The moduli space interpretation of modular curves

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What are the modular curves over $\mathbb{C}$?

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We call $SL_2(\mathbb{Z})$ the modular group. Every element in the modular group can be viewed as a fractional linear transformation on $\mathcal{H}$. Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and } \tau \in \mathcal{H}.$$

Then

$$\gamma(\tau) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = \begin{cases} \frac{a\tau+b}{c\tau+d} & \text{, if } \tau \in \mathcal{H} \setminus \{-\frac{d}{c}\} \\ \infty & \text{, if } \tau = -\frac{d}{c} \end{cases}$$

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Moreover, we have

$$\mathrm{Im}(\gamma(\tau)) = \frac{\mathrm{Im}(\tau)}{\mid c\tau + d \mid^2}.$$

So the modular group acting on $\mathcal{H}$ is indeed an automorphism.

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Definition

Modular curves, denoted by $Y(\Gamma)$, are the quotient of $\mathcal{H}$ by the action of certain subgroups Γ of the modular group $SL_2(\mathbb{Z})$.

$$Y(\Gamma) := \Gamma \setminus \mathcal{H} = \{\Gamma \cdot \tau \mid \tau \in \mathcal{H}\}$$

Those subgroups Γ are called congruence subgroups. Different congruence subgroups define different modular curves.

What are the modular curves over $\mathbb{C}$?

Example

$$\Gamma_1(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod N\}$$

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What is the moduli space interpretation of modular curves?

The modular curves parametrises elliptic curves with additional structures, and we call such parameterisation the moduli space interpretation of modular curves.

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An elliptic curve over a field K is a non-singular projective cubic curve defined over K with a specified K-rational point $\mathcal{O}$.

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An elliptic curve over a field K is a non-singular projective cubic curve defined over K with a specified K-rational point $\mathcal{O}$.

If $\mathrm{char}(K) \neq 2, 3$, any elliptic curve defined over K is birationally equivalent over K to a curve of the form $y^2 = x^3 + Ax + B$ with $A, B \in K$.

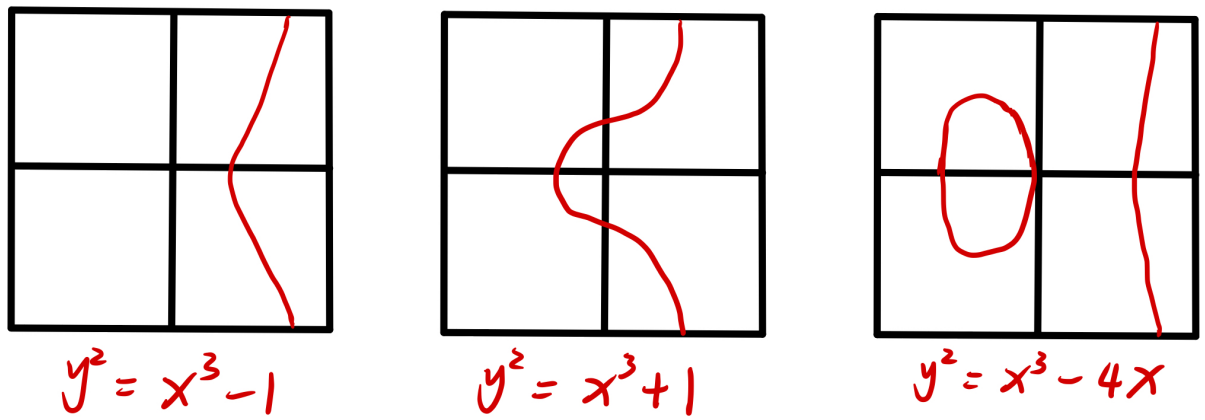


Figure: Examples of Elliptic Curves

What is the moduli space interpretation of modular curves?

Example

The moduli space for $Y_1(N)$ over $\mathbb{C}$ is

$$S_1(N) := \{[E, P] \mid E \text{ is a complex elliptic curve and } P \text{ is an } N\text{-torsion point on } E(\mathbb{C})\}.$$

There is a bijection between $Y_1(N)$ over $\mathbb{C}$ and $S_1(N)$.

How do we prove it?

Every complex elliptic curve E can be written as a complex torus $\mathbb{C}/\Lambda_\tau$ with $\tau \in \mathcal{H}$.

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$$S_1(N) := \{[\mathbb{C}/\Lambda_\tau, \frac{1}{N} + \Lambda_\tau] \mid \tau \in \mathcal{H}\}$$

and it is reduced to prove $f : Y_1(N) \to S_1(N)$ such that

$$\Gamma_1(N)\tau \mapsto [\mathbb{C}/\Lambda_\tau, \frac{1}{N} + \Lambda_\tau]$$

is bijective.

How do we prove it?

Problems

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Answer

Build the integral models

How do we prove it?

Definition

An elliptic curve over a scheme S is a smooth proper scheme E with a section $0 : S \to E$ such that each geometric fibre is a connected genus 1 curve.

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Definition

Let $N \geq 1$ be an integer. Let E be an elliptic curve over S. A $\Gamma_1(N)$-structure on E is a section $P \in E(S)$ of exact order N.

That is, for all geometric points $s : \operatorname{Spec} k \to S$, $P \circ s$ has order N in $E(k)$, where k is the algebraically closed field.

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Remark

$\Gamma_1(N) : \mathbf{Ell} \to \mathbf{Set}$ can be viewed as a moduli problem defined by

$$E/S \mapsto \Gamma_1(N)(E/S) := \{P \in E(S) \mid P \text{ has exact order } N\}.$$

How do we prove it?

Let us define the contravariant functor $\mathcal{F}_{\Gamma_1(N)}$ from $\mathbb{Z}[1/N]$-**scheme** to **Set** as follows:
For any scheme S over $\mathbb{Z}[1/N]$,

$$\mathcal{F}_{\Gamma_1(N)}(S) := \{[E, P] \mid E \text{ is an elliptic curve over } S \text{ and } P \in E(S) \text{ has exact order } N\}$$

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It is reduced to prove the functor $\mathcal{F}_{\Gamma_1(N)}$ is representable and its representing object $\mathcal{Y}_1(N)$ is the required integral model of $Y_1(N)$.
That is, we need to prove that for all scheme S over $\mathbb{Z}[1/N]$,

$$\mathcal{F}_{\Gamma_1(N)}(S) \cong \mathrm{Hom}_{Sch/\mathbb{Z}[1/N]}(S, \mathcal{Y}_1(N))$$

How do we prove it?

We can reduce it further to prove the moduli problem $\Gamma_1(N)$ is representable.

Theorem

Let $\mathcal{P}$ be a moduli problem for elliptic curves. If $\mathcal{P}$ is relatively representable and affine over Ell, *then*

$$\mathcal{P} \text{ is representable if and only if } \mathcal{P} \text{ is rigid.}$$

It suffices to prove $\Gamma_1(N)$ is affine, rigid and relatively representable.

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Let $S = \mathrm{Spec}(\bar{\mathbb{Q}})$. By the functor of points view, every morphism $\mathrm{Spec}(\bar{\mathbb{Q}}) \to \mathcal{Y}_1(N)$ is a point of $\mathcal{Y}_1(N)$ over $\bar{\mathbb{Q}}$.

Thank you!