

① QUANTUM STATE CONVERSION FACTOR

② EXAMPLE calc. of $g(E)$ Lbr. 21

Via complex inversion

③ ORDER MAG. ANALYSIS OF $g(E)$

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I) Important Asides:

Conversion factor = no. of

QUANTUM STATES per unit

Volume, ^{dw,} of classical

Phase Space.

INFORMATION

A) ITS SHOWN, VIA A COUPLE

OF EXAMPLE calculations

IN chapter I (as well as

IN EARLY COURSE COURSE

NOTES) THAT WITHIN

A CLASSICAL PHASE SPACE
VOLUME ———

space
coords
ptcl. 1

space
coords
ptcl. 2

$$dw = dq_1 dq_2 dq_3 dq_4 dq_5 dq_6 \dots$$

$$dq_1 dq_2 dq_3 \dots dq_{3N-1} dq_{3N} \cdot dp_1 dp_2 dp_3 \dots$$

coords
ptcl. N

MOMENTUM
components
ptcl. 1

$$\dots dp_{3N-2} dp_{3N-1} dp_{3N}$$

MOMENTUM
components
ptcl. N

THAT THE NUMBER OF
QUANTUM STATES WITH dw IS

$$(2.1) \left[\frac{dw}{N! h^{3N}} \right] \equiv \begin{array}{l} \text{NO. OF QUANTUM} \\ \text{STATES WITH} \\ \text{VOLUME } dw \end{array}$$

B) SINCE WE COUNT ^{PAY ATTENTION TO} QUANTUM
STATES IN STAT THERMO,
THEN WHEN USING THE
PHASE SPACE APPROACH FOR
STUDYING CLASSICAL SYSTEMS
($\equiv$ CALCULATING THEIR THERMODYNAMICS)
WE SHOULD, STRICTLY SPEAKING
INCLUDE THE CONVERSION FACTOR (2.1).

C) YOU CAN SHOW THAT
OMITTING THE CONVERSION
FACTOR (2.1) FROM A CLASSICAL

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$$\text{AND } \frac{V}{10^3} = \frac{V}{\left(\frac{2\pi m k T}{h}\right)^3} \gtrsim 10^{24}$$
$$\lambda_0 = 10 \text{ \AA}, \quad \frac{V}{\lambda_0^3} = \frac{10^{-6} \text{ m}^3 (10^3)}{10^{-27}} = 10^{24}$$

SAFELY WORKING (2.1).

$$\text{Log of } Q_v(v, T) = \frac{1}{N_i!} \left[\frac{v}{h^3} (2\pi m k T)^{3/2} \right]^N$$

$$O(2.3(10^{23})) \quad O(10^{23})$$

$$\ln Q_N (= -\frac{A}{kT}) = -N \ln N + N + N \ln (4/10^3)$$

$$\downarrow \qquad \qquad \downarrow$$

$$O(10^{23}) \quad O(24)$$

D) AS AN EXERCISE, you should ④
 verify EQS. (3.5.8) - (3.5.14) in
 the text, starting w/ EQS. (3.5.5)
 and (3.5.6).

II) QUICK review of forms
 of PARTITION FN. Q.

text

3.4 and
 3.5

A) FOR QUANTUM SYSTEM DESCRIBABLE
 IN TERMS OF DISCRETE

ENERGY STATES: (i.e. FOR $\lambda_D = \frac{h}{\sqrt{2\pi m kT}}$
 $= 0.1 \text{ \AA}$) (i.e., LOW TEMPS):

$$(4.1) \quad Q = \sum_i g_i e^{-\beta E_i} = f_n(V, T, N)$$

B) FOR QUANTUM SYSTEMS WHERE QUANTUM
 ENERGY LEVELS ARE CLOSELY SPACED

effectively continuous. eg., for a quantum
 ideal gas where $\frac{8\pi m^{3/2} E}{h^2} \equiv \frac{2(n_x^2 + n_y^2 + n_z^2)}{\lambda_D^2}$
 total qss $\frac{8\pi m^{3/2} E}{h^2} \approx \frac{N E^2}{\lambda_D^2} \gg 1$, then

$$(4.2) \quad Q = \int_0^\infty g(E) e^{-\beta E} dE = f_n(V, T, N)$$

(explanation: we showed that when $L/\lambda_D \gg 1$
 then the no. of ^{quantum} states on/near the surface of g

3N-dim
 Q^n hypersphere (defined by

$$R^2 = \sum_{i=1}^N (n_{xi}^2 + n_{yi}^2 + n_{zi}^2) = \frac{8m^{3/2} E_{\text{kin}} n_1 n_2}{h^2}$$

is very large. This fact suggests use of ^{cont.} continuous $g(E)$.

c) CLASSICAL SYSTEMS: (i.e., one where $\lambda_0 \gg \sim 1 \text{ \AA}$)

$$Q = \frac{1}{N! h^{3N}} \int e^{-B H(p, q)} \left(\prod_i d^3 q_i d^3 p_i \right)$$

D) SIGNIFICANCE OF $Q = Q(N, V, T)$:
 COMES VIA THE BRIDGE RELATION

$$A = -KT \ln Q$$

which we derived before

III)

Relationship Between
 $Q(N, \beta, T)$ and $g(\epsilon)$.

A) AS SHOWN ON PAPER (sec. 3.4),
we can infer / calculate
the density of quantum states, $g(\epsilon)$,
given $Q(N, \beta, T)$ using the
inversion theorem for
LAPLACE TRANSFORMS.

i) SINCE

$$Q(N, \beta, T) = \int_0^{\infty} g(\epsilon) e^{-\beta \epsilon} d\epsilon$$

then Q IS THE LAPLACE
TRANSFORM OF $g(\epsilon)$ (WITH
 β PLAYING THE ROLE OF
 s IN THE LT.)

ii) THUS,

INVERSION
THEOREM

$$g(\epsilon) = \frac{1}{2\pi i} \int_{\beta' - i\infty}^{\beta' + i\infty} Q(\beta', N, T) e^{\beta' \epsilon} d\beta'$$

IV) EXAMPLE CALCULATION OR
 SEP 15 ⑦
 $g(E)$ FOR THE CASE OF

A QUANTUM IDEAL GAS OF N PARTICLES
 IN A BOX OF VOLUME, $V = L^3$.

[see SEC. 14 FOR MORE/BACKGROUND]
 FOR THE ^{PROTON} CALCN., see SEC 3.5 text.

i) FROM ^{BQ} (3.5.17)

$$Q(B) = Q(N, V, T) = \frac{1}{N!} \left(\frac{V}{h^3} \right)^N \left(\frac{2\pi m}{\beta} \right)^{3N/2}$$

ii) $\therefore$

$$g(E) = \frac{1}{2\pi i} \int_{B' - i\infty}^{B' + i\infty} \frac{1}{N!} \left(\frac{V}{h^3} \right)^N \left(\frac{2\pi m}{\beta} \right)^{3N/2} \frac{e^{\beta E}}{\beta^{3N/2}} d\beta$$

iii) use and verify Kubo's
 approach for evaluating the
 above integral:

write

$$\int_{s'-i\infty}^{s'+i\infty} \frac{e^{sE}}{B^{3N/2}} dB$$

$$e^{sE}$$

$$B^{3N/2}$$

$$dB$$

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a)

$$\int_{s'-i\infty}^{s'+i\infty} \frac{e^{sE}}{s^{n+1}} ds$$

(so that $n+1 = 3N/2$)

b) From the Residue theorem of the theory of complex vbls)

$$\int_{s'-i\infty}^{s'+i\infty} \frac{e^{sE}}{s^{n+1}} ds = \frac{2\pi i}{n!} \lim_{s \rightarrow 0} \left[\frac{d^n}{ds^n} \left(s^{n+1} \frac{e^{sE}}{s^{n+1}} \right) \right]$$

$$b) \left. \frac{d^n}{ds^n} e^{sE} \right|_{s \rightarrow 0}$$

there is a pole of order $n+1$ at $s=0$

$$\frac{d^2}{ds^2} e^{sE} = \frac{d}{ds} (E e^{sE}) = E^2 e^{sE}$$

$$\therefore \frac{d^n}{ds^n} e^{sE} = E^n e^{sE}$$

$$\therefore \left. \frac{d^n}{ds^n} e^{sE} \right|_{s \rightarrow 0} = E^n$$

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$$\Rightarrow \int_{s'-i\infty}^{s'+i\infty} \frac{e^{s\epsilon}}{s^{n+1}} ds = \frac{2\pi i}{n!} \epsilon^n \quad (n = \frac{3N}{2} - 1)$$

$$c) \therefore \boxed{g(E) = \frac{1}{N_0!} \left(\frac{V}{h^3}\right)^N (2\pi m)^{3N/2} \frac{E^{(\frac{3N}{2}-1)}}{(\frac{3N}{2}-1)!}}$$

d) LET'S DO A QUICK ORDER
OF MAGNITUDE ESTIMATE FOR
 $g(E)$

$$i) E = N \frac{3}{2} kT \quad \left(\begin{array}{l} \text{equipartition} \\ \text{theorem for} \\ N\text{-ptcl. \& GAS} \end{array} \right)$$

$$ii) \therefore g(E) \approx \frac{1}{N! N!} \left(\frac{V}{h_0^3}\right)^N N^{\frac{3N}{2}}$$

iii) Choose $\frac{V}{h_0^3} = O(1)$; Also take
Log of ii):

$$\begin{aligned} \ln g(E) &\approx N \ln(V/h_0^3) + \\ &\quad + (-N \ln N + N)^2 + \frac{3}{2} N \ln N \\ &\approx N^2 (\ln N)^2 - 2N^2 \ln N + N^2 \\ &\quad + \frac{3}{2} N \ln N \end{aligned}$$

$$\ln g(\varepsilon) \approx \oplus N^2 (\ln N)^2$$

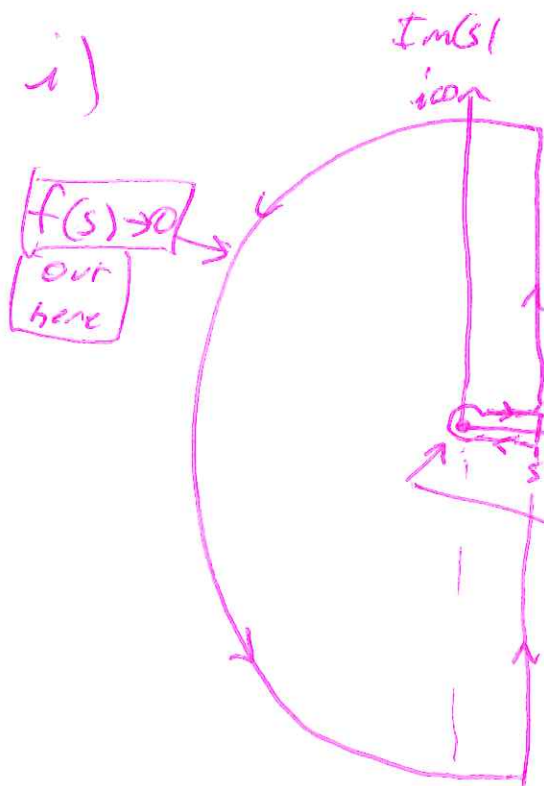
$$\therefore g(\varepsilon) \approx e^{N^2 (\ln N)^2} \Rightarrow g(\varepsilon) \text{ is } \underline{\underline{\text{huge}}}$$

e) Note on integration of

let $I \equiv \int_{s' - i\infty}^{s' + i\infty} \frac{e^{s\varepsilon}}{s^{n+1}} ds$:

let $f(s) = \frac{e^{s\varepsilon}}{s^{n+1}}$

i)



ii) $\therefore$ From the Residue theorem

$$* \quad \boxed{I = 2\pi i \times (\text{residue at } s=0)} \\ = 2\pi i \times r_1$$

iii) For a pole of order $n+1$ @ $s=0$,

$$r_1 = \frac{1}{n!} \lim_{s \rightarrow 0} \frac{d^n}{ds^n} \left[\frac{s^{n+1} e^{s\varepsilon}}{s^{n+1}} \right] = \frac{\varepsilon^n}{n!}$$