_2 &= x_3 + \varphi_2\big(x_3(t), ..., x_n(t), \\ &x_3(t-\tau_{23}(t)), ..., x_n(t-\tau_{2n}(t)), f, t\big), \\ &\vdots \\ \dot{x}_n(t) &= bu(t) + \varphi_n\big(f, t\big), \end{aligned} \tag{1}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^{\mathrm{T}}$ is a state vector, $u(t) \in R$ is an input, $\varphi_i(\cdot) \in R$, $i = 1, 2, \ldots, n$ are unknown functions, f is a vector of unknown parameters, $b > 0$ is unknown gain, $[f^{\mathrm{T}}, b]^{\mathrm{T}} \in \Xi$, where Ξ is a known bounded set, $\tau_{ij}(t)$, $i = 1, \ldots, n-1, j = 2, \ldots, n$ are unknown time-varying delays.

We assume that the state vector $x(t)$ is available for measurement. Additionally, we assume that the functions $\varphi_i(\cdot) \in R$, $i = 1, 2, \ldots, n$ are bounded or bounded on t and f, and Lipchitz in $x(t)$, $x_2(t-\tau_{ij}(t))$, ..., $x_n(t-\tau_{ij}(t))$ $i = 1, \ldots, n-1$, $j = 2, \ldots, n$.

The problem is to design a continuous control law such that the goal condition holds

$$\big|x(t)\big| < \delta \quad \text{for } t > T, \tag{2}$$

where $\delta > 0$ is a prespecified required accuracy, $T > 0$ is a transient time, $|\cdot|$ is Euclidean norm of the corresponding vector.

III. Main Result

According to the backstepping method, the synthesis of the control system is split into n steps. We will introduce control laws in the each steps for compensation of unknown mismatched disturbances.

Step 1. Let us introduce the auxiliary loop in the form

$$\dot{z}_1(t) = -c_1 z_1(t) + x_2(t), \tag{3}$$

where $c_1 > 0$ is the coefficient chosen by a designer and it influences on convergence rate of z_1. Auxiliary loop (3) allows to get the unknown function φ_1.

Taking into account the first equation of system (1) and equation (3), consider the function $\xi_1(t) = x_1(t) - z_1(t)$ in the form

$$\dot{\xi}_1(t) = c_1 z_1(t) + \varphi_1. \tag{4}$$

Rewriting equation (4) with respect to φ_1, we have

$$\varphi_1 = \dot{\xi}_1(t) - c_1 z_1(t). \tag{5}$$

Substituting (5) to the first equation of (1), we get

$$\dot{x}_1(t) = x_2(t) + \dot{\xi}_1(t) - c_1 z_1(t). \tag{6}$$

Assume that the function $x_2(t)$ is a virtual control signal in (6). Then, introduce $x_2(t)$ in the form $x_2(t) = v_1(t)$, where $v_1(t)$ is the first auxiliary control law in the form

$$v_1(t) = -c_1 \xi_1(t) - \hat{\dot{\xi}}_1(t). \tag{7}$$

Here $\hat{\dot{\xi}}_1(t)$ is an estimate of the function $\dot{\xi}_1(t)$. Taking into account $x_2(t) = v_1(t)$, substitute (7) into (6)

$$\dot{x}_1(t) = -c_1 x_1(t) + \eta_1(t), \tag{8}$$

where $\eta_1(t) = \dot{\xi}_1(t) - \hat{\dot{\xi}}_1(t)$ is an error estimate.

Step 2. According to the backstepping procedure, we consider the first error function $e_1(t) = x_2(t) - v_1(t)$. Taking into account the second equation of (1) and (7), we get

$$\dot{e}_1(t) = x_3(t) + \widetilde{\varphi}_2(t), \tag{9}$$

where $\widetilde{\varphi}_2(t) = \varphi_2 - \dot{v}_1(t)$.

Let us introduce the auxiliary loop in the form

$$\dot{z}_2(t) = -c_2 z_2(t) + x_3(t), \tag{10}$$

where $c_2 > 0$ is a coefficient chosen by a designer.

Taking into account (9) and (10), rewrite the function $\xi_2(t) = e_1(t) - z_2(t)$ in the form

$$\dot{\xi}_2(t) = c_2 z_2(t) + \widetilde{\varphi}_2(t). \tag{11}$$

Rewriting equation (11) with respect to $\widetilde{\varphi}_2(t)$, we get

$$\widetilde{\varphi}_2(t) = \dot{\xi}_2(t) - c_2 z_2(t). \tag{12}$$

Substituting (12) to (9), we have

$$\dot{e}_1(t) = x_3(t) + \dot{\xi}_2(t) - c_2 z_2(t). \tag{13}$$

Assume that $x_3(t)$ is a control signal in (13). Introduce $x_3(t)$ in the form $x_3(t) = v_2(t)$, where $v_2(t)$ is defined as follows

$$v_2(t) = -c_2 \xi_2(t) - \hat{\dot{\xi}}_2(t). \tag{14}$$

Here $\hat{\dot{\xi}}_2(t)$ is an estimate of the function $\dot{\xi}_2(t)$. Substituting (14) to (13), we get

$$\dot{e}_1(t) = -c_2 e_1(t) + \eta_2(t), \tag{15}$$

where $\eta_2(t) = \dot{\xi}_2(t) - \hat{\dot{\xi}}_2(t)$ is an error estimate.

Step i ($3 \le i \le n-1$). Since $v_{i-1}(t)$ is not a real control law, we consider the i-th error function $e_{i-1}(t) = x_i(t) - v_{i-1}(t)$ as follows

$$\dot{e}_{i-1}(t) = x_{i+1}(t) + \widetilde{\varphi}_i(t), \tag{16}$$

where $\widetilde{\varphi}_i(t) = \varphi_i - \dot{v}_{i-1}(t)$.

Consider the i-th auxiliary loop in the form

$$\dot{z}_i(t) = -c_i z_i(t) + x_{i+1}(t), \quad (17)$$

where $c_i > 0$ is the coefficient chosen by a designer.

Taking into account (16) and (17), rewrite the function $\xi_i(t) = e_{i-1}(t) - z_i(t)$ as follows

$$\dot{\xi}_i(t) = c_i z_i(t) + \tilde{\varphi}_i(t). \quad (18)$$

Rewriting equation (18) with respect to $\tilde{\varphi}_i(t)$, we get

$$\tilde{\varphi}_i(t) = \dot{\xi}_i(t) - c_i z_i(t). \quad (19)$$

Substituting (19) to (18), we have

$$\dot{e}_{i-1}(t) = x_{i+1}(t) + \dot{\xi}_i(t) - c_i z_i(t). \quad (20)$$

Assume that function $x_{i+1}(t)$ is a control signal in (20). Therefore, introduce $x_{i+1}(t)$ as $x_{i+1}(t) = v_i(t)$, where $v_i(t)$ is defined by

$$v_i(t) = -c_i \xi_i(t) - \hat{\xi}_i(t), \quad (21)$$

where $\hat{\xi}_i(t)$ is an estimate of the function $\dot{\xi}_i(t)$. Substituting (21) to (20), we get

$$\dot{e}_{i-1}(t) = -c_i e_{i-1}(t) + \eta_i(t), \quad (22)$$

where $\eta_i(t) = \dot{\xi}_i(t) - \hat{\xi}_i(t)$ is an error estimate.

Step n. Since $v_n(t)$ is not real control law, consider the $n-1$-the error function $e_{n-1}(t) = x_n(t) - v_{n-1}(t)$ in the form

$$\dot{e}_{n-1}(t) = bu(t) + \tilde{\varphi}_n(t), \quad (23)$$

where $\tilde{\varphi}_n(t) = \varphi_n - \dot{v}_{n-1}(t)$.

Introduce the n-th auxiliary loop in the form

$$\dot{z}_n(t) = -c_n z_n(t) + \alpha u(t), \quad (24)$$

where $c_n > 0$ and $\alpha > 0$ are coefficients chosen by a designer.

Taking into account (23) and (24), rewrite the function $\xi_n(t) = e_{n-1}(t) - z_n(t)$ as follows

$$\dot{\xi}_n(t) = c_n z_n(t) + \tilde{\varphi}_n(t) + (b - \alpha)u(t), \quad (25)$$

Rewriting equation (25) with respect to $\tilde{\varphi}_n(t)$, we get

$$\tilde{\varphi}_n(t) = \dot{\xi}_n(t) - c_n z_n(t) - (b - \alpha)u(t). \quad (26)$$

Substituting (26) to (22), we have

$$\dot{e}_{n-1}(t) = \dot{\xi}_n(t) - c_n z_n(t) + \alpha u(t). \quad (27)$$

Introduce the control law $u(t)$ in the form

$$u(t) = -\frac{1}{\alpha}\left(c_n \xi_n(t) + \hat{\xi}_n(t)\right), \quad (28)$$

where $\hat{\xi}_n(t)$ is an estimate of the function $\dot{\xi}_n(t)$. Substituting (28) to (27), we get

$$\dot{e}_{n-1}(t) = -c_n e_{n-1}(t) + \eta_n(t), \quad (29)$$

where $\eta_n(t) = \dot{\xi}_n(t) - \hat{\xi}_n(t)$ is an error estimate.

According to Section II, the signals $\dot{\xi}_i(t)$, $i = 1, 2, ..., n$ are not available for measurement, because it depends on derivatives of $x(t)$. Therefore, we introduced the estimate functions $\hat{\xi}_i(t)$, $i = 1, 2, ..., n$ of the functions $\dot{\xi}_i(t)$, $i = 1, 2, ..., n$ at each step. For implementation of $\hat{\xi}_i(t)$, $i = 1, 2, ..., n$ we consider the following observers

$$\dot{\hat{\xi}}_{i-1}(t) = -\mu^{-1}\hat{\xi}_{i-1}(t) + \mu^{-1}\dot{\xi}_{i-1}(t), \; i = 1, 2, ..., n, \quad (30)$$

where $\mu > 0$ is a small enough constant. Note, that equation (30) can be implemented as a dirty derivative filter $\hat{\xi}_{i-1}(s) = \frac{s}{\mu s + 1}\xi_{i-1}(s)$, where s is complex variable in Laplace domain.

Theorem 1. There exist constants $c_i > 0$, $i = 1, 2, ..., n$ and $\mu_0 > 0$ such that for $\mu \le \mu_0$ the control system consisting of auxiliary loops (3), (10), (17), (24), auxiliary control laws (7), (14), (21), control law (28), observers (30) provides goal (2) for plant (1).

The proof of Theorem is given in Appendix. It is seen from proof of Theorem 1 that the function $|x(t)|$ does not exceed the value δ after transient time T. Moreover, value δ can be reduced, for example, by means of decrease of small parameter μ in observers.

IV. EXAMPLE

Consider the plant model with varying time-delay in the form

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + \varphi_1\left(x_2(t), x_2(t - \tau_{12}(t)), f, t\right), \\ \dot{x}_2(t) &= bu(t) + \varphi_2\left(f, t\right), \end{aligned} \quad (31)$$

where $f = [f_1, f_2]^{\mathrm{T}}$. The set Ξ is defined by the following inequalities:

$$-15 \le f_1 \le 15, \qquad -15 \le f_2 \le 15.$$

The problem is to design the control system providing inequality (2). Introduce the auxiliary loops in the following forms

$$\dot{z}_1(t) = -c_1 z_1(t) + x_2(t), \quad \dot{z}_2(t) = -c_2 z_2(t) + \alpha u(t), \quad (32)$$

where $c_1 = c_2 = 1$ and $\alpha = 1$.

Consider the control laws $v_1(t)$ and $u(t)$ in the forms

$$v_1(t) = -c_1\xi_1(t) - \hat{\xi}_1(t), \quad u(t) = -\frac{1}{\alpha}\left(c_2\xi_2(t) + \hat{\xi}_2(t)\right) \quad (33)$$

Here $\xi_1(t) = x_1(t) - z_1(t)$, $\xi_2(t) = e_1(t) - z_2(t)$ and $e_1(t) = x_2(t) - v_1(t)$. Define the observers in the forms

$$\hat{\xi}_1(t) = \frac{p}{\mu p + 1}\xi_1(t), \quad \hat{\xi}_2(t) = \frac{p}{\mu p + 1}\xi_2(t), \quad (34)$$

where $p = d / dt$, $\mu = 0.01$.

Let all initial conditions be zero in the control system. We see that algorithm (32)-(34) does not depend on functions φ_i, $i = 1, 2$ nor of the coefficient b. Let us choose the functions φ_i, $i = 1, 2$ in (31) as follows

$$\begin{aligned} \varphi_1 &= f_1\left(1 + 0.5\sin t\right)x_2(t)x_2^2(t - \tau_{11}), \\ \tau_{12} &= 1 + \sin t \text{ (sec)}, \quad \varphi_2 = f_2\left(\sin(1.3t) + 1\right). \end{aligned} \quad (35)$$

For comparison we also consider the synthesis of control systems using the backstepping algorithm [1, 11] and the auxiliary loop algorithm [2]. According to [1] the backstepping algorithm is presented by the following equations

$$v_1 = -c_1x_1 - \varphi_1, \quad u = -c_2e_1 - \frac{1}{b}\left[\varphi_2 + c_1(x_2 + \varphi_1) + \dot{\varphi}_1\right], \quad (36)$$

where $e_1(t) = x_2(t) - v_1(t)$. Algorithm (36) depends on functions φ_i, $i = 1, 2$. Therefore, implementation of (36) requires that functions φ_i, $i = 1, 2$ must be known. According to [2], the auxiliary loop algorithm is presented by the following equations:

equation of the auxiliary loop

$$\dot{x}_a(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}x_a(t) + u(t), \quad (37)$$

equation of the control law

$$u(t) = \begin{bmatrix} -1 & -2 \end{bmatrix}\varepsilon(t) - \frac{p}{\mu p + 1}\begin{bmatrix} 0 & 1 \end{bmatrix}\varepsilon(t), \quad (38)$$

where $\varepsilon(t) = x(t) - x_a(t)$. Algorithm (37), (38) compensates only function φ_2 and coefficient b and cannot compensate function φ_1. Therefore, algorithm (37), (38) may be unstable for appropriate values of the function φ_1.

Consider two cases.

Case 1. Let $f_1 = 1$ and $f_2 = 1$ in (35) and functions (35) are known.

Case 2. Let $f_1 = 2$ and $f_2 = 3$ in (35) and functions (35) are unknown.

In Fig. 1-6 the transients are presented for the state vector $x(t)$ which are obtained by proposed MBADC, the backstepping algorithm and the auxiliary loop algorithm for each of two cases. In Fig.1-6 black curve and red curves correspond to the signals $x_1(t)$ and $x_2(t)$ respectively.

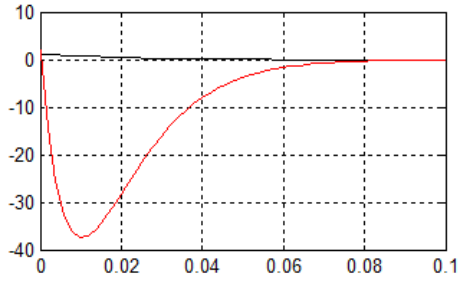


Fig. 1. The transients of the vector state $x(t)$ with MBADC for case 1.

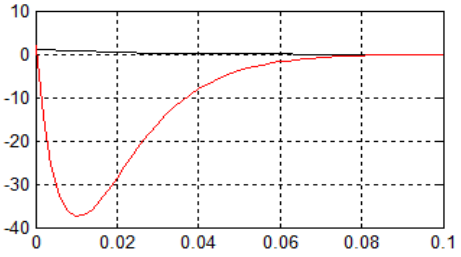


Fig. 2. The transients of the vector state $x(t)$ with MBADC for case 2.

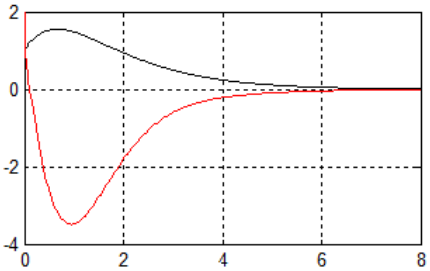


Fig. 3. The transients of the vector state $x(t)$ with the backstepping algorithm for case 1.

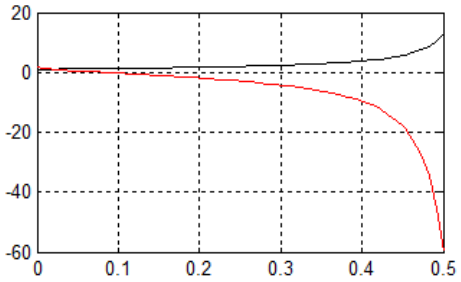


Fig. 4. The transients of the vector state $x(t)$ with the backstepping algorithm for case 2.

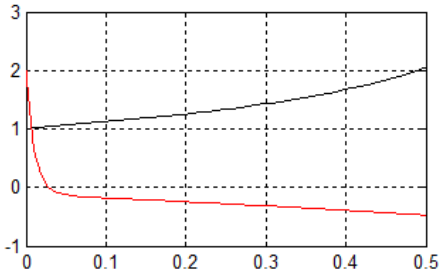


Fig. 5. The transients of the vector state $x(t)$ with the auxiliary loop algorithm for case 1.

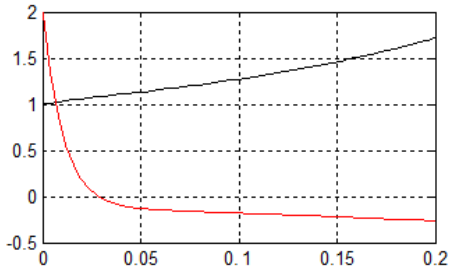


Fig. 6. The transients of the vector state $x(t)$ with the auxiliary loop algorithm for case 2.

The analysis of simulation results shows that the proposed MBADC guarantees the fulfillment of goal (2). The backstepping algorithm is stable if all functions in (31) are known, but the control system is sensitive to parametric uncertainty and external disturbances. The auxiliary loop algorithm may be unstable even when we known all functions in the plant model. We also can say that the proposed algorithm (32)-(34) compensates parametric uncertainties and external disturbances with the required accuracy $\delta = 0.05$ achieved after 1 s for any uncertainties from the set Ξ.

V. Conclusion

The paper describes the robust algorithm with compensation of mismatched disturbances depending on a varying time-delay. The synthesis of a control system based on the backstepping algorithm and the auxiliary loop algorithm. The proposed algorithm guarantees stabilization of the plant state vector with the required accuracy. We also compared the proposed algorithm with the backstepping algorithm and auxiliary loop algorithm. The simulation results illustrate the efficiency and robustness of the suggested control system. Obviously, the proposed results can be directly applied for network systems with nonlinear agents with nonlinear links and mismatched disturbances, because system (1) has been considered as network system with nth agents.

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Appendix

Proof of Theorem 1. Taking into account (30), rewrite the equations for the error estimates $\eta_i(t) = \dot{\xi}_i(t) - \hat{\xi}_i(t)$, $i = 1, 2, ..., n$ as follows

$$\dot{\eta}_i(t) = -\mu^{-1}\eta_i(t) + \ddot{\xi}_i(t),\ i = 1, 2, ..., n. \qquad (39)$$

Rewrite (9), (15), (22), (29) and (39) as the following system

$$\begin{aligned} \dot{x}_1(t) &= -c_1 x_1(t) + \eta_1(t), \\ \dot{e}_j(t) &= -c_{j+1} e_j(t) + \eta_{j+1}(t), \quad j = 1, 2, ..., n-1, \\ \mu_1 \dot{\eta}_i(t) &= -\eta_i(t) + \mu_2 \ddot{\xi}_i(t), \ i = 1, 2, ..., n, \end{aligned} \qquad (40)$$

where $\mu_1 = \mu_2 = \mu$. To analyze system (40) the following Lemma [9, 10] is needed.

Lemma 1. Let the system be described by the following differential equation

$$\dot{x} = f(x, \mu_1, \mu_2, t), \quad (41)$$

where $x \in R^{s_1}$, $\mu = \mathrm{col}(\mu_1, \mu_2) \in R^{s_2}$, $f(x, \mu_1, \mu_2, t)$ is Lipchitz continuous function in x. Let (41) have a bounded closed set of attraction $\Omega = \{x \mid P(x) \le C\}$ for $\mu_2 = 0$, where $P(x)$ is piecewise-smooth, positive definite function in R^{s_1}. In addition let there exist some constants $C_1 > 0$ and $\overline{\mu}_1 > 0$ such that the following condition holds

$$\sup_{|\mu_1| \le \overline{\mu}_1} \left[\left\langle \left[\frac{\partial P(x)}{\partial x} \right]^{\mathrm{T}}, f(x, \mu_1, 0, t) \right\rangle \middle| \quad P(x) = C \right] \le -C_1.$$

Then there exists some $\mu_0 > 0$ such that the system (41) has the same set of attraction Ω for $\mu_2 \le \mu_0$.

Let us check conditions of Lemma 1. Consider system (40) for $\mu_2 = 0$. Let $P(x) = V(t)$, where $V(t)$ is Lyapunov function defined in the form

$$V(t) = 0.5x_1^2(t) + 0.5\sum_{j=1}^{n-1} e_i^2(t) + 0.5\sum_{i=1}^{n} \eta_i^2(t). \quad (42)$$

Take the derivative of $V(t)$ along the trajectories (40), we get

$$\dot{V}(t) = -c_1 x_1^2(t) + x_1(t)\eta_1(t) + \sum_{j=1}^{n-1}\left[-c_{j+1} e_j^2(t) + e_j(t)\eta_{j+1}(t) \right] - \sum_{i=1}^{n} \mu_1^{-1}\eta_i^2(t). \quad (43)$$

Find upper bounds for the fourth term of (43):

$$e_j\eta_{j+1} \le 0.5\mu_0^{-1}e_j^2 + 0.5\mu_0\eta_{j+1}^2, \quad j = 1, 2, ..., n-1. \quad (44)$$

By substituting (44) to (43), we get

$$\dot{V} \le -c_1 x_1^2 - \sum_{j=1}^{n-1} \hat{c}_{j+1} e_j^2 - \mu_1^{-1}\eta_1^2 - \sum_{i=2}^{n} \hat{d}_i \eta_i^2, \quad (45)$$

where $\hat{c}_{j+1} = c_{j+1} - 0.5\mu_0^{-1}$ and $\hat{d}_i = \mu_1^{-1} - 0.5\mu_0$. Obviously, there exist coefficients c_{j+1}, $j = 1, 2, ..., n\text{-}1$, μ_1, and μ_0 such that $\hat{c}_{j+1} > 0$, $\hat{d}_i > 0$ and system (40) is asymptotically stable.

Taking into account to (42), rewrite (45) as follows

$$\dot{V}(t) \le -\beta V(t), \quad (46)$$

where $\beta = 2\min\left\{c_1, \hat{c}_2, ..., \hat{c}_n, \mu_1^{-1}, \hat{d}_2, ..., \hat{d}_n\right\}$. Solving inequality (46) with respect to $V(t)$, we have

$$V(t) \le V(0)e^{-\beta t}. \quad (47)$$

From (47) it follows that solutions of system (40) are exponentially tend to zero.

Let us proof boundendess of all signals in the closed-loop system. Taking into account (30), rewrite the first equation of (40) in the following form $\mu_1 p^2 \xi_1(t) = (p + c_1)(\mu_1 p + 1)x_1(t)$. Since the function $x_1(t)$ is asymptotically stable then the functions $\xi_1(t)$, $\dot{\xi}_1(t)$ and $\ddot{\xi}_1(t)$ are bounded. From boundedness of $\eta_1(t)$ it follows that the signal $\hat{\xi}_1(t)$ is bounded. Taking into account (30) and system (40), the proof of boundedness of the signals $\xi_i(t)$, $\dot{\xi}_i(t)$, $\ddot{\xi}_i(t)$ and $\hat{\xi}_i(t)$, $i = 2, 3, ..., n$ is same. Therefore, from (7), (14), (21) and (28) the signals $v_i(t)$, $i = 1, 2, ..., n-1$ and $u(t)$ are bounded. Hence, the functions $x(t)$ and $x_i(t - \tau_{ij}(t))$, $i, j = 1, 2, ..., n$ are bounded. From (3), (10), (17) and (24) it follows that the functions $z_i(t)$, $i = 1, 2, ..., n$ are bounded. Therefore, the functions φ_i and $\widetilde{\varphi}_i$ are bounded. Consequently, all signals in the closed-loop system are bounded.

According to Lemma 1 there exist some $\mu_0 > 0$ such that for $\mu_1 \le \mu_0$ and $\mu_2 \le \mu_0$ the attraction set is the same as for $\mu_2 = 0$. However, system (40) is not asymptotically stable for $\mu_2 \ne 0$. It will be has some attraction set. Let us find the set of attraction of system (40) for $\mu_2 \ne 0$. Taking into account result (45), take derivative in time of (42) along trajectories (40) for $\mu_1 = \mu_2 = \mu_0$

$$\dot{V}(t) \le -c_1 x_1^2 - \sum_{j=1}^{n-1} \hat{c}_{j+1} e_j^2 - \mu_0^{-1}\eta_1^2 - \sum_{i=2}^{n} \widetilde{d}_i \eta_i^2 + \sum_{l=1}^{n} \eta_l \ddot{\xi}_l, \quad (48)$$

where $\hat{d}_i = \mu_0^{-1} - 0.5\mu_0$.

Use the following upper bounds:

$$\eta_l \ddot{\xi}_l \le 0.5\mu_0^{-1}\eta_l^2(t) + 0.5\mu_0 \ddot{\xi}_l^2 \le 0.5\mu_0^{-1}\eta_l^2(t) + \mu_0\chi, \quad (49)$$

where $\chi = 0.5 \sup_{t, l = 1, 2, ..., n} \left\{\ddot{\xi}_l^2\right\}$.

Taking into account (49), rewrite (48) in the form

$$\dot{V}(t) \le -c_1 x_1^2 - \sum_{j=1}^{n-1} \hat{c}_{j+1} e_j^2 - 0.5\mu_0^{-1}\eta_1^2 - d\sum_{i=2}^{n} \eta_i^2 + n\mu_0\chi, \quad (50)$$

where $d = \mu_0^{-1} - \mu_0$

Taking into account to (42), rewrite (50) as follows

$$\dot{V}(t) \le -\gamma V(t) + n\mu_0\chi, \quad (51)$$

where $\gamma = 2\min\left\{c_1, \hat{c}_2, ..., \hat{c}_n, 0.5\mu_0^{-1}, d\right\}$.

Solving inequality (51) with respect to $V(t)$, we get

$$V(t) \le e^{-\gamma t}V(0) + \left(1 - e^{-\gamma t}\right)\gamma^{-1} n\mu_0\chi. \quad (52)$$

From (52) we can note that goal (2) holds. The theorem 1 is proved.