

8.1 - Sequences

★ 8.1 & 8.2 are very important for understanding the rest of the chapter

A sequence is a list of numbers, in order.

e.g. $\{1, 2, 4, 8, 16, 32, \dots\}$

↑ goes on forever

Terminology:

- Numbers in the sequence are terms
- We use the notation a_n ("a sub n") for the n th term in the sequence
- n is the index of the term a_n (tells you the position)

ex: $\{1, 2, 4, 8, 16, 32, \dots\} = \{2^{n-1}\}_{n=1}^{\infty}$

$$a_1 = 1 = 2^0$$

$$a_2 = 2 = 2^1$$

$$a_3 = 4 = 2^2$$

$$a_4 = 8 = 2^3$$

$$a_5 = 16 = 2^4$$

$\vdots$

$$a_n = 2^{n-1}$$

compare the
index to
what's changing
in the term

Notation: $\{a_n\}_{n=1}^{\infty}$ ← "stopping" index

← starting index

(doesn't have to be 1)

↑
curly braces
indicate it's
a seq.

ex: $\left\{ \frac{n+1}{n+2} \right\}_{n=2}^{\infty} = \left\{ \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots \right\}$

$\uparrow \quad \quad \uparrow \quad \quad \uparrow$
 $a_2 \quad \quad a_3 \quad \quad a_4$

ex: Fibonacci sequence

Start w/ 0 and 1. To get the next term, you sum the previous two terms.

$$\{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$$

We say this sequence is recursively defined because the value of the next term depends on terms that came before it.

recursive
definition :

$$a_0 = 0, a_1 = 1$$

$$\text{and } a_n = a_{n-1} + a_{n-2}$$

$$\text{for } n \geq 2$$

ex: $\{(-1)^n\}$

if no starting index,
assume it starts at 1

$$\begin{aligned} \hookrightarrow &= \{-1, (-1)^2, (-1)^3, (-1)^4, \dots\} \\ &= \{-1, 1, -1, 1, -1, 1, \dots\} \end{aligned}$$

ex: $\{(-1)^{n+1}\} = \{1, -1, 1, -1, 1, \dots\}$

ex: $\{(-1)^n \cdot \frac{1}{n}\}_{n=1}^{\infty}$

$$= \{(-1) \cdot \frac{1}{1}, (-1)^2 \frac{1}{2}, (-1)^3 \cdot \frac{1}{3}, (-1)^4 \frac{1}{4}, \dots\}$$

$$= \{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots\}$$

Defn: A sequence whose terms alternate positive and negative is called alternating

(Note: $\{1, 1, -2, -2, 3, 3, -4, -4, \dots\}$
is NOT alternating)

Fact: If $\{a_n\}$ is alternating, then a_n CAN be written as either ★

$$a_n = (-1)^n b_n \quad \text{or} \quad a_n = (-1)^{n+1} b_n$$

where $b_n \geq 0$.

ex: $\{ \underbrace{(-1)^n}_{\uparrow b_n = \frac{1}{n}} \frac{1}{n} \}$

ex: Find a formula for a_n for

$$\left\{ \underset{n=1}{\frac{1}{3}}, \underset{n=2}{-\frac{1}{9}}, \underset{n=3}{\frac{1}{27}}, \underset{n=4}{-\frac{1}{81}}, \dots \right\}$$

$$= \left\{ \frac{1}{3^1}, -\frac{1}{3^2}, \frac{1}{3^3}, -\frac{1}{3^4}, \dots \right\}$$

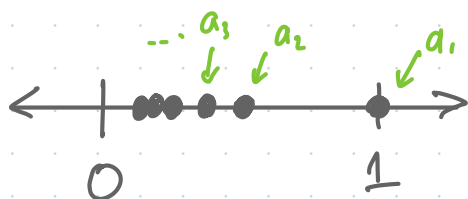
$$a_n = \underbrace{(-1)^{n+1}}_{(-1)^n \text{ or } (-1)^{n+1}} \cdot \frac{1}{3^n} = (-1)^{n+1} \frac{1}{3^n}$$

neg. ↗ ↖ pos.

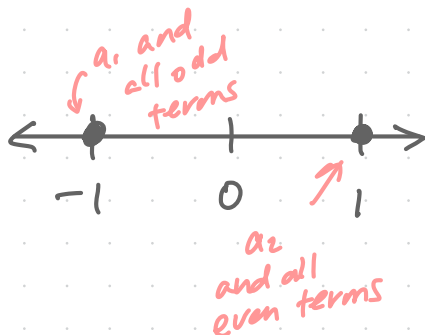
Two ways to Visualize a Sequence

① As points on a number line

ex 1: $\left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$



ex 2: $\left\{ (-1)^n \right\}_{n=1}^{\infty}$

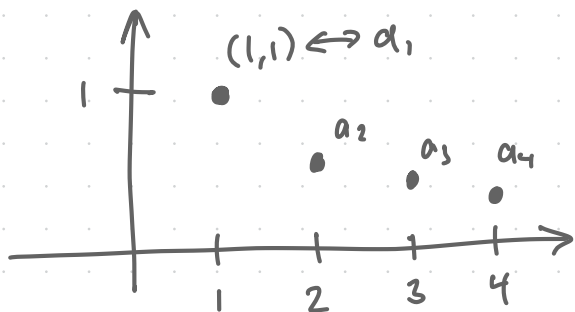


② as functions whose domain is

$$\{1, 2, 3, 4, \dots\} = \mathbb{N} \text{ or } \mathbb{N}$$

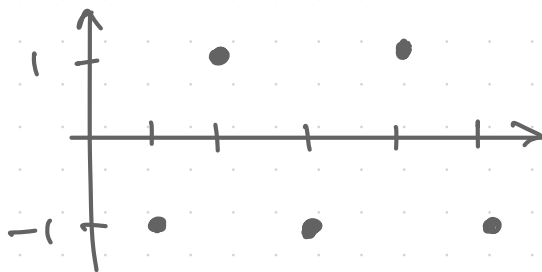
(natural numbers)

ex 1: $\left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$



* plot (n, a_n)

ex 2: $\left\{ (-1)^n \right\}_{n=1}^{\infty}$



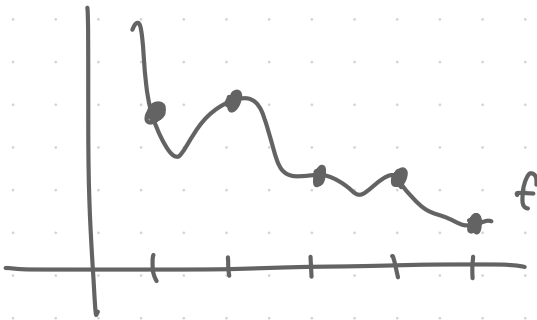
Limits of Sequences (as $n \rightarrow \infty$)

Works like limits of functions

Fact: If $f(x)$ is a function on $\mathbb{R}$ such that $f(n) = a_n$ at each natural number n , then

$$\lim_{n \rightarrow \infty} a_n = \lim_{x \rightarrow \infty} f(x)$$

* IF $\lim_{x \rightarrow \infty} f(x)$ exists *



ex: $\left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$

$f(x) = \frac{1}{x}$ lines up w/ $\frac{1}{n}$ at each n .

$$\lim_{n \rightarrow \infty} \frac{1}{n} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

ex: $\lim_{n \rightarrow \infty} \frac{n^2 + 2n - 1}{5n^2 + 3}$

* delete lower order terms

$$= \lim_{n \rightarrow \infty} \frac{n^2}{5n^2}$$

$$= \frac{1}{5}$$

{ L'Hospital's
L'Hôpital's
→ $\frac{\infty}{\infty}$ or $\frac{0}{0}$

ex: $\lim_{n \rightarrow \infty} \frac{2n^2 + n}{\sqrt{3n^2 + 7n^4}}$

$$= \lim_{n \rightarrow \infty} \frac{2n^2}{\sqrt{7n^4}}$$

$$= \lim_{n \rightarrow \infty} \frac{2}{\sqrt{7} \cdot n^2}$$

$$= \frac{2}{\sqrt{7}}$$

ex: $\lim_{n \rightarrow \infty} \frac{2^n}{n^2} \leftarrow \frac{\infty}{\infty}$

$$= \lim_{x \rightarrow \infty} \frac{2^x}{x^2} \stackrel{LH}{=} \lim_{x \rightarrow \infty} \frac{2^x \cdot \ln(2)}{2x}$$

$$\stackrel{LH}{=} \lim_{x \rightarrow \infty} \frac{2^x \cdot (\ln 2)^2}{2}$$

$$= \frac{(\ln 2)^2}{2} \lim_{x \rightarrow \infty} 2^x$$
$$= \infty$$

If $\lim_{n \rightarrow \infty} a_n$ exists and is finite, we say the sequence $\{a_n\}$ converges.
Otherwise, we say it diverges.

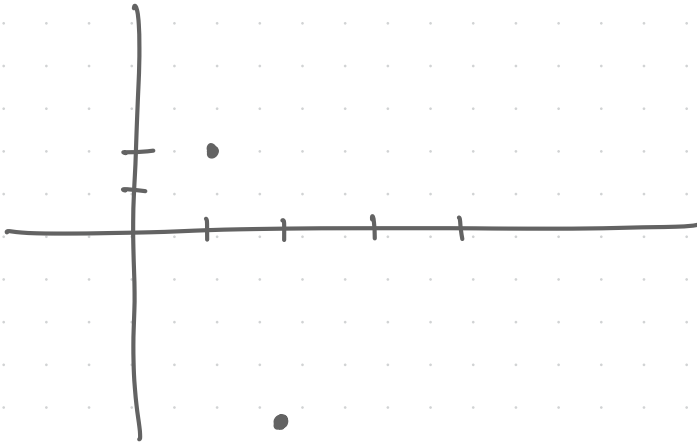
↑ $\pm \infty$ or DNE

[If asked if a sequence converges or diverges
just take the limit.]

example where the limit does not exist

$$\{(-1)^{n+1} 2^n\}_{n=1}^{\infty}$$

$$\lim_{n \rightarrow \infty} \underbrace{(-1)^{n+1}}_{\text{alternates}} \cdot \underbrace{2^n}_{\text{goes to } \infty}$$



↙

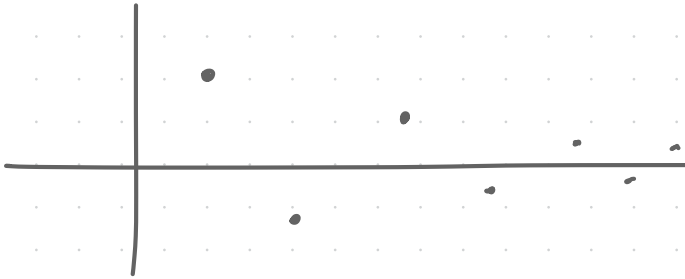
~~$$f(x) = (-1)^x 2^x$$
$$f\left(\frac{1}{2}\right) = \underbrace{(-1)^{1/2}}_{\text{not defined}} 2^{1/2}$$~~

terms bounce
pos. → neg & back
and get larger

⇒ limit DNE

ex: $\{ (-1)^{n+1} \cdot \frac{1}{2^n} \}$

alternating goes to 0



$$\lim_{n \rightarrow \infty} (-1)^{n+1} \frac{1}{2^n} = 0$$

positive & negative terms can agree at 0

Moral: an alternating sequence can only converge to 0 (no other value)