

8.6 - Representations of Functions as Power Series

Given a function $f(x)$, we want a "power series representation" of f (i.e. a power series that equals f on its IOC)

ex: $\sum_{n=0}^{\infty} x^n \leftarrow \text{cogs in } (-1, 1)$

Also $\sum_{n=0}^{\infty} x^n = \frac{d}{1-r} = \boxed{\frac{1}{1-x}}$

$\uparrow$ P.S. representation $\uparrow$ function

We can use this to find P.S. representations of functions similar to $\frac{1}{1-x}$

ex: $\frac{1}{1-(x^3)} = \sum_{n=0}^{\infty} (x^3)^n$

$$= \sum_{n=0}^{\infty} x^{3n}$$

← this converges if $|x^3| < 1$
 $-1 < x < 1$

ex: $\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n$

$$= \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

get in form

" $\frac{1}{1-r}$ "

converges if $|-x^3| < 1$
 $-1 < x < 1$

ex: $\frac{1}{2+x^3} = \frac{1/2}{1+\frac{x^3}{2}} = \frac{1}{2} \left(\frac{1}{1+\frac{x^3}{2}} \right)$

$$= \frac{1}{2} \left(\frac{1}{1-\left(-\frac{x^3}{2}\right)} \right)$$

↓ $r = -\frac{x^3}{2}$

geometric
series test

cvgs if

$$\left| -\frac{x^3}{2} \right| < 1$$

$$\left| \frac{x^3}{2} \right| < 1$$

$$|x^3| < 2$$

$$\sqrt[3]{|x|^3} < \sqrt[3]{2}$$

$$|x| < \sqrt[3]{2}$$

$$-\sqrt[3]{2} < x < \sqrt[3]{2}$$

↑
IOC



$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x^3}{2} \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} (-1)^n \frac{x^{3n}}{2^n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{2^{n+1}}$$

ex: $\frac{x^2}{2+x^3} = x^2 \cdot \left(\frac{1}{2+x^3} \right)$ ← from prev. example

$$= x^2 \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{3n}}{2^{n+1}}$$

← Same IOC

insert into the series

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{3n+2}}{2^{n+1}}$$

$(-\sqrt[3]{2}, \sqrt[3]{2})$

ex: $\frac{x^3}{1+x^4} = x^3 \cdot \left(\frac{1}{1+x^4} \right)$

★ establish your 1's

$$= x^3 \left(\frac{1}{1 - (-x^4)} \right)$$

$$= x^3 \cdot \sum_{n=0}^{\infty} (-x^4)^n$$

$$= x^3 \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

$$= \sum_{h=0}^{\infty} (-1)^n x^{4n+3}$$

$$x^3 \cdot \sum_{h=0}^{\infty} (-1)^n x^{4n}$$

$$= x^3 (1 - x^4 + x^8 - x^{12} + \dots)$$

$$= x^3 - x^7 + x^{11} - x^{15} + \dots$$

Differentiating & Integrating Power Series (term by term)

Given

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1 (x-a) + C_2 (x-a)^2 + \dots$$

with a radius of convergence R , then

$f(x)$ is differentiable on $(a-R, a+R)$

$$\text{and } f'(x) = \frac{d}{dx} (C_0 + C_1 (x-a) + C_2 (x-a)^2 + \dots)$$

$$= 0 + C_1 + C_2 \cdot 2(x-a) + C_3 \cdot 3(x-a)^2 + \dots$$

$$= \sum_{n=1}^{\infty} C_n \cdot n (x-a)^{n-1}$$

↑
b/c we lose the constant term
when we differentiate

Can also integrate term by term:

$$\int \left(\sum_{n=0}^{\infty} c_n (x-a)^n \right) dx = \left(\sum_{n=0}^{\infty} c_n \cdot \frac{1}{n+1} (x-a)^{n+1} \right) + C$$
$$= C + \sum_{n=0}^{\infty} c_n \cdot \frac{(x-a)^{n+1}}{n+1}$$

a new P.S. representation
for $\int f(x) dx$

The P.S. representations for $f'(x)$ and $\int f(x) dx$ have the SAME radius of convergence as $f(x)$. BUT the interval can change (i.e. behavior at the end points might be different)

$$\boxed{(a-R, a+R] \xrightarrow{\frac{d}{dx}} (a-R, a-R)}$$

Tricks so far: try to make f look like

$$\frac{1}{1-r}$$

ex: $f(x) = \ln(3-x)$

$$f'(x) = \frac{1}{3-x} \cdot (-1) = \frac{-1}{3-x}$$

$$= - \left(\frac{1}{3-x} \right)$$

$$= - \left(\frac{1/3}{1 - x/3} \right)$$

$$= - \frac{1}{3} \left(\frac{1}{1 - x/3} \right)$$

$$= - \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3} \right)^n$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{3} \right) \frac{x^n}{3^n}$$

$$= \sum_{n=0}^{\infty} \left(-\frac{x^n}{3^{n+1}} \right)$$

cvgs if $\left| \frac{x}{3} \right| < 1 \rightarrow$

$$|x| < 3$$

$$-3 < x < 3$$

↑
P.S. rep. for $f'(x)$
but we want f

$$\begin{aligned} f(x) &= \int \sum_{n=0}^{\infty} \left(-\frac{x^n}{3^{n+1}} \right) dx = C + \sum_{n=0}^{\infty} \left(-\frac{\frac{1}{n+1} \cdot x^{n+1}}{3^{n+1}} \right) \\ &= C + \sum_{n=0}^{\infty} -\frac{x^{n+1}}{(n+1)3^{n+1}} \end{aligned}$$

↑
need to find C

Remember :

$$f(x) = \ln(3-x) = C + \sum_{n=0}^{\infty} \left(-\frac{x^{n+1}}{(n+1)3^{n+1}} \right)$$

↓ plug in $x=0$ (base point)

$$\ln(3-0) = C + 0$$

$$\text{so } \boxed{C = \ln(3)}$$

P.S. representation for

$$f(x) = \ln(3) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+1}}{(n+1)3^{n+1}} \right)$$

and IOC is $(-3, 3)$

↑ ↗
we know
endpoints
are not included,
don't have to
check (because we
used geometric
series test)