

5.10 - Improper Integrals

↙ finitely many jump discty's are ok

FTC (part 2) : If f is continuous on $[a, b]$

then
$$\int_a^b f(x) dx = F(b) - F(a)$$

where F is antiderivative of f .

(cty - continuity)

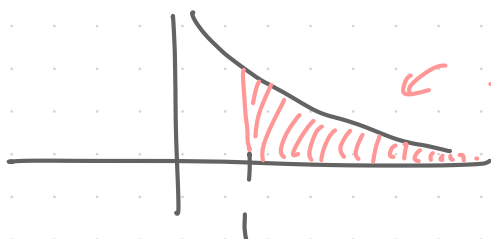
For FTC to apply:

- a finite interval
- f is continuous
(or has only jump discontinuities)
on $[a, b]$

If one of these is not satisfied, we have an improper integral.

ex: $\int_1^{\infty} \frac{1}{x^2} dx$

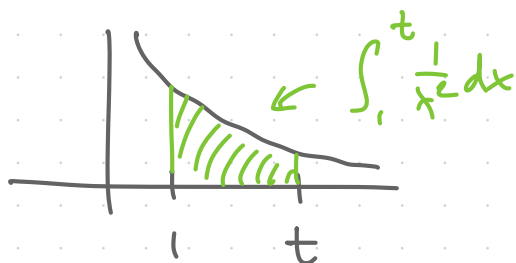
interval is $[1, \infty)$ so
this is improper



$\int_1^{\infty} \frac{1}{x^2} dx = \text{area under the curve}$

We know how to compute $\int_1^2 \frac{1}{x^2} dx$

or $\int_1^{100} \frac{1}{x^2} dx$ or $\int_1^t \frac{1}{x^2} dx$



Let $t \rightarrow \infty$.

Solution:

$$\underbrace{\int_1^{\infty} \frac{1}{x^2} dx}_{\text{improper}} = \lim_{t \rightarrow \infty} \underbrace{\int_1^t \frac{1}{x^2} dx}_{\text{proper int.}}$$

$$= \lim_{t \rightarrow \infty} \left(\int_1^t x^{-2} dx \right)$$

$$= \lim_{t \rightarrow \infty} \left(-x^{-1} \Big|_1^t \right)$$

$$= \lim_{t \rightarrow \infty} \left(-\frac{1}{x} \Big|_1^t \right)$$

$$= \lim_{t \rightarrow \infty} \left(-\frac{1}{t} - \left(-\frac{1}{1} \right) \right)$$

$$= \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + 1 \right)$$

$$= \boxed{1}$$

converges

3 types of infinite intervals:

① (a, ∞)

$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

② $(-\infty, a)$

$$\int_{-\infty}^a f(x) dx = \lim_{t \rightarrow -\infty} \int_t^a f(x) dx$$

③ $(-\infty, \infty)$

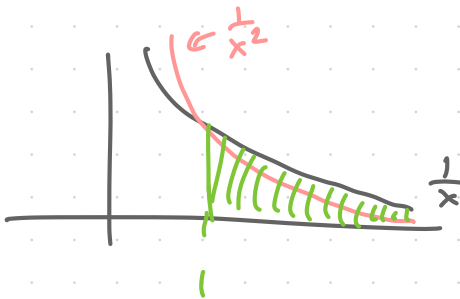
$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$$

(usually choose $c = 0$)

If the limit exists and is finite, we say that the improper integral converges. Otherwise (if limit DNE or equals $\pm\infty$) we say the improper integral diverges.

"Determine if the improper integral converges or diverges. If it converges, compute the value."

ex: $\int_1^{\infty} \frac{1}{x} dx$



$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx$$

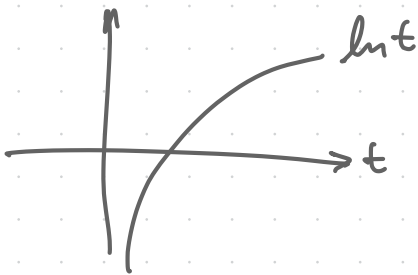
$$= \lim_{t \rightarrow \infty} \left(\ln|x| \Big|_1^t \right)$$

$$= \lim_{t \rightarrow \infty} \left(\ln|t| - \cancel{\ln|1|}^0 \right)$$

$$= \lim_{t \rightarrow \infty} \ln|t|$$

$$= \infty$$

imp. int.
diverges



$\ln t$ goes to ∞
as $t \rightarrow \infty$

ex: $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

$$= \underbrace{\int_{-\infty}^0 \frac{1}{1+x^2} dx}_{(1)} + \underbrace{\int_0^{\infty} \frac{1}{1+x^2} dx}_{(2)}$$

If either (1) or (2) diverges, the whole integral diverges.

Now the answer is (1) + (2).

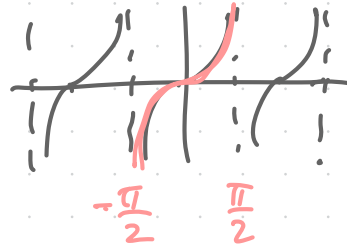
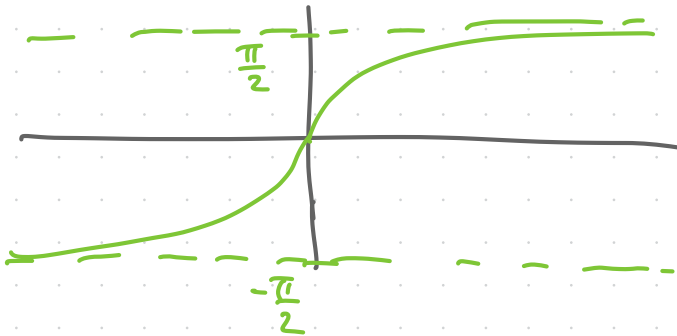
$$(1) = \int_{-\infty}^0 \frac{1}{1+x^2} dx = \lim_{t \rightarrow -\infty} \int_t^0 \frac{1}{1+x^2} dx$$

$$= \lim_{t \rightarrow -\infty} \left(\arctan x \Big|_t^0 \right)$$

$$= \lim_{t \rightarrow -\infty} \left(\arctan(0) - \arctan(t) \right)$$

$$= \lim_{t \rightarrow -\infty} \left(0 - \arctan(t) \right)$$

$$= \lim_{t \rightarrow -\infty} (-\arctan(t)) = \frac{\pi}{2}$$



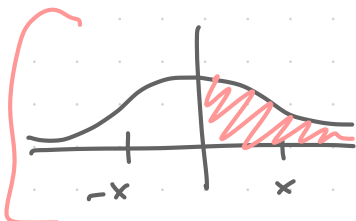
Fact: $\lim_{t \rightarrow \infty} \arctan t = \frac{\pi}{2}$

$$\lim_{t \rightarrow -\infty} \arctan t = -\frac{\pi}{2}$$

Can check that $\textcircled{2} = \int_0^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2}$

Answer: $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

converges



$\frac{1}{1+x^2}$ is even b/c $\frac{1}{1+(-x)^2} = \frac{1}{1+x^2}$

Showing your work:

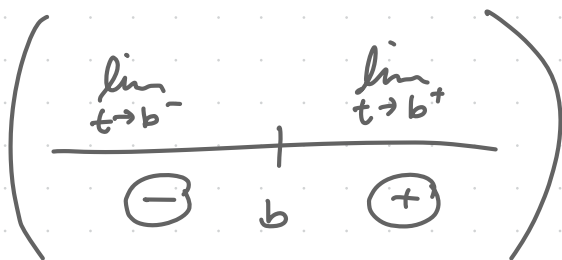
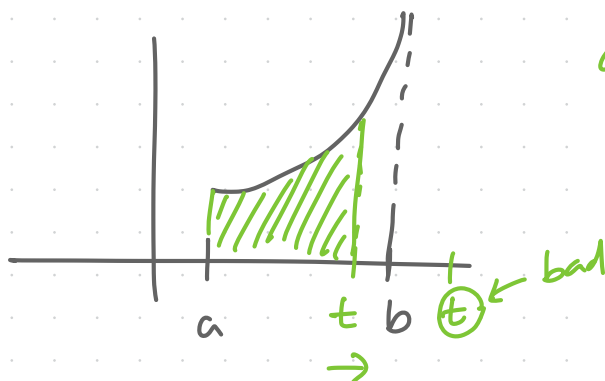
- ① Write improper integral as a limit of proper integrals
- ② Compute the integral (ignore the limit)
- ③ Evaluate the limit.

Note: make sure to use proper limit notation throughout

Improper Integrals of Discontinuous Functions

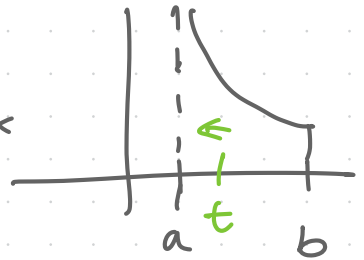
① If f is continuous on $[a, b)$ but not at $x=b$

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$



② If f is continuous on $(a, b]$ but not at $x=a$,

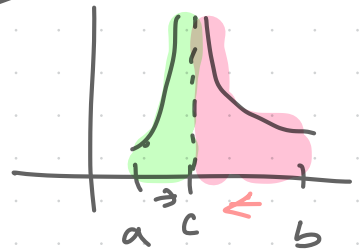
$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$



③ If f is continuous on $[a, b]$ except at $x=c$ where $a < c < b$,

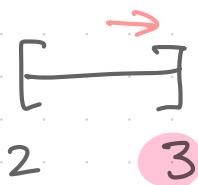
$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

← split at c



$$\rightarrow = \lim_{t \rightarrow c^-} \int_a^t f(x) dx + \lim_{s \rightarrow c^+} \int_s^b f(x) dx$$

ex: $\int_2^3 \frac{1}{\sqrt{3-x}} dx$



from the left

discty at $x=3$
 b/c plugging in gives
 0 in the denominator
 of the integrand
 (function)

$$\int_2^3 \frac{1}{\sqrt{3-x}} dx = \lim_{t \rightarrow 3^-} \int_2^t \frac{1}{\sqrt{3-x}} dx$$

$$\boxed{\begin{array}{l} u = 3-x \\ du = -dx \end{array}}$$

$$x=2 \Rightarrow u=1$$

$$x=t \Rightarrow u=3-t$$

$$= \lim_{t \rightarrow 3^-} \left(\int_1^{3-t} \frac{1}{\sqrt{u}} \cdot (-1) du \right)$$

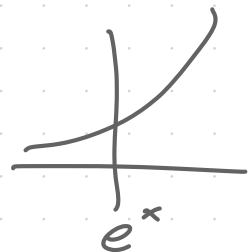
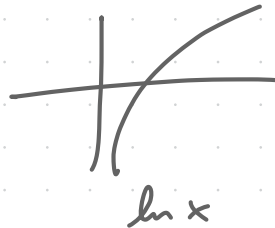
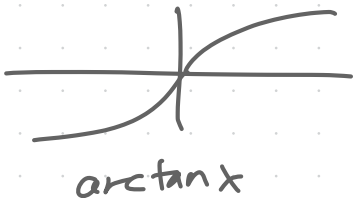
$$= - \lim_{t \rightarrow 3^-} \int_1^{3-t} u^{-1/2} du$$

$$\begin{aligned}
 &= - \lim_{t \rightarrow 3^-} \left(2 u^{1/2} \Big|_1^{3-t} \right) \\
 &= - \lim_{t \rightarrow 3^-} (2 \sqrt{3-t} - 2) \\
 &= - (0 - 2) \\
 &= 2
 \end{aligned}$$

$$\lim_{x \rightarrow 2^-} \frac{1}{x-2} = -\infty \quad \text{vs} \quad \lim_{x \rightarrow 2^+} \frac{1}{x-2} = \frac{1}{0^+} = \infty$$



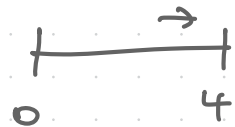
$\frac{1}{0^-}$ blows up to $\pm \infty$



ex: $\int_0^6 \frac{1}{(x-4)^{2/3}} dx$ discontinuity at $x=4$

$$= \underbrace{\int_0^4 \frac{1}{(x-4)^{2/3}} dx}_{(1)} + \underbrace{\int_4^6 \frac{1}{(x-4)^{2/3}} dx}_{(2)}$$

$$(1) = \int_0^4 \frac{1}{(x-4)^{2/3}} dx$$



$$= \lim_{t \rightarrow 4^-} \int_0^t \frac{1}{(x-4)^{2/3}} dx$$

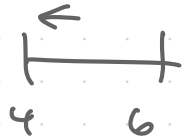
$$= \lim_{t \rightarrow 4^-} \left(\int_0^t (x-4)^{-2/3} dx \right)$$

$$= \lim_{t \rightarrow 4^-} \left(3 (x-4)^{1/3} \Big|_0^t \right)$$

$$= \lim_{t \rightarrow 4^-} \left(3 (t-4)^{1/3} - 3 (-4)^{1/3} \right)$$

$$= 0 - 3 (-1)^{1/3} 4^{1/3} = 3 \sqrt[3]{4}$$

$$\textcircled{2} = \int_4^6 \frac{1}{(x-4)^{2/3}} dx$$



$$= \lim_{t \rightarrow 4^+} \int_{x=t}^{x=6} \frac{1}{(x-4)^{2/3}} dx$$

$$= \lim_{t \rightarrow 4^+} \left(3(x-4)^{1/3} \Big|_{x=t}^{x=6} \right)$$

$$= \lim_{t \rightarrow 4^+} \left(3(2)^{1/3} - 3(t-4)^{1/3} \right)$$

$$= 3\sqrt[3]{2}$$

$$\int_0^6 \frac{1}{(x-4)^{2/3}} dx = \underbrace{3\sqrt[3]{4}}_{\textcircled{1}} + \underbrace{3\sqrt[3]{2}}_{\textcircled{2}}$$

$$\int_1^{\infty} \frac{1}{x^2} dx$$

$p=2$

vs.

$$\int_1^{\infty} \frac{1}{x} dx$$

$p=1$

p-integral:

$$\int_1^{\infty} \frac{1}{x^p} dx$$