

# Convergence of Random Walks to the Wiener Process

Chris Mount

May 7

# Goal: Model the Wiener Process

- ▶ Rescaled random walk  $\rightarrow$  convergence in distribution on compact time intervals. (Donsker)

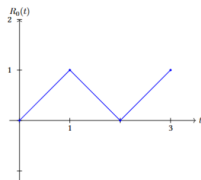
$$\forall \epsilon > 0, \quad \lim_{n \rightarrow \infty} P \left( \left| \frac{S(\lfloor nt \rfloor)}{\sqrt{n}} - W_t \right| > \epsilon \right) = 0.$$

- ▶ Bridges and hats  $\rightarrow$  a.s. convergence in  $L^2$  on compact time intervals. (Fatkullin, 565A)

$$P \left( \lim_{n \rightarrow \infty} \left\| \sum_i^n \sum_j \sqrt{T} \mathcal{N}_{i,j} h_{i,j}(t) - W_t \right\|_2 = 0 \right) = 1.$$

- ▶ (Goal)  $\rightarrow$  a.s. uniform convergence on compact time intervals. [3]

# Constructing the Process



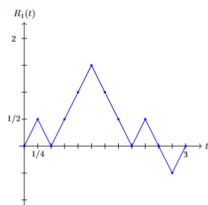
- ▶ Build a stochastic process  $(\Omega_\infty, \mathcal{F}_\infty, P_\infty)$  from a family of probability spaces  $\{(\Omega_k, \mathcal{F}_k, P_k)\}_{k \geq 0}$  [2].
- ▶ Define

$$R_k(t) = \frac{1}{2^k} \sum_{i=1}^{m2^{2k}} X_i,$$

$$X_i \sim \text{Uni}\{-1, +1\}, \quad \frac{m}{2^{2k}} \leq t < \frac{m+1}{2^{2k}}.$$

- ▶ Each  $R_k(t)$  is an event in a probability space  $\Omega_k$  (called refinement level  $k$ ).

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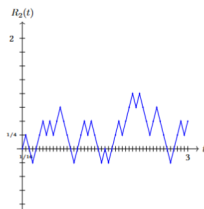
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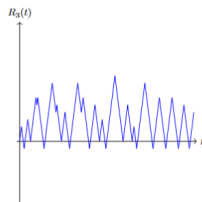
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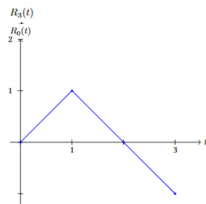
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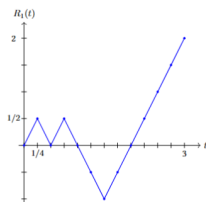
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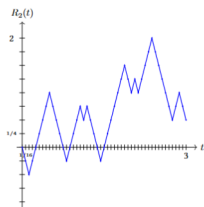
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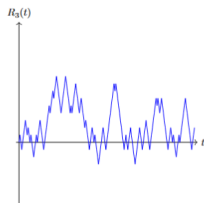
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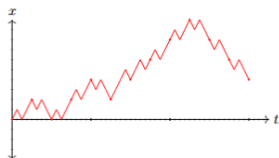
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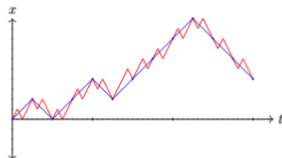


- ▶  $M_k : \Omega_k \rightarrow \Omega_{k-1}$  maps between levels by eliminating all points not on the  $k - 1$  grid and then eliminating any remaining points that are equal to their immediate predecessor.
- ▶  $M_k$  has an inverse, but it is not unique.
- ▶ Require that

$$P_{k-1}(E) = P_k(M_k^{-1}(E))$$

$$\forall E \in \mathcal{F}_{k-1}, \quad \forall k \geq 0.$$

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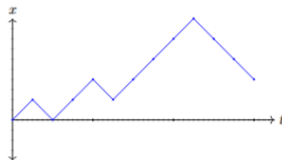


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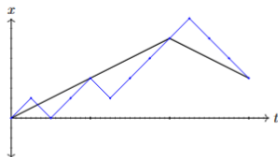


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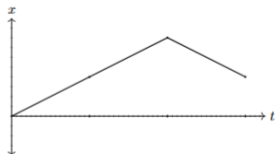


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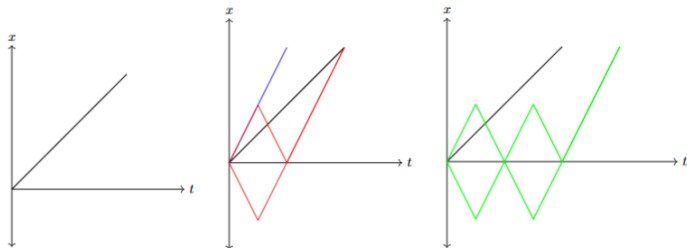


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## Constructing the Process, cont.



Consider the inverse image of a single step at grid size  $k = 0$ . There is one path, in blue, that makes it to 1 in two steps on the finer grid. There are two paths, in red, that make it to 1 in four steps in the finer grid. There are four paths, in green, that make it to 1 in six steps in the finer grid.

$$\frac{1}{2^2} + \frac{2}{2^4} + \frac{4}{2^6} + \cdots = \frac{1}{2}.$$

# Analyzing the Time Lag

- Time lag from level  $k$  to  $r$  across  $m2^{-2k}$  number of steps is

$$L_{k,r}(m2^{-2k}) = E_{r-1} \cdots E_k(m)2^{-2r} - m2^{-2k}, \quad r > k,$$

$E_{k-1}(m)$  is the random least integer such that for  $M_k \omega_k = \omega_{k-1}$ , the coarser path  $\subset$  the finer path.

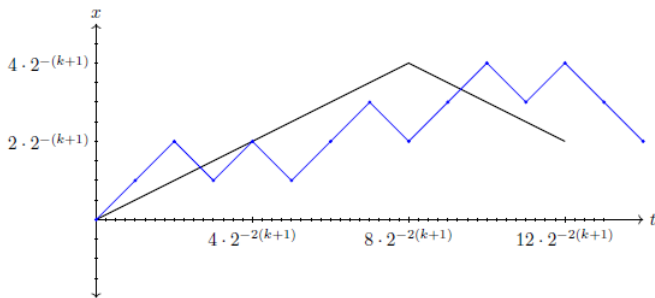


Figure 1:  $L_{k,k+1}(3) = (E_k(3)/4 - 3)2^{-2k} = (\frac{2+8+4}{4} - 3)2^{-2k} = \frac{1}{2} \cdot 2^{-2k}$ .

## Analyzing the Time Lag, cont.

- ▶ For a single step, across one level of refinement, the time lag is

$$L_{k,k+1}(2^{-2k}) = E_k(1)2^{-2(k+1)} - 2^{-2k}.$$

- ▶  $E_k(1) \sim 2(X+1)$ , with  $X \sim \text{Geo}(1/2) \implies$  an a.s. and uniform bound on the time lag for time  $t \in [0, m2^{-2k}]$ : For all  $\delta > 0$ ,

$$P_\infty \left[ \max_{0 \leq t \leq m2^{-2k}} |L_{k,r}(t)| > \delta \right] < \frac{m2^{-4k+1}}{3\delta^2}.$$

- ▶ We can get uniformity w.r.t.  $r$  for the bound on the time lag: For fixed  $k$  and  $m$ , and  $\delta = 2^{-2k+1}$ ,

$$P_\infty \left[ \sup_{0 \leq t \leq m2^{-2k}, k \leq r} |L_{k,r}(t)| > \delta \right] < C$$

where  $C$  depends on  $k$  and  $m$ .

# Defining the Limiting Process

- ▶  $\lim_{r \rightarrow \infty} L_{k,r}(m2^{-2k})$  exists a.s.

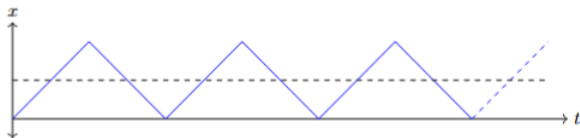
- ▶ The times

$$m2^{-2k} + \lim_{r \rightarrow \infty} L_{k,r}(m2^{-2k})$$

are a.s. dense in  $(0, \infty)$ .

- ▶ Define

$$R_{\infty}(t) = \begin{cases} \lim_{k \rightarrow \infty} R_k(t), & \text{if time lag limit exists} \\ 0, & \text{otherwise} \end{cases}$$



# Main Result

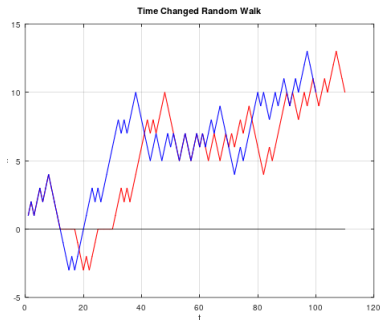
- ▶  $R_\infty(t)$  is a.s. continuous on dense times, hence uniformly continuous on finite intervals.
- ▶  $R_\infty(t)$  is a separable Brownian motion.
- ▶ For all  $\epsilon, \delta > 0$ , and  $k$  large:

$$P_\infty \left[ \max_{0 \leq t \leq T, r \geq k} |R_\infty(t) - R_r(t)| > \delta \right] < D,$$

where  $D$  depends on  $T, k, \delta$ .

# Implications and Extensions

- ▶ This method is not efficient for simulation, but it is useful for studying path-dependent functionals (local time) of Brownian motion.
- ▶ The work can be extended to diffusion processes with speed measures that are AC w.r.t. Lebesgue measure.
- ▶ With a time change, this work applies to diffusions with speed measures with a point mass, i.e., a sticky Brownian motion [1].



# References

- [1] Madjid Amir. “Sticky Brownian motion as the strong limit of a sequence of random walks”. In: *Stochastic Processes and their Applications* 39.2 (Dec. 1991), pp. 221–237. URL: <https://ideas.repec.org/a/eee/spapps/v39y1991i2p221-237.html>.
- [2] Salomon Bochner. *Harmonic Analysis and the Theory of Probability*. California Monographs in Mathematical Sciences. Berkeley and Los Angeles: University of California Press, 1955.
- [3] Frank B. Knight. “On the Random Walk and Brownian Motion”. In: *Transactions of the American Mathematical Society* 103.2 (1962), pp. 218–228. DOI: 10.2307/1993657. URL: <https://www.jstor.org/stable/1993657>.



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