

Plasma Physics Primer Part I

Statistical Behavior of Plasmas

1. Distribution Functions
2. Averages & Fluxes
3. Isotropic Equilibrium Distribution Function
4. Collision Frequency
5. Mean Free Path
6. Boltzmann's Equation
7. Conservation Laws

1. Distribution Functions: Since there are on the order of 10^{12} to 10^{24} gas atoms, molecules and ions per m^3 in a plasma, it is most convenient to look at the behavior of those particles from a statistical standpoint. The particles in any given volume in general, have a distribution of velocities that arise due to numerous collisions between particles. The distribution function f describes the distribution of particle velocities at any given point $\bar{x}$ and any given instant t . In particular, the number of particles having velocities between (c_1, c_2, c_3) and $((c_1 + dc_1, c_2 + dc_2, c_3 + dc_3))$ in a volume dx_1, dx_2, dx_3 centered at $\bar{x}$, at time t is:

$$dN = f(\bar{c}, \bar{x}, t) d\bar{x} d\bar{c} \quad (1)$$

Where

$$d\bar{c} = dc_1, dc_2, dc_3 \quad (2)$$

$$d\bar{x} = dx_1, dx_2, dx_3 \quad (3)$$

[Note: we will often denote a vector $\bar{b}$ as (b_1, b_2, b_3) .]

In a plasma, there are several species which interact (collide) with one another and with other species. These include electrons, ions, and neutral particles. Each specie will in general have its own distribution function:

$$\text{Electron distribution function} = f_e = f_e(\bar{c}, \bar{x}, t) \quad (4a)$$

$$\text{Ion distribution function} = f_i = f_i(\bar{c}, \bar{x}, t) \quad (4b)$$

$$\text{Neutral specie distribution function} = f_o = f_o(\bar{c}, \bar{x}, t) \quad (4c)$$

Or, in general, the distribution function per specie s is:

$$f_s = f_s(\bar{c}, \bar{x}, t) \quad (4d)$$

We can either determine the distribution function for all species or we can use the typically unknown distribution functions to derive useful continuum conservation laws (conservation of mass, momentum, energy) for each specie and the overall plasma. f_s can only be determined (in closed form) for relatively simple problems. Typically, we have to use Boltzmann's equation, which describes the temporal and spatial evolution of any given f_s , to derive continuum conservation principals.

$$\text{Number density (of species)} \equiv dn_s \equiv \frac{dN_s}{d\bar{x}} \equiv f_s(\bar{c}, \bar{x}, t) d\bar{c} \quad (5)$$

Thus, dn_s is the number of particles of species s per unit volume having velocities between (c_1, c_2, c_3) and $(c_1 + dc_1, c_2 + dc_2, c_3 + dc_3)$. The total number density is found from (5) by integration over all possible velocities:

$$\begin{aligned} n_s &= n_s(\bar{x}, t) = \iiint_{-\infty}^{\infty} f_s dc_1 dc_2 dc_3 \\ &= \iiint_{-\infty}^{\infty} f_s d\bar{c} \end{aligned} \quad (6)$$

Note, physically and mathematically, (6) requires that $f_s(\bar{c}, \bar{x}, t) \rightarrow 0$ as $|\bar{c}| \rightarrow \pm\infty$.

Mass Partial Density of Species $s \equiv \rho_s \equiv \rho_s(\bar{x}, t) \equiv n_s(\bar{x}, t)m_s$

Where m_s is $\frac{\text{mass}}{\text{particle}}$

$$\text{Total Mass Density} \equiv \rho = \sum_s \rho_s = \sum_s n_s m_s \quad (8)$$

Temperature of species s is defined based on the kinetic energy of s :

$$\frac{\text{(total random kinetic energy)/s}}{\text{Unit volume}} = \frac{3}{2} n_s k T_s \quad (9)$$

Where K =Boltzmann's Constant

This definition arises from the equipartition of energy theorem which states that each particle degree of freedom has an energy $\frac{1}{2} kT$ associated with it. Thus, since kinetic energy is the energy of translation and since there are 3 translational degrees of freedom (1 for each coordinate direction), then the kinetic energy of each particle s is $3(\frac{1}{2} kT_s) = \frac{3}{2} kT_s$. Hence the total kinetic energy (per unit volume) of specie s @ position $\bar{x}$ and time t is:

$$n_s \left(\frac{3}{2} kT_s \right) = \frac{kE_s}{\text{volume}} \quad (10)$$

We can relate T_s to f_s by using definition of kinetic energy:

$$\text{Kinetic Energy of a single particle} = \frac{m_s}{2} c^2$$

Since the number of particles (at $\bar{x}$ and t) having velocities between (c_1, c_2, c_3) and $(c_1 + dc_1, c_2 + dc_2, c_3 + dc_3)$ is $f_s d\bar{c}$, then the kinetic energy of these particles is:

$$(f_s d\bar{c}) \frac{m_s}{2} c^2 = \frac{m_s}{2} c^2 f_s dc_1 dc_2 dc_3 \quad (12)$$

Now the total thermal (i.e. translational) kinetic energy of species s at $\bar{x}$ and time t (per unit volume) is:

$$n_s \left(\frac{3}{2} k T_s \right) = \iiint_{-\infty}^{\infty} \frac{m_s}{2} c^2 f_s dc_1 dc_2 dc_3 \quad (13)$$

Where we have used equation (10) and have integrated equation (12). (Note that $c^2 = (c_1^2 + c_2^2 + c_3^2)$.) The average value of any property A ($A = (c_1, c_2, c_3)$) n (@ position $\bar{x}$, time t) per unit volume is defined as:

$$n_s \langle A \rangle = \iiint_{-\infty}^{\infty} A(c_1, c_2, c_3) f_s dc_1 dc_2 dc_3 \quad (14)$$

Thus (13) can also be written as:

$$n_s \left\langle \frac{m_s}{2} c^2 \right\rangle = n_s \left(\frac{3}{2} k T_s \right) \quad (13a)$$

Fluxes

Let θ_s be some property of species s [mass, m_s ; electric charge, ez_s ; momentum, $m_s \bar{w}$; energy, $\varepsilon_{\text{int},s} + \frac{m_s}{2} w^2$, where z_s = free charge (integer), e = charge = $1.6(10)^{-23}$, $\bar{w}$ = velocity relative to lab frame, $\varepsilon_{\text{int},s}$ = internal energy associated with rotation, vibration, electron motion.] The flux of θ_s

For a single particle through an element of area $d\bar{s}$ is:

$$\theta_s \bar{w}_o d\bar{s} = \text{Flux of } \theta_s \text{ due to motion of single particle through } d\bar{s}$$

The total flux for all particles through $d\bar{s}$ is given by:

$$\left(\iiint_{-\infty}^{\infty} (\theta_s \bar{w} f_s d\bar{w}) \right) d\bar{s} = \text{total flux of } \theta_s \text{ through } d\bar{s} = \text{'flux'} \quad (16)$$

Where we have express f_s in terms of $\bar{w}$ (the relationship between $\bar{c}$ and $\bar{w}$ will be given below). In (16), the term in () is just the sum of all fluxes due to particles of all velocities (where position $\bar{x}$ in $f_s(\bar{w}, \bar{x}, t)$ is at or near $d\bar{s}$).

From (14), (16) can be expressed as:

$$\text{Flux} = \left(\iiint_{-\infty}^{\infty} (\theta_s \bar{w} f_s d\bar{w}) \right) d\bar{s} = n_s \langle \theta_s \bar{w} \rangle d\bar{s} \quad (17)$$

We will call $n_s \langle \theta_s \bar{w} \rangle$ the flux vector, where:

$$n_s \langle \theta_s \bar{w} \rangle = \iiint_{-\infty}^{\infty} (\theta_s \bar{w} f_s d\bar{w}) \quad (18)$$

Particle flux ($\theta_s = 1$):

$$\iiint (\bar{w} f_s d\bar{w}) = n_s \langle \bar{w} \rangle I_s = \text{Particle flux vector species s} \quad (19)$$

Mass flux ($\theta_s = m_s$):

$$\begin{aligned}
 \iint \int (m_s \bar{w} f_s d\bar{w}) &= n_s \langle \bar{w} m_s \rangle I_s \\
 &= n_s m_s \langle \bar{w} \rangle I_s && \text{= Mass flux vector species s} \\
 &= \rho_s \langle \bar{w} \rangle I_s \longrightarrow \text{Since } n_s m_s = \rho_s \\
 &= \dot{m}_s
 \end{aligned} \tag{20}$$

Note that total mass flux is given by:

$$\dot{m} = \sum_s \dot{m}_s = \sum_s \rho_s n_s \langle \bar{w} \rangle I_s \tag{21}$$

We will define average velocity (relative to lab frame) of species s as:

$$\bar{v}_s = \langle \bar{w}_s \rangle = \frac{1}{n_s} \iint \int (\bar{w} f_s d\bar{w}) \tag{22}$$

Thus from (21) and (22):

$$\dot{m} = \sum_s \dot{m}_s = \sum_s \rho_s \langle \bar{w} \rangle I_s = \sum_s \rho_s \bar{v}_s \tag{23}$$

Mass average velocity=

$$\begin{aligned}
 \bar{v} &= \frac{\dot{m}}{\rho} = \frac{\sum_s \dot{m}_s}{\sum_s \rho_s} \\
 &= \frac{\sum_s \rho_s \bar{v}_s}{\sum_s \rho_s} && \text{'bulk velocity'} \\
 &= \frac{\sum_s \rho_s \bar{v}_s}{\sum_s \rho_s}
 \end{aligned} \tag{24}$$

Thermal velocities (velocity relative to bulk velocity) we can define a particles thermal velocity relative to the mass average velocity:

$$\bar{c}_s = \bar{w}_s - \bar{v} \tag{25}$$

Or relative to the average velocity of species s:

$$\bar{c}'_s = \bar{w}_s - \bar{v}_s \tag{26}$$

(Where $\bar{w}_s$ is particles velocity relative to lab frame). The distribution function f_s can be expressed in terms of $\bar{w}$, $\bar{c}'$, or $\bar{c}$; all are equivalent.

Exercise 1:

Show that $\langle \bar{c}'_s \rangle = 0$:

$$\begin{aligned}
 \langle \bar{c}'_s \rangle &= \frac{1}{n_s} \iint \int \bar{c}'_s f_s(\bar{c}'_s, \bar{x}, t) d\bar{c}'_1 d\bar{c}'_2 d\bar{c}'_3 \\
 &= \frac{1}{n_s} \iint \int (\bar{w}_s - \bar{v}_s) f_s d\bar{c}'(*)
 \end{aligned}$$

Now

$$\bar{v}_s = \langle \bar{w}_s \rangle = \frac{1}{n_s} \iiint \bar{w}_s \hat{f}_s d\bar{w}$$

Where $\hat{f}_s$ is f_s expressed in terms of $\bar{w}$ (and where $\bar{c}'_s = \bar{w}_s - \bar{v}_s$).

Now:

$$\begin{aligned} dc'_1 &= d(w_1 - v_{s1}) \\ &= dw_1 \longrightarrow \boxed{\text{Since } v_{s1} \text{ is fixed}} \\ dc'_2 &= dw_2 \\ dc'_3 &= dw_3 \end{aligned}$$

Thus, (*) can be expressed as:

$$\bar{c}'_s = \frac{1}{n_s} \iiint (\bar{w}_s - \bar{v}_s) \hat{f}_s d\bar{w}$$

$$\begin{aligned} \langle \bar{c}'_s \rangle &= \frac{1}{n_s} \iiint \bar{w}_s \hat{f}_s d\bar{w} - \frac{1}{n_s} \iiint \bar{v}_s \hat{f}_s d\bar{w} \\ &= \langle \bar{w}_s \rangle I_s - \bar{v}_s \left(\frac{1}{n_s} \iiint \hat{f}_s d\bar{w} \right) \\ &= \bar{v}_s - \bar{v}_s \end{aligned}$$

$$\langle \bar{w}_s \rangle I_s = \bar{v}_s$$

Since and

$$\langle \bar{c}'_s \rangle = 0$$

$$\iiint \hat{f}_s d\bar{w} = n_s$$

Exercise 2: Show that $\langle \bar{c}'_s \rangle$ is the difference between the species average velocity ($\bar{v}_s$) and the mass average velocity ($\bar{v}$).

$$\begin{aligned} \langle \bar{c}'_s \rangle &= \frac{1}{n_s} \iiint \bar{c}'_s \hat{f}_s d\bar{c} \\ &= \frac{1}{n_s} \iiint (\bar{w} - \bar{v}) \hat{f}_s(\bar{c}'_s, \bar{x}, t) d\bar{c} \\ &= \frac{1}{n_s} \iiint (\bar{w} - \bar{v}) \hat{f}_s(\bar{c}'_s, \bar{x}, t) d\bar{w} \\ &= \frac{1}{n_s} \iiint \bar{w} \hat{f}_s d\bar{w} - \bar{v}_s \\ \langle \bar{c}'_s \rangle &= \bar{v}_s - \bar{v} \end{aligned}$$

Note that $\hat{f}_s(\bar{c}'_s, \bar{x}, t) = \hat{f}_s(\bar{w} - \bar{v}, \bar{x}, t) = \hat{f}_s(\bar{w}, \bar{x}, t)$

$\langle \bar{c}_s \rangle$ is called the diffusion velocity. This definition recognizes that any difference between the average velocity of s ($\bar{v}_s$) and the average bulk velocity ($\bar{v}$) has to be due to diffusion:

$$\langle \bar{c}_s \rangle = \bar{v}_s = \text{Diffusion velocity of specie s (} \bar{x} \text{ \& } \bar{t} \text{)} \quad (27)$$

Exercise 3: Show that $\dot{m} = \sum_s \rho_s \bar{v}_s + \bar{v} \sum_s \rho_s$

Exercise 4: Show that since $\dot{m} = \bar{v} \sum_s \rho_s = \bar{v} \rho$, $\sum_s \rho_s \bar{v}_s = 0$ (refer to exercise 3).

Thus the sum of all the specie diffusion fluxes is always zero.

Electrical current= charge flux $\Rightarrow \theta_s = e z_s$; z_s = free charge of species s
(example: for Na^+ : $z_{\text{Na}} = 1, e = 1.602(10^{-19})$; for an electron (e^-):
 $z_e = -1, e = 1.602(10^{-19})$; for Ar: $z_{\text{Ar}} = 0$).

$$\bar{J}_s = n_s \bar{v}_s e z_s = \text{Charge flux due to motion of species s} \quad (28)$$

$$\bar{J} = \sum_s \bar{J}_s = \sum_s n_s \bar{v}_s e z_s$$

$$= \text{Total charge flux (current) due to motion of all charged species} \quad (29)$$

Alternative form of (28):

$$\bar{J}_s = n_s \bar{v}_s e z_s = n_s e z_s (\bar{v}_s + \bar{v}) \quad (28a)$$

Alternative form of (29):

$$\bar{J} = \sum_s \bar{J}_s = \sum_s n_s e z_s (\bar{v}_s) + \sum_s n_s e z_s \bar{v}$$

$$\bar{J} = \sum_s n_s e z_s (\bar{v}_s) + \sum_s n_s e z_s \bar{v}$$

$$= \bar{J} + \bar{v} \rho e$$

$$= \text{current due to diffusion (conduction) + current due to convection} \quad (29a)$$

Where:

$$\bar{J} = \sum_s n_s e z_s (\bar{v}_s) = \text{conduction current}$$

$$= \text{current due to specie diffusion}$$

$$\rho e = \sum_s n_s e z_s = \text{free charge density}$$

Momentum Flux $\theta_s = m_s \bar{w}$

Exercise 5: Starting with the definition momentum flux of s = $\iiint m_s \bar{w} \bar{w} \hat{f}_s d\bar{w}$ show

that $\iiint m \rho_s \bar{w} \bar{w} \hat{f}_s d\bar{w} = \rho_s (\bar{v} \bar{v} + \bar{v}_s \bar{v} + \bar{v} \bar{v}_s) = \rho_s \langle \bar{c} \bar{c} \rangle_s$ where the term on the right is a dyadic.

For simplicity, let's use shorthand notation to represent the triple integral ($\iiint$):

$$\int = \iiint$$

Now recall that $\bar{c} = \bar{w} - \bar{v}$ thus:

$$\int m \rho_s \bar{w} \bar{w} \hat{f}_s d\bar{w} = \int m_s (\bar{v} + \bar{c})(\bar{v} + \bar{c}) \hat{f}_s d\bar{w}$$

Also,

$$\hat{f}_s(\bar{w} - \bar{v}, \bar{x}, t) = \hat{f}_s(\bar{v} + \bar{c}, \bar{x}, t) = \hat{f}_s(\bar{c}, \bar{x}, t) \text{ (since } \bar{v} \text{ isn't volume)}$$

And

$$\begin{aligned} d\bar{w} &= dw_1 dw_2 dw_3 = d(v_1 + c_1) d(v_2 + c_2) d(v_3 + c_3) \\ &= dc_1 dc_2 dc_3 \end{aligned}$$

$$\begin{aligned} \int m_s \bar{w} \bar{w} \hat{f}_s d\bar{w} &= \int m_s (\bar{v} + \bar{c})(\bar{v} + \bar{c}) \hat{f}_s d\bar{c} \\ &= \int m_s \bar{v} \bar{v} \hat{f}_s d\bar{c} + \int m_s \bar{v} \bar{c} \hat{f}_s d\bar{c} + \int m_s \bar{c} \bar{v} \hat{f}_s d\bar{c} + \int m_s \bar{c} \bar{c} \hat{f}_s d\bar{c} \\ &= m_s \bar{v} \bar{v} n_s + m_s \bar{v} \langle \bar{c} \rangle n_s + m_s \left(\int \bar{c} \hat{f}_s d\bar{c} \right) \bar{v} + m_s \langle \bar{c} \bar{c} \rangle n_s \end{aligned}$$

$$\downarrow$$

$\bar{v}_s$

$$\rho_s \bar{v} \bar{v} + \rho_s \bar{v} \bar{v}_s + \rho_s \bar{v}_s \bar{v} + \rho_s \langle \bar{c} \bar{c} \rangle \quad (30^*)$$

We can attach physical meaning to each term in the last equation:

$\rho_s \bar{v} \bar{v}$ = partial fraction of total mass-average momentum (= $\rho \bar{v} \bar{v}$)

$\rho_s \bar{v} \bar{v}_s$ = diffusion at moment flux

$\rho_s \bar{v}_s \bar{v}$ = moment flux due to diffusion

$\rho_s \langle \bar{c} \bar{c} \rangle$ = see below

Note: word definitions of 2nd and 3rd terms are somewhat arbitrary. These will drop out when we sum all specie momentum fluxes to obtain overall moment flux.

The last term is the momentum flux component due to the random motion of species s; it is thus interpreted as the partial stress tensor of species s.

$$\rho_s \langle \bar{c}\bar{c} \rangle = [P_s] = \rho_s \begin{pmatrix} (c_1 c_1) & (c_1 c_2) & (c_1 c_3) \\ (c_2 c_1) & (c_2 c_2) & (c_2 c_3) \\ (c_3 c_1) & (c_3 c_2) & (c_3 c_3) \end{pmatrix}$$

Or in index notation:

$$P_{ijs} = \rho_s (c_i c_j)$$

We will identify the trace of $[P_s]$ with the partial mechanical pressure P_s (this will be shown to be true when we derive conservation of momentum equation).

Thus:

$$P_s = \frac{1}{3} (P_{11s} + P_{22s} + P_{33s}) = \frac{1}{3} P_s \sum_{i=1}^3 \langle c_i c_i \rangle_s \quad (33)$$

If we assume that species s behaves as an ideal gas (i.e. no forces between individual particles negligible overall particle volume) then:

$$P_s = n_s k T_s \quad (34)$$

Total moment flux is obtained by summing (30*) over all species in plasma:

$$\text{T.M.F.} = \rho_s \bar{v} \bar{v} = \sum_s [P_s] \quad (35)$$

$$\rho_s \bar{v} \bar{v} + [P]$$

Where $[P]$ = total stress tensor.

$$[J] = \text{Shear stress tensor} \equiv [P] - P[I] \quad (36)$$

Or

$$J_{ij} = P_{ij} - P \delta_{ij} = \rho_s (c_i c_j) \quad (36a)$$

Where $[I]$ is the identity matrix (all off-diagonal terms =0, all on-diagonal terms =1).

$$\text{Energy Flux: } \theta_s = \frac{1}{2} m_s w^2 + \varepsilon_{\text{int}_s}$$

Energy flux = $\theta_s = \int (\frac{1}{2} m_s w^2 + \varepsilon_{\text{int}_s}) \bar{w} f_s d\bar{w}$ to obtain (due to motion of s) a

physically meaningful expressions replace $\bar{w}$ with $\bar{v} + \bar{c}$ (and $d\bar{w}$ with $d\bar{c}$):

$$= \int (\frac{1}{2} m_s (\bar{v} + \bar{c})(\bar{v} + \bar{c})(\bar{v} + \bar{c}) f_s d\bar{c} + \theta_s = \int \varepsilon_{\text{int}_s} (\bar{v} + \bar{c}) f_s d\bar{c}$$

$$(\hat{f}_s = \hat{f}_s(\bar{c}, \bar{x}, t))$$

$$\text{Energy Flux:} = \frac{1}{2} m_s v^2 \bar{v} n_s + \frac{1}{2} m_s v^2 \langle \bar{c} \rangle I_s n_s + m_s \bar{v} \int (\bar{v} \bar{c}) f_s d\bar{c} + m_s \int (\bar{v} \bar{c}) \bar{c} \hat{f}_s d\bar{c}$$

$$+ \frac{1}{2} m_s \int (\bar{c} \bar{c}) \bar{v} \hat{f}_s d\bar{c} + \frac{1}{2} m_s \int (\bar{c} \bar{c}) \bar{c} \hat{f}_s d\bar{c} + \bar{v} \int \varepsilon_{\text{int}_s} \hat{f}_s d\bar{c} + \int \varepsilon_{\text{int}_s} \bar{c} \hat{f}_s d\bar{c}$$

$$\begin{aligned}
&= \left(\frac{1}{2}\rho_s v^2\right)\bar{v} + \frac{1}{2}\bar{v}\rho_s v^2\bar{v}_s + \bar{v}m_s\bar{v}n_s\langle\vec{c}\rangle I_s + m_s\bar{v}\int\bar{c}\hat{c}\hat{f}_s d\bar{c} \\
&+ \bar{v}\frac{m_s}{2}\mu_s\langle c^2\rangle I_s + \frac{1}{2}m_s\int(\bar{c}\hat{c})\bar{c}\hat{f}_s d\bar{c} + \bar{v}\langle\epsilon_{\text{int}_s}\rangle n_s + \int\epsilon_{\text{int}_s}\bar{c}\hat{f}_s d\bar{c} \\
&= \bar{v}\left(\frac{1}{2}\rho_s v^2 + \rho_s v^2\bar{v}_s + \frac{\rho_s}{2}\langle c^2\rangle I_s + \langle\epsilon_{\text{int}_s}\rangle n_s\right) + \frac{1}{2}\rho_s v^2\bar{v}_s + \bar{v}[P] \\
&+ \int\bar{c}\left(\frac{1}{2}m_s c^2 + \epsilon_{\text{int}_s}\right)\hat{f}_s d\bar{c}
\end{aligned}$$

Since $P_s = [J_s] + P_s[I]$ and $m_s\langle c^2\rangle I_s \equiv \frac{3}{2}kT_s$, then R.H.S. becomes:

$$\begin{aligned}
\text{R.H.S.} &= \rho_s\bar{v}\left(\frac{v^2}{2} + \bar{v}\bar{v}_s + \frac{3}{2}\frac{kT_s}{m_s} + \frac{P_s}{\rho_s} + \frac{\langle\epsilon_{\text{int}_s}\rangle}{m_s}\right) + \frac{1}{2}\rho_s v^2\bar{v}_s + \bar{v}[J_s] \\
&+ \int\bar{c}\left(\frac{1}{2}m_s c^2 + \epsilon_{\text{int}_s}\right)\hat{f}_s d\bar{c}
\end{aligned}$$

(Note that $\bar{v}[P_s[I]] = P_s\bar{v}$)

Now,

$$\frac{3}{2}\frac{kT_s}{m_s} + \frac{P_s}{\rho_s} + \frac{\langle\epsilon_{\text{int}_s}\rangle}{m_s} = u_s + \frac{P_s}{\rho_s} = \tilde{h}_s$$